

Nonlinear Spline Approximation in $BMO(\mathbb{R})$

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We study nonlinear approximation in BMO from splines generated by a hierarchy of B-splines over regular multilevel nested partitions of \mathbb{R} . Companion Jackson and Bernstein estimates are established that allow to completely characterize the associated approximation spaces.

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1. Introduction

The space of functions of bounded mean oscillation (BMO) is a natural replacement of L^∞ in many problems in Analysis and, in particular, in Approximation theory. Here we consider nonlinear approximation from regular splines generated by a nested hierarchy of B-splines in BMO on \mathbb{R} .

As usual the space BMO is defined as the set of all functions $f \in L^1_{loc}(\mathbb{R})$ such that

$$\|f\|_{BMO} := \sup_I \frac{1}{|I|} \int_I |f(x) - \text{Avg}_I f| dx < \infty, \quad \text{Avg}_I f := \frac{1}{|I|} \int_I f(x) dx,$$

where the sup is over all intervals $I \subset \mathbb{R}$. As is well known the set $C_0(\mathbb{R})$ of all continuous functions with compact support is not dense in BMO. Therefore, it is natural to approximate in the BMO-norm functions that belong to the space VMO, the closure of the space $C_0(\mathbb{R})$ in the BMO-norm, see [2].

In this article we consider nonlinear n -term approximation from B -splines generated by multilevel nested partitions of \mathbb{R} . More specifically, let $\{\mathcal{I}_m\}_{m \in \mathbb{Z}}$ be a sequence of families of intervals such that each level \mathcal{I}_m is a partition of \mathbb{R} into compact intervals with disjoint interiors and a refinement of the previous level \mathcal{I}_{m-1} . We consider regular partitions of this sort, i.e. we require

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that the intervals from each level \mathcal{I}_m be of comparable length. Define $\mathcal{I} := \cup_{m \in \mathbb{Z}} \mathcal{I}_m$. Each such multilevel partition \mathcal{I} generates a ladder of spline spaces $\cdots \subset S_{-1}^k \subset S_0^k \subset S_1^k \subset \cdots$ of degree $k - 1$, where S_m^k is spanned by B-splines $\{\varphi_Q\}$. We denote by \mathcal{Q}_m the supports of the m th level B-splines φ_Q and set $\mathcal{Q} := \cup_{m \in \mathbb{Z}} \mathcal{Q}_m$.

We are interested in approximating from the nonlinear set Σ_n of all splines of the form

$$g = \sum_{Q \in \Lambda_n} c_Q \varphi_Q, \quad \Lambda_n \subset \mathcal{Q}, \quad \#\Lambda_n \leq n.$$

Here the index set Λ_n is allowed to vary with g . The approximation error is defined by

$$\sigma_n(f)_{\text{BMO}} := \inf_{g \in \Sigma_n} \|f - g\|_{\text{BMO}}.$$

The Besov spaces $\dot{B}_\tau^{\alpha,k}$ with $\alpha > 0$, $\tau := 1/\alpha$, and $k \in \mathbb{N}$, play a crucial role here. In the case when $\tau \geq 1$ the space $\dot{B}_\tau^{\alpha,k}$ consists of all functions $f \in L_{\text{loc}}^\tau(\mathbb{R})$ such that $\Delta_h^k f \in L^\tau(\mathbb{R})$, $\forall h \in \mathbb{R}$, and

$$\|f\|_{\dot{B}_\tau^{\alpha,k}} := \left(\int_0^\infty [t^{-\alpha} \omega_k(f, t)_\tau]^\tau \frac{dt}{t} \right)^{1/\tau} < \infty, \quad \omega_k(f, t)_\tau := \sup_{|h| \leq t} \|\Delta_h^k f(\cdot)\|_{L^\tau(\mathbb{R})}.$$

In the case when $\tau < 1$ the above definition the Besov space $\dot{B}_\tau^{\alpha,k}$ is not quite satisfactory to us. We believe that in principle the use of $\omega_k(f, t)_\tau$ with $\tau < 1$, as well as polynomial approximation in L^τ , $\tau < 1$, should be avoided when possible. For this reason we define the Besov space $\dot{B}_\tau^{\alpha,k}$ when $\tau < 1$ via local polynomial approximation in L^q , $q \geq 1$, (see Definition 3.2).

Clearly, $\|f + P\|_{\dot{B}_\tau^{\alpha,k}} = \|f\|_{\dot{B}_\tau^{\alpha,k}}$ for all $P \in \Pi_k$, the set of all algebraic polynomials of degree $k - 1$. Hence, $\dot{B}_\tau^{\alpha,k}$ consists of equivalence classes modulo Π_k . As will be shown (Theorem 3.1) for any function $f \in \dot{B}_\tau^{\alpha,k}$ there exists a polynomial $P \in \Pi_k$ such that $f - P \in \text{VMO}$. We shall be assuming that each $f \in \dot{B}_\tau^{\alpha,k}$ is the *canonical representative* $f - P \in \text{VMO}$ of the equivalence class modulo Π_k generated by f .

Our primary goal in the article is to prove the following Jackson and Bernstein estimates (Theorems 4.1 and 4.2): Let $\alpha > 0$, $\tau := 1/\alpha$, $k \geq 2$. If $f \in \dot{B}_\tau^{\alpha,k}$, then $f \in \text{VMO}$ and

$$\sigma_n(f)_{\text{BMO}} \leq cn^{-\alpha} \|f\|_{\dot{B}_\tau^{\alpha,k}}, \quad n \geq 1,$$

and for any $g \in \Sigma_n$

$$\|g\|_{\dot{B}_\tau^{\alpha,k}} \leq cn^\alpha \|g\|_{\text{BMO}}.$$

As is well known these two estimates imply a complete characterization of the approximation spaces associated to this sort of approximation (see Theorem 4.3 below).

To achieve our objectives we first develop in sufficient detail the theory of the Besov spaces $\dot{B}_\tau^{\alpha,k}$. In particular we derive representations of these

spaces in terms of B-splines, local polynomial approximation, and spline quasi-interpolants.

We also compare the nonlinear spline approximation in BMO with the spline approximation in the uniform norm and in L^p , $1 \leq p < \infty$.

There is a considerable difference between nonlinear spline approximation in BMO in dimension $d = 1$ and in dimension $d > 1$. A detailed discussion and clarification of this phenomenon will be given in a followup article.

Observe that nonlinear approximation in $\text{BMO}(\mathbb{R}^d)$ from dyadic piecewise polynomial functions has been studied by I. Irodova, see [12] and the references therein. In this case dyadic Besov spaces are used for characterization of the rates of approximation.

There is a close relationship between nonlinear approximation from splines and wavelets in BMO. We develop the nonlinear n -term wavelet approximation in BMO in [14].

Outline. This article is organized as follows. In Section 2 we introduce our setting and collect all facts we need concerning splines, local polynomial approximation, spline quasi-interpolants. The Besov spaces involved in nonlinear spline approximation in BMO are studied in Section 3. In Section 4 we state our main results on spline approximation in BMO and we compare them in Section 5 with the existing results on spline approximation in the uniform norm and in L^p , $1 \leq p < \infty$. Sections 6 and 7 contain the proofs of our Jackson and Bernstein estimates. Section 8 is an appendix, where we place the proofs of some statements from previous sections.

Notation. We shall use the notation $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$. $C_0(\mathbb{R})$ will stand for the set of all continuous and compactly supported functions on \mathbb{R} . Given a measurable set $A \subset \mathbb{R}$ we denote by $|A|$ its Lebesgue measure and by $\mathbb{1}_A$ its characteristic function. For an interval J and $\lambda > 0$ we denote by λJ the interval of length $\lambda|J|$ which is co-centric with J . As usual \mathbb{Z} will denote the set of all integers, \mathbb{N} will be the set of all positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Also, Π_k will stand for the set of all univariate algebraic polynomials of degree $k - 1$. Unless specified otherwise, all functions are complex-valued. Positive constant will be denoted by c, c', c_1, \dots and they may vary at every occurrence; $a \sim b$ will stand for $c_1 \leq a/b \leq c_2$.

2. Preliminaries

In this section we collect all facts regarding the BMO space, B-splines, local approximation, quasi-interpolants, and other results that will be needed in the sequel.

2.1. BMO and VMO Spaces

As usual the space BMO on \mathbb{R} is defined as the set of all locally integrable functions on \mathbb{R} such that

$$\|f\|_{BMO} := \sup_J \frac{1}{|J|} \int_J |f(x) - \text{Avg}_J f| dx < \infty, \quad \text{Avg}_J f := \frac{1}{|J|} \int_J f(x) dx, \quad (2.1)$$

where the sup is over all compact intervals J and $|J|$ is the length of J . Note that $\|\cdot\|_{BMO}$ is not a norm because $\|g\|_{BMO} = 0$ if $g = \text{constant}$ ($g \in \Pi_1$). For this reason we identify each $f \in BMO$ with $f + a$, $a = \text{constant}$, and view BMO as a subset of $L^1_{loc}(\mathbb{R})/\Pi_1$. Then $\|\cdot\|_{BMO}$ is a norm on BMO.

From the well known John-Nirenberg inequality [15] it follows that any function $f \in BMO$ is in $L^p_{loc}(\mathbb{R})$, $1 < p < \infty$, and

$$\sup_J \left(\frac{1}{|J|} \int_J |f(x) - \text{Avg}_J f|^p dx \right)^{1/p} \sim \sup_J \frac{1}{|J|} \int_J |f(x) - \text{Avg}_J f| dx = \|f\|_{BMO} \quad (2.2)$$

with constants of equivalence depending only on p .

As is well known BMO is *not a separable space*. However, the space VMO, defined as the closure of $C_0(\mathbb{R})$ in the BMO norm (see Section 4 in [2]), is a *separable* Banach space. As in BMO the elements of VMO are classes of equivalence modulo constants. We shall consider spline approximation of functions in VMO.

2.2. Nested Partitions of \mathbb{R}

We say that

$$\mathcal{I} = \cup_{m \in \mathbb{Z}} \mathcal{I}_m$$

is a *regular multilevel partition* of \mathbb{R} into subintervals with levels $\{\mathcal{I}_m\}$ if the following three conditions are obeyed:

(a) Each level \mathcal{I}_m is a partition of \mathbb{R} , i.e. $\mathbb{R} = \cup_{I \in \mathcal{I}_m} I$, and \mathcal{I}_m consists of compact intervals with disjoint interiors.

(b) The levels $\{\mathcal{I}_m\}$ of \mathcal{I} are nested, i.e. \mathcal{I}_{m+1} is a refinement of \mathcal{I}_m , and each $I \in \mathcal{I}_m$ has at least two and at most M_0 children in \mathcal{I}_{m+1} , where $M_0 \geq 2$ is a constant independent of m .

(c) There exists a constant $\lambda \geq 1$ such that

$$|I'| \leq \lambda |I''|, \quad \forall I', I'' \in \mathcal{I}_m, \quad \forall m \in \mathbb{Z}. \quad (2.3)$$

The dyadic intervals $\mathcal{D} = \cup_{m \in \mathbb{Z}} \mathcal{D}_m$, $\mathcal{D}_m = \{[2^{-m}k, 2^{-m}(k+1)] : k \in \mathbb{Z}\}$, are an example of a regular multilevel partition of \mathbb{R} with $\lambda = 1$ and $M_0 = 2$.

Other scenarios. Another version of the above setting is when the multilevel partition \mathcal{I} of \mathbb{R} is of the form $\mathcal{I} = \cup_{m=0}^{\infty} \mathcal{I}_m$, where the levels \mathcal{I}_m satisfy conditions (a)–(c) from above.

Yet a third version of this setting is when we have a regular multilevel partition $\mathcal{I} = \cup_{m \geq 0} \mathcal{I}_m$ of a fixed compact interval J obeying conditions similar to (a)–(c) above and J is the only element of \mathcal{I}_0 .

Our theory can readily be adjusted to each of these settings. We shall stick to the first setting on \mathbb{R} from above.

A couple of remarks are in order.

(1) There is a natural tree structure in \mathcal{I} induced by the inclusion relation.

(2) Conditions (a)–(c) imply that there exist constants $0 < r < \rho < 1$ such that if $I \in \mathcal{I}_m$, $I' \in \mathcal{I}_{m+1}$, and $I' \subset I$, then

$$r \leq |I'|/|I| \leq \rho. \quad (2.4)$$

In fact, inequality (2.4) is derived with bounds $1/(M_0\lambda - \lambda + 1) \leq r \leq 1/2$, $\rho \leq \lambda/(\lambda + 1)$. Note that the upper bound for ρ and the lower bound for r cannot be simultaneously achieved provided $M_0 \geq 3$.

We denote by $x_{m,j}$, $j \in \mathbb{Z}$, the knots of \mathcal{I}_m , i.e. $\mathcal{I}_m = \{[x_{m,j}, x_{m,j+1}] : j \in \mathbb{Z}\}$, where

$$\cdots < x_{m,-1} < x_{m,0} < x_{m,1} < \cdots.$$

With this notation the nested condition (b) can be described as

$$\{x_{m,j} : j \in \mathbb{Z}\} \subset \{x_{m+1,j} : j \in \mathbb{Z}\}, \quad m \in \mathbb{Z}.$$

Observe that inequality (2.4) implies that every regular multilevel partition \mathcal{I} may have either zero or one common knot for all levels \mathcal{I}_m . Dyadic intervals \mathcal{D} are example with one common knot – the origin.

For a fixed integer $k \geq 2$ we denote by \mathcal{Q}_m the collection of all unions of k consecutive intervals from \mathcal{I}_m , i.e.

$$\mathcal{Q}_m = \{Q = [x_{m,j}, x_{m,j+k}] : j \in \mathbb{Z}\}, \quad m \in \mathbb{Z}. \quad (2.5)$$

Further, set $\mathcal{Q}(\mathcal{I}) = \mathcal{Q} := \cup_{m \in \mathbb{Z}} \mathcal{Q}_m$. The intervals from \mathcal{Q} are the supports of the B -splines of degree $k - 1$ defined below in §2.3. We shall use both \mathcal{I} and \mathcal{Q} as index sets of the objects discussed in this article.

To every $I \in \mathcal{I}$ we associate the set

$$\Omega_I := \cup\{Q : Q \in \mathcal{Q}_m, Q \supset I\}, \quad I \in \mathcal{I}_m. \quad (2.6)$$

In other words, if $I = [x_{m,j}, x_{m,j+1}]$ then $\Omega_I = [x_{m,j+1-k}, x_{m,j+k}]$. In view of condition (c) of the regular multilevel partition \mathcal{I} we always have

$$|I| \sim |Q| \sim |\Omega_I|, \quad \forall I \in \mathcal{I}_m, Q \in \mathcal{Q}_m, m \in \mathbb{Z}.$$

Given a regular multilevel partition \mathcal{I} , we define *the level of a compact interval J* as the largest $\nu \in \mathbb{Z}$ such that \mathcal{I}_ν has no more than one knot in the interior of J . Observing that $\mathcal{I}_{\nu+1}$ has at least two knots in the interior of J we infer from (2.4) that $|J| \sim |I|$ for every $I \in \mathcal{I}_\nu$. Also for every $m \leq \nu$ we have two adjacent intervals I_1 and I_2 in \mathcal{I}_m such that $J \subset I_1 \cup I_2$ and $|I_1| \sim |I_2| \geq c|J|$.

2.3. Splines over Nested Partitions of \mathbb{R}

Piecewise polynomials and splines. Assuming that \mathcal{I} is a regular multilevel partition of \mathbb{R} , we denote by $\tilde{\mathcal{S}}_m^k := \tilde{\mathcal{S}}^k(\mathcal{I}_m)$, $m \in \mathbb{Z}$, $k \geq 1$, the set of all piecewise polynomial (possibly discontinuous) functions over \mathcal{I}_m of degree $k - 1$, i.e.

$$S \in \tilde{\mathcal{S}}_m^k \quad \text{if} \quad S = \sum_{I \in \mathcal{I}_m} \mathbb{1}_I \cdot P_I,$$

where $\mathbb{1}_I$ is the characteristic function of I and $P_I \in \Pi_k$. S is assumed to be right-continuous at the knots of \mathcal{I}_m . Then the space of the m th level splines is defined by

$$\mathcal{S}_m^k := \tilde{\mathcal{S}}_m^k \cap C^{k-2}, \quad k \geq 2. \quad (2.7)$$

B-splines. Given $Q := [x_{m,j}, x_{m,j+k}] \in \mathcal{Q}_m$, $m \in \mathbb{Z}$, (see (2.5)) we denote by φ_Q the B -spline of degree $k - 1$ with knots $x_{m,j}, \dots, x_{m,j+k}$; these are $k + 1$ consecutive knots of \mathcal{I}_m . For the precise definition of φ_Q , see e.g. [6, Chapter 5, (2.7)]. Note that: (i) Q is the support of φ_Q and (ii) $\|\varphi_Q\|_\infty \sim 1$ with constants of equivalence depending only on k and λ . We denote by $V_Q := \{x_{m,j}, \dots, x_{m,j+k}\}$ the knots of φ_Q .

It will be convenient to index the B -splines of degree $k - 1$ by their supports. Thus, given a regular multilevel partition \mathcal{I} , the collection of all B -splines of degree $k - 1$ is

$$\Phi = \Phi(\mathcal{I}) := \{\varphi_Q : Q \in \mathcal{Q}(\mathcal{I})\}.$$

As is well known $\{\varphi_Q : Q \in \mathcal{Q}_m\}$ is a basis for \mathcal{S}_m^k . Each $S \in \mathcal{S}_m^k$ has a unique representation

$$S = \sum_{Q \in \mathcal{Q}_m} \beta_Q(S) \varphi_Q.$$

According to de Boor – Fix theorem (see e.g. [6, Chapter 5, Theorem 3.2]) the coefficient $\beta_Q(S)$ is a linear functional given by:

$$\beta_Q(S) := \sum_{\nu=0}^{k-1} (-1)^\nu \varpi_Q^{(k-\nu-1)}(\xi_Q) S^{(\nu)}(\xi_Q), \quad (2.8)$$

$$\varpi_Q(x) := \frac{1}{(k-1)!} \prod_{\nu=j+1}^{j+k-1} (x - x_{m,\nu}), \quad Q = [x_{m,j}, x_{m,j+k}],$$

where ξ_Q is an arbitrary point from $(x_{m,j}, x_{m,j+k})$. The value of $\beta_Q(S)$ in (2.8) is independent of the choice of ξ_Q .

From condition (c) on the multilevel partition \mathcal{I} it readily follows that

$$|\beta_Q(S)| \leq c \|S\|_{L^\infty(Q)}, \quad Q \in \mathcal{Q}_m.$$

This implies that (see e.g. [4, Lemma 2.3] in the case \mathbb{R}^2) if $S = \sum_{Q \in \mathcal{Q}_m} b_Q \varphi_Q$, where $\{b_Q\}$ is an arbitrary sequence of complex numbers, then

$$\|S\|_p \sim \left(\sum_{Q \in \mathcal{Q}_m} \|b_Q \varphi_Q\|_p^p \right)^{1/p}, \quad 0 < p \leq \infty.$$

Moreover, for any $0 < p, \tau \leq \infty$

$$\left(\sum_{I \in \mathcal{I}_m} [\|S\|_{L^p(I)}]^\tau \right)^{1/\tau} \sim \left(\sum_{Q \in \mathcal{Q}_m} [\|b_Q \varphi_Q\|_p]^\tau \right)^{1/\tau} \quad (2.9)$$

with the usual modification when $\tau = \infty$. Note that $\beta_Q(S) = b_Q$.

2.4. Local Polynomial Approximation

Recall first some simple properties of polynomials that will be frequently used.

Lemma 2.1. *Let $P \in \Pi_k$, $k \geq 1$, and $0 < p, q \leq \infty$.*

(a) *For any compact interval J*

$$|J|^{-1/p} \|P\|_{L^p(J)} \sim |J|^{-1/q} \|P\|_{L^q(J)} \quad (2.10)$$

with constants of equivalence depending only on k and $\min\{p, q\}$.

(b) *Let $J' \subset J$ be two intervals such that $|J| \leq (1 + \delta)|J'|$, $\delta > 0$. Then*

$$\|P\|_{L^p(J)} \leq c \|P\|_{L^p(J')}$$

with $c = c(p, k, \delta)$.

(c) *For any compact interval J*

$$\|P'\|_{L^p(J)} \leq c |J|^{-1} \|P\|_{L^p(J)}$$

with $c = c(p, k)$.

Applied to B-splines Lemma 2.1 (a), (c) imply

$$|Q|^{-1/q} \|\varphi_Q\|_q \sim \|\varphi_Q\|_\infty \sim 1, \quad \|\varphi'_Q\|_p \sim |Q|^{-1} \|\varphi_Q\|_p.$$

For a function $f \in L^p(J)$, defined on a compact interval $J \subset \mathbb{R}$, $1 \leq p \leq \infty$, and $k \geq 1$, we define

$$E_k(f, J)_p := \inf_{P \in \Pi_k} \|f - P\|_{L^p(J)}. \quad (2.11)$$

Also, we denote by $\omega_k(f, J)_p$ the k -th modulus of smoothness of f on J :

$$\omega_k(f, J)_p := \sup_{h \in \mathbb{R}} \|\Delta_h^k(f, \cdot, J)\|_{L^p(J)}, \quad (2.12)$$

where

$$\Delta_h^k(f, x, J) := \begin{cases} \sum_{j=0}^k (-1)^{k+j} \binom{k}{j} f(x + jh), & \text{if } [x, x + kh] \subset J; \\ 0, & \text{otherwise.} \end{cases} \quad (2.13)$$

Lemma 2.2 (Whitney). *If $f \in L^p(J)$, $1 \leq p \leq \infty$, and $k \geq 1$, then*

$$E_k(f, J)_p \leq c\omega_k(f, J)_p.$$

From above and $\Delta_h^k(P, x, J) = 0$ for all $P \in \Pi_k$ it readily follows that

$$E_k(f, J)_p \sim \omega_k(f, J)_p, \quad \forall f \in L^p(J), \quad 1 \leq p \leq \infty. \quad (2.14)$$

The following useful property of moduli of smoothness is well known:

$$\omega_k(f, J)_p^p \sim \frac{1}{|J|} \int_0^{|J|} \int_J |\Delta_h^k(f, x, J)|^p dx dh, \quad 1 \leq p < \infty. \quad (2.15)$$

For proofs of the above claims and further details, see e.g. [20, § 7.1].

We find useful the concept of *near best approximation*. A polynomial $P_J(f) \in \Pi_k$ is said to be a polynomial of near best $L^p(J)$ -approximation to f from Π_k with constant $A \geq 1$ if

$$\|f - P_J(f)\|_{L^p(J)} \leq AE_k(f, J)_p.$$

Note that since $p \geq 1$ a near best $L^p(J)$ -approximation $P_J(f)$ (with an appropriate A) can be easily realized by a linear projector, see below.

Lemma 2.3. [6, Chapter 12, Lemma 6.2] *Suppose $1 \leq q < p \leq \infty$ and P_J is a polynomial of near best $L^q(J)$ -approximation to $f \in L^p(J)$ from Π_k . Then P_J is a polynomial of near best $L^p(J)$ -approximation to f .*

Local projectors onto polynomials. Given $1 \leq p \leq \infty$ and a compact interval J , we let $P_{J,p} : L^p(J) \rightarrow \Pi_k$ be a *linear projector* such that

$$\|f - P_{J,p}(f)\|_{L^p(J)} \leq AE_k(f, J)_p, \quad \forall f \in L^p(J), \quad (2.16)$$

where the constant $A \geq 1$ is independent of J and f . Note that (2.16) implies $\|P_{J,p}\|_{L^p(J) \rightarrow L^p(J)} \leq 1 + A$, i.e. $P_{J,p}$ is a bounded linear operator. $P_{J,p}$ can be defined via the averaged Taylor polynomial, see e.g. [1, Section 4.1], which is a linear operator for fixed J, k that does not depend on p (cf. Lemma 2.3).

Remark 2.1. As is well know most of the above approximation results are valid in the wider range $p > 0$. The restriction $p \geq 1$ reflects our believe that polynomial approximation of functions in L^p , $p < 1$, is not natural. There is no linear operator realization of polynomial near best approximation in L^p , $p < 1$, because there are no continuous linear functional in L^p , $p < 1$. Hence, the polynomial approximation in L^p , $p < 1$, is *nonconstructive*. It should be replaced by approximation in the Hardy space H^p , $p < 1$. However, as will be seen later on the use of polynomial approximation in L^p or H^p , $p < 1$, can be avoided completely in nonlinear spline approximation in BMO .

2.5. Quasi-interpolant

We define the linear operator $T_m : \tilde{\mathcal{S}}_m^k \rightarrow \mathcal{S}_m^k$ by

$$T_m(S) := \sum_{Q \in \mathcal{Q}_m} \beta_Q(S) \varphi_Q, \quad \forall S \in \tilde{\mathcal{S}}_m^k,$$

where $\beta_Q(S)$ are defined in (2.8) with the additional requirement ξ_Q does not coincide with a knot of \mathcal{Q}_m . As observed in § 2.3 the identity $T_m(S) = S$ for every $S \in \mathcal{S}_m^k$ holds independently of the choice of ξ_Q . But for $S \in \tilde{\mathcal{S}}_m^k \setminus \mathcal{S}_m^k$ the value of $\beta_Q(S)$ may depend on the interval $(x_{m,\nu}, x_{m,\nu+1})$, $\nu = j, \dots, j+k-1$, containing ξ_Q . In order to have well defined operator T_m from now on for every $Q \in \mathcal{Q}_m$ we fixed the interval $I \in \mathcal{I}_m$, $I \subset Q$, containing ξ_Q in its interior.

From (2.8) it follows that

$$|\beta_Q(S)| \leq c \|S\|_{L^\infty(Q)}, \quad \forall S \in \tilde{\mathcal{S}}_m^k, \quad Q \in \mathcal{Q}_m,$$

which easily leads to

$$\|T_m(S)\|_{L^p(I)} \leq c \|S\|_{L^p(\Omega_I)}, \quad \forall S \in \tilde{\mathcal{S}}_m^k, \quad I \in \mathcal{I}_m, \quad 1 \leq p \leq \infty, \quad (2.17)$$

where Ω_I is given in (2.6).

We next extend T_m to $L_{\text{loc}}^p(\mathbb{R})$. Let $P_{J,p} : L^p(J) \rightarrow \Pi_k$ be the projector from (2.16). Define

$$\mathcal{P}_{m,p}(f) := \sum_{I \in \mathcal{I}_m} \mathbb{1}_I \cdot P_{I,p}(f), \quad f \in L_{\text{loc}}^p(\mathbb{R}), \quad (2.18)$$

(the precise values of $\mathcal{P}_{m,p}(f)$ at the knots of \mathcal{I}_m are not of importance for this study) and set

$$T_{m,p}(f) := T_m(\mathcal{P}_{m,p}(f)), \quad f \in L_{\text{loc}}^p(\mathbb{R}). \quad (2.19)$$

Lemma 2.4. *If $f \in L_{\text{loc}}^p(\mathbb{R})$, $1 \leq p \leq \infty$, $m \in \mathbb{Z}$, then*

$$\|f - T_{m,p}(f)\|_{L^p(I)} \leq c E_k(f, \Omega_I)_p, \quad \forall I \in \mathcal{I}_m.$$

Proof. Let $R \in \Pi_k$ be such that $\|f - R\|_{L^p(\Omega_I)} \leq c E_k(f, \Omega_I)_p$. Then

$$\begin{aligned} \|f - T_{m,p}(f)\|_{L^p(I)} &= \|f - T_m(\mathcal{P}_{m,p}(f))\|_{L^p(I)} \\ &\leq \|f - R\|_{L^p(I)} + \|R - T_m(\mathcal{P}_{m,p}(f))\|_{L^p(I)} \\ &= \|f - R\|_{L^p(I)} + \|T_m[\mathcal{P}_{m,p}(R - f)]\|_{L^p(I)} \\ &\leq \|f - R\|_{L^p(\Omega_I)} + c \|\mathcal{P}_{m,p}(R - f)\|_{L^p(\Omega_I)} \\ &\leq c \|f - R\|_{L^p(\Omega_I)} \leq c E_k(f, \Omega_I)_p. \end{aligned}$$

Here we used (2.17) and the boundedness of $\mathcal{P}_{m,p}$. □

2.6. Embedding of Sequence Type B-spline Spaces into BMO

The sequence $\ell^\tau = \ell^\tau(\mathcal{Q})$ space indexed by \mathcal{Q} , $0 < \tau \leq \infty$, is the set of all sequence $\{a_Q\}_{Q \in \mathcal{Q}}$ of complex numbers such that

$$\|\{a_Q\}\|_{\ell^\tau} := \left(\sum_{Q \in \mathcal{Q}} |a_Q|^\tau \right)^{1/\tau} < \infty.$$

The embedding of the Besov spaces of interest to us in BMO will play a crucial role in this article. This embedding is in essence contained in the following

Theorem 2.1. *Let $\{a_Q\}_{Q \in \mathcal{Q}} \in \ell^\tau(\mathcal{Q})$, $0 < \tau < \infty$.*

(a) *If $\tau > 1$, then $\sum_{Q \in \mathcal{Q}} a_Q \varphi_Q$ converges unconditionally in $BMO(\mathbb{R})$ and*

$$\left\| \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \right\|_{BMO} \leq c \|\{a_Q\}\|_{\ell^\tau}. \quad (2.20)$$

Consequently, $\sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \in VMO(\mathbb{R})$.

(b) *If $0 < \tau \leq 1$, then obviously $\sum_{Q \in \mathcal{Q}} a_Q \varphi_Q$ converges absolutely and unconditionally in $L^\infty(\mathbb{R})$ to a continuous function and*

$$\left\| \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \right\|_{BMO} \leq \left\| \sum_{Q \in \mathcal{Q}} |a_Q \varphi_Q| \right\|_\infty \leq c \|\{a_Q\}\|_{\ell^\tau}.$$

The proof of this theorem depends on the following

Lemma 2.5. *Let $0 < p, \tau < \infty$. Then for any sequence $\{a_Q\}_{Q \in \mathcal{Q}}$ and any compact interval $J \subset \mathbb{R}$ we have*

$$\left\| \frac{1}{|J|^{1/p}} \sum_{Q \in \mathcal{Q}, Q \subset J} |a_Q \varphi_Q| \right\|_p \leq c \left(\sum_{Q \in \mathcal{Q}, Q \subset J} |a_Q|^\tau \right)^{1/\tau}. \quad (2.21)$$

This lemma is a consequence of the following well known embedding result (see e.g. [16, Theorem 3.3]).

Proposition 2.1. *If $0 < \tau < p < \infty$, then for any sequence $\{a_Q\}_{Q \in \mathcal{Q}}$ of complex number one has*

$$\left\| \sum_{Q \in \mathcal{Q}} |a_Q \varphi_Q| \right\|_p \leq c \left(\sum_{Q \in \mathcal{Q}} \|a_Q \varphi_Q\|_p^\tau \right)^{1/\tau},$$

where the constant $c > 0$ depends only on p , τ , and the parameter λ from condition (c) on \mathcal{I} .

To streamline our presentation we defer the proofs of Lemma 2.5 and Theorem 2.1 to §8.1 in the appendix.

3. Homogeneous Besov Spaces

In this section, we introduce and discuss the Besov spaces $\dot{B}_\tau^{\alpha,k}$ that will be used for characterization of nonlinear n -term spline approximation in BMO. For the theory of Besov spaces we refer the reader to [18, 22, 9, 10, 11].

Throughout the section we assume that

$$\alpha > 0, \quad k \geq 2, \quad \text{and} \quad \tau := 1/\alpha.$$

3.1. The Homogeneous Besov Space $\dot{B}_\tau^{\alpha,k}$ in the Case $\tau \geq 1$

Definition 3.1. The homogeneous Besov space $\dot{B}_\tau^{\alpha,k} = \dot{B}_\tau^{\alpha,k}(\mathbb{R})$, $1 \leq \tau < \infty$ ($0 < \alpha \leq 1$), is defined as the collection of all functions $f \in L_{\text{loc}}^\tau(\mathbb{R})$ such that $\Delta_h^k f \in L^\tau(\mathbb{R})$ for all $h \in \mathbb{R}$ and

$$\|f\|_{\dot{B}_\tau^{\alpha,k}} := \left(\int_0^\infty [t^{-\alpha} \omega_k(f, t)_\tau]^\tau \frac{dt}{t} \right)^{1/\tau} < \infty, \quad (3.1)$$

where

$$\omega_k(f, t)_\tau := \sup_{|h| \leq t} \|\Delta_h^k f\|_\tau, \quad \Delta_h^k f(x) = \sum_{j=0}^k (-1)^{k+j} \binom{k}{j} f(x + jh).$$

Notice the different definition and notation of finite differences (2.13) and moduli (2.12) on compact interval. Observe that $\|f + P\|_{\dot{B}_\tau^{\alpha,k}} = \|f\|_{\dot{B}_\tau^{\alpha,k}}$ for each polynomial $P \in \Pi_k$.

From the properties of $\omega_k(f, t)_\tau$ it readily follows that

$$\|f\|_{\dot{B}_\tau^{\alpha,k}} \sim \left(\sum_{\nu \in \mathbb{Z}} (2^{\alpha\nu} \omega_k(f, 2^{-\nu})_\tau)^\tau \right)^{1/\tau}. \quad (3.2)$$

It is also easy to see that

$$\|f\|_{\dot{B}_\tau^{\alpha,k}} \sim \left(\sum_{I \in \mathcal{I}} (|I|^{-\alpha} \omega_k(f, \Omega_I)_\tau)^\tau \right)^{1/\tau} \sim \left(\sum_{I \in \mathcal{I}} (|I|^{-\alpha} E_k(f, \Omega_I)_\tau)^\tau \right)^{1/\tau}. \quad (3.3)$$

Recall that $E_k(f, \Omega_I)_\tau \sim \omega_k(f, \Omega_I)_\tau$, see (2.14). For reader's convenience we give a simple proof of equivalence (3.3) in §8.2 in the appendix.

Subsequently, in Theorem 3.2 we shall show the following key equivalence: For any $1 \leq q < \infty$, $\tau \geq 1$,

$$\|f\|_{\dot{B}_\tau^{\alpha,k}} \sim \left(\sum_{I \in \mathcal{I}} (|I|^{-\frac{1}{q}} \omega_k(f, \Omega_I)_q)^\tau \right)^{1/\tau} \sim \left(\sum_{I \in \mathcal{I}} (|I|^{-\frac{1}{q}} E_k(f, \Omega_I)_q)^\tau \right)^{1/\tau}. \quad (3.4)$$

3.2. The Besov Space $\dot{B}_\tau^{\alpha,k}$ in the General Case When $0 < \tau < \infty$

As was alluded to in Definition 2.1 to us polynomial approximation in L^τ , $\tau < 1$, is not normal and should be avoided. Furthermore, we think that even when $\tau \geq 1$ the Besov spaces $\dot{B}_\tau^{\alpha,k}$ are most naturally defined via local polynomial approximation in L^q with $q \geq 1$. The equivalence (3.4) is a motivation for making the following

Definition 3.2. Let \mathcal{I} be a regular multilevel partition of \mathbb{R} (see §2.2). The Besov space $\dot{B}_\tau^{\alpha,k}(E, q)$, $1 \leq q < \infty$, $0 < \tau < \infty$, is defined as the collection of all functions $f \in L_{\text{loc}}^q(\mathbb{R})$ such that

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)} := \left(\sum_{I \in \mathcal{I}} (|I|^{-\frac{1}{q}} \omega_k(f, \Omega_I)_q)^\tau \right)^{1/\tau} \sim \left(\sum_{I \in \mathcal{I}} (|I|^{-\frac{1}{q}} E_k(f, \Omega_I)_q)^\tau \right)^{1/\tau} \quad (3.5)$$

is finite. Here $\omega_k(f, \Omega_I)_q$ and $E_k(f, \Omega_I)_q$ are defined in (2.12) and (2.11) with Ω_I from (2.6).

Lemma 3.1. *The Besov space $\dot{B}_\tau^{\alpha,k}(E, q)$ introduced above is independent of the particular selection of the multilevel partition \mathcal{I} of \mathbb{R} being used.*

Proof. Let \mathcal{I}' be another multilevel partition of \mathbb{R} with the properties of \mathcal{I} . It is readily seen that for every level \mathcal{I}'_m of \mathcal{I}' there exists a level \mathcal{I}_n of \mathcal{I} with these properties: (a) The intervals in \mathcal{I}'_m and \mathcal{I}_n are comparable in length, and (b) For each interval $I' \in \mathcal{I}'_m$ there exists an interval $I \in \mathcal{I}_n$ such that $\Omega_{I'} \subset \Omega_I$. For example, condition (b) is satisfied whenever $\max_{I' \in \mathcal{I}'_m} |I'| \leq 3/2 \min_{I \in \mathcal{I}_n} |I|$. Hence,

$$\sum_{I' \in \mathcal{I}'_m} (|I'|^{-\frac{1}{q}} \omega_k(f, \Omega_{I'})_q)^\tau \leq c \sum_{I \in \mathcal{I}_n} (|I|^{-\frac{1}{q}} \omega_k(f, \Omega_I)_q)^\tau.$$

Clearly, each level \mathcal{I}_n of \mathcal{I} can serve in this capacity for only uniformly bounded number of levels from \mathcal{I}' . The claim follows. \square

In light of (3.3) the spaces $\dot{B}_\tau^{\alpha,k}$ and $\dot{B}_\tau^{\alpha,k}(E, \tau)$ are the same with equivalent norms when $\tau \geq 1$. Furthermore, as will be shown in Theorem 3.2 (3.4) is valid and hence the spaces $\dot{B}_\tau^{\alpha,k}(E, q)$ are the same space for all $1 \leq q < \infty$ with equivalent norms when $\tau \geq 1$. The same theorem extends the equivalence of these spaces for $0 < \tau < \infty$. To achieve this we need some preparation.

3.3. Norms via Projectors

We define

$$\mathfrak{q}_{m,q} := T_{m,q} - T_{m-1,q} \quad \text{for } m \in \mathbb{Z}, \quad (3.6)$$

where $T_{m,q}$ is defined in (2.19). For a given function $f \in L_{\text{loc}}^q(\mathbb{R})$, $1 \leq q < \infty$, clearly $\mathfrak{q}_{m,q}(f) \in \mathcal{S}_m^k$ and we define uniquely the sequence $\{b_{Q,q}(f)\}_{Q \in \mathcal{Q}_m}$ by

$$\mathfrak{q}_{m,q}(f) =: \sum_{Q \in \mathcal{Q}_m} b_{Q,q}(f) \varphi_Q. \quad (3.7)$$

Since the near-best approximant in (2.16) is realized as a linear operator, then, evidently, $\{b_{Q,q}(\cdot)\}$ are linear functionals.

We introduce the following (quasi-)norms for functions $f \in L_{\text{loc}}^q(\mathbb{R})$, $1 \leq q < \infty$, $0 < \tau < \infty$:

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(Q,q)} := \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-1/q} \|b_{Q,q}(f) \varphi_Q\|_q)^\tau \right)^{1/\tau}.$$

By (2.9) we have

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(Q,q)} \sim \left(\sum_{I \in \mathcal{I}} (|I|^{-1/q} \|\mathfrak{q}_{m,q}(f)\|_{L^q(I)})^\tau \right)^{1/\tau}, \quad (3.8)$$

and, since $\|\varphi_Q\|_p \sim |Q|^{1/p}$,

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(Q,q)} \sim \left(\sum_{Q \in \mathcal{Q}} |b_{Q,q}(f)|^\tau \right)^{1/\tau}. \quad (3.9)$$

Lemma 3.2. *If $f \in \dot{B}_\tau^{\alpha,k}(E, q)$, $1 \leq q < \infty$, then*

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(Q,q)} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)}. \quad (3.10)$$

Proof. Let $f \in \dot{B}_\tau^{\alpha,k}(E, q)$. If $I \in \mathcal{I}_j$ and $J \in \mathcal{I}_{j-1}$ is the unique parent of I ($I \subset J$), then by Lemma 2.4

$$\begin{aligned} \|\mathfrak{q}_{j,q}(f)\|_{L^q(I)} &\leq c \|f - T_{j,q}(f)\|_{L^q(I)} + c \|f - T_{j-1,q}(f)\|_{L^q(J)} \\ &\leq c E_k(f, \Omega_I)_q + c E_k(f, \Omega_J)_q, \quad 1 \leq q < \infty. \end{aligned} \quad (3.11)$$

This, (3.8) and (3.5) imply (3.10). \square

As will be shown later $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}(Q,q)}$ is another equivalent norm in $\dot{B}_\tau^{\alpha,k}(E, q)$.

3.4. Decomposition of $\dot{B}_\tau^{\alpha,k}(E, q)$ and Embedding in VMO

Our next step is to derive a representation of the functions in $\dot{B}_\tau^{\alpha,k}(E, q)$ via the quasi-intepolant from (2.19). We first show that the Besov space $\dot{B}_\tau^{\alpha,k}(E, q)$ is embedded in BMO modulo polynomials of degree $k - 1$.

We define the BMO type space $\text{BMO}^{q,k}(\mathbb{R})$, $1 \leq q < \infty$, as the set of all functions $f \in L_{\text{loc}}^q(\mathbb{R})$ such that

$$\|f\|_{\text{BMO}^{q,k}} := \sup_{I \in \mathcal{I}} |I|^{-1/q} \omega_k(f, \Omega_I)_q \sim \sup_{I \in \mathcal{I}} |I|^{-1/q} E_k(f, \Omega_I)_q < \infty.$$

Proposition 3.1. *For any $f \in \text{BMO}^{q,k}(\mathbb{R})$, $1 \leq q < \infty$, there exists a polynomial $P \in \Pi_k$ such that $f - P \in \text{BMO}$ and*

$$\|f - P\|_{\text{BMO}} \leq c\|f\|_{\text{BMO}^{q,k}}. \quad (3.12)$$

Proof. If $\|f\|_{\text{BMO}^{q,k}} = 0$ then f coincides with its polynomial of best approximation on every of the over-lapping intervals Ω_I and (3.12) follows trivially.

Let $\|f\|_{\text{BMO}^{q,k}} > 0$. Denote $J_\nu := [-2^\nu, 2^\nu]$, $\nu \in \mathbb{N}_0$. Evidently, for each $\nu \in \mathbb{N}_0$ there exists an interval $I_\nu \in \mathcal{I}$ such that $J_\nu \subset \Omega_{I_\nu}$ and $|J_\nu| \sim |I_\nu|$. In light of Whitney's theorem (Lemma 2.2), there exists a polynomial $P_\nu \in \Pi_k$ such that

$$\frac{1}{|J_\nu|} \int_{J_\nu} |f(x) - P_\nu(x)|^q dx \leq c|J_\nu|^{-1} \omega_k(f, J_\nu)_q^q \leq c\|f\|_{\text{BMO}^{q,k}}^q. \quad (3.13)$$

Denote $\Upsilon_\nu := P_\nu - P_{\nu-1}$. Since $\Upsilon_\nu \in \Pi_k$ we have for $x \in J_n$, $n \geq 0$, and $\nu \geq n$

$$|\Upsilon_\nu(x) - \Upsilon_\nu(0)| \leq |J_n| \|\Upsilon'_\nu\|_{L^\infty(J_n)} \leq |J_n| \|\Upsilon'_\nu\|_{L^\infty(J_\nu)} \leq c|J_n| |J_\nu|^{-1} \|\Upsilon_\nu\|_{L^\infty(J_\nu)}$$

and using Lemma 2.1 and (3.13)

$$\begin{aligned} \|\Upsilon_\nu\|_{L^\infty(J_\nu)} &\leq |J_\nu|^{-1/q} \|\Upsilon_\nu\|_{L^q(J_\nu)} \leq |J_\nu|^{-1/q} \|\Upsilon_\nu\|_{L^q(J_{\nu-1})} \\ &\leq c|I_\nu|^{-1/q} \omega_k(f, \Omega_{I_\nu})_q + c|I_{\nu-1}|^{-1/q} \omega_k(f, \Omega_{I_{\nu-1}})_q \leq c\|f\|_{\text{BMO}^{q,k}}. \end{aligned}$$

Then for any $n, m \in \mathbb{N}$, $n < m$, we have

$$\begin{aligned} \sum_{\nu=n+1}^m \|\Upsilon_\nu - \Upsilon_\nu(0)\|_{L^\infty(J_n)} &\leq c|J_n| \sum_{\nu=n+1}^m |J_\nu|^{-1} \|\Upsilon_\nu\|_{L^\infty(J_\nu)} \\ &\leq c\|f\|_{\text{BMO}^{q,k}}. \end{aligned} \quad (3.14)$$

From above with $n = 0$ it follows that the series $\sum_{\nu=1}^{\infty} (\Upsilon_\nu - \Upsilon_\nu(0))$ converges uniformly on $J_0 = [-1, 1]$ to some polynomial in Π_k . Hence, there exists a polynomial $P \in \Pi_k$ such that

$$\|P_m - P_m(0) - (P - P(0))\|_{L^\infty(J_0)} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

From this and (3.14) it follows that for any (fixed) $n \in \mathbb{N}$

$$\|P_m - P_m(0) - (P - P(0))\|_{L^\infty(J_n)} \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (3.15)$$

We shall next show that (3.12) holds with the polynomial P from above. Let J be an arbitrary compact interval. Then there exists an interval $\tilde{I}_0 \in \mathcal{I}$ such that $J \subset \Omega_{\tilde{I}_0}$ and $|\tilde{I}_0| \sim |J|$. Let $n \in \mathbb{N}$ be the minimal positive integer such that $J \subset J_n$. Let $\{\tilde{I}_j\}_{j=0}^\ell$ be intervals from consecutive levels of \mathcal{I} such that $\tilde{I}_0 \subset \tilde{I}_1 \subset \dots \subset \tilde{I}_\ell$, \tilde{I}_j is a parent of \tilde{I}_{j-1} , $\tilde{I}_\ell \cap J_n \neq \emptyset$, and $|\tilde{I}_\ell| \sim |J_n|$.

As $n \in \mathbb{N}$ is already fixed we choose $m > n$ so that

$$\|P_m - P_m(0) - (P - P(0))\|_{L^\infty(J_n)} < \|f\|_{\text{BMO}^{q,k}}. \quad (3.16)$$

This is possible because of (3.15).

Just as in (3.13), applying Whitney's theorem (Lemma 2.2) there exist polynomials $\tilde{P}_j \in \Pi_k$, $j = 0, 1, \dots, \ell$, such that

$$\frac{1}{|\tilde{I}_j|} \int_{\Omega_{\tilde{I}_j}} |f(x) - \tilde{P}_j(x)|^q dx \leq c |\tilde{I}_j|^{-1} \omega_k(f, \Omega_{\tilde{I}_j})_q^q \leq c \|f\|_{\text{BMO}^{q,k}}^q. \quad (3.17)$$

At this point we select several constants. We choose $\tilde{c} := \tilde{P}_0(y) - \tilde{P}_\ell(y)$, where $y \in J$ is fixed, $c^\diamond := P_n(0) - P_m(0)$, and $c^{**} := P_m(0) - P(0)$. We also set $c^\diamond = \tilde{c} + c^\diamond + c^{**}$.

Using the above polynomials and constants we get

$$\begin{aligned} \frac{1}{|J|} \int_J |f(x) - P(x) - c^\diamond|^q dx &\leq \frac{c}{|J|} \int_J |f(x) - \tilde{P}_0(x)|^q dx \\ &\quad + c \|\tilde{P}_0 - \tilde{P}_\ell - \tilde{c}\|_{L^\infty(J)}^q + c \|\tilde{P}_\ell - P_n\|_{L^\infty(J)}^q + c \|P_n - P_m - c^\diamond\|_{L^\infty(J)}^q \\ &\quad + c \|P_m - P - c^{**}\|_{L^\infty(J)}^q =: W_1 + W_2 + W_3 + W_4 + W_5. \end{aligned}$$

To estimate W_1 we use (3.17) and obtain

$$W_1 \leq \frac{1}{|\tilde{I}_0|} \int_{\Omega_{\tilde{I}_0}} |f(x) - \tilde{P}_j(x)|^q dx \leq c |\tilde{I}_0|^{-1} \omega_k(f, \Omega_{\tilde{I}_0})_q^q \leq c \|f\|_{\text{BMO}^{q,k}}^q.$$

We proceed just as in (3.14) to obtain

$$W_2 = c \|\tilde{P}_0 - \tilde{P}_\ell - \tilde{c}\|_{L^\infty(J)}^q \leq c \|f\|_{\text{BMO}^{q,k}}^q.$$

We now estimate W_3 . Observe that because $\tilde{I}_\ell \cap J_n \neq \emptyset$ and $|\tilde{I}_\ell| \sim |J_n|$ we have $|\Omega_{\tilde{I}_\ell} \cap J_n| \sim |\Omega_{\tilde{I}_\ell}| \sim |J_n|$. We use this, (3.13), and (3.17) to obtain

$$\begin{aligned} W_3 &\leq c \|\tilde{P}_\ell - P_n\|_{L^\infty(J_n \cap \Omega_{\tilde{I}_\ell})}^q \leq c |J_n \cap \Omega_{\tilde{I}_\ell}|^{-1} \|\tilde{P}_\ell - P_n\|_{L^q(J_n \cap \Omega_{\tilde{I}_\ell})}^q \\ &\leq c |J_n|^{-1} \|f - P_n\|_{L^q(J_n)}^q + c |\Omega_{\tilde{I}_\ell}|^{-1} \|f - \tilde{P}_\ell\|_{L^q(\Omega_{\tilde{I}_\ell})}^q \leq c \|f\|_{\text{BMO}^{q,k}}^q. \end{aligned}$$

To estimate W_4 we use (3.14) and obtain $W_4 \leq c \|f\|_{\text{BMO}^{q,k}}^q$. We also have from (3.16) that $W_5 \leq c \|f\|_{\text{BMO}^{q,k}}^q$.

Putting the above estimates together we obtain that for any interval J there is a constant c^\diamond such that

$$\frac{1}{|J|} \int_J |f(x) - P(x) - c^\diamond|^q dx \leq c \|f\|_{\text{BMO}^{q,k}}^q$$

and (3.12) follows in view of (2.2). \square

From the trivial embedding $\dot{B}_\tau^{\alpha,k}(E, q) \subset \text{BMO}^{q,k}$, which is a consequence of $\|\cdot\|_{\ell^\infty} \leq \|\cdot\|_{\ell^\tau}$, and Proposition 3.1 we obtain

Proposition 3.2. *For any $f \in \dot{B}_\tau^{\alpha,k}(E, q)$, $1 \leq q < \infty$, $0 < \tau < \infty$, there exists a polynomial $P \in \Pi_k$ such that $f - P \in \text{BMO}$ and*

$$\|f - P\|_{\text{BMO}} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E, q)}.$$

The following decomposition result will play a central role in our theory of Besov spaces.

Theorem 3.1. *For any $f \in \dot{B}_\tau^{\alpha,k}(E, q)$, $1 \leq q < \infty$, $0 < \tau < \infty$, there exists a polynomial $P \in \Pi_k$ such that $f - P \in \text{VMO}$ and*

$$f - P = \sum_{m \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_m} b_{Q,q}(f) \varphi_Q, \quad (3.18)$$

where the convergence is unconditional in BMO . Here the coefficients $\{b_{Q,q}(f)\}$ are from (3.7). Furthermore,

$$\left(\sum_{Q \in \mathcal{Q}} |b_{Q,q}(f)|^\tau \right)^{1/\tau} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E, q)}. \quad (3.19)$$

We divert the long and tedious proof of this theorem to §8.3 in the appendix.

3.5. Norm in $\dot{B}_\tau^{\alpha,k}$ via B-splines

Theorems 2.1 and 3.1 are the motivation for the following

Definition 3.3. The Besov space $\dot{B}_\tau^{\alpha,k}(\Phi)$ is defined as the collection of all functions f on \mathbb{R} for which there exists a polynomial $P \in \Pi_k$ such that $f - P \in \text{VMO}$ and $f - P$ can be represented in the form $f - P = \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q$, where the convergence is unconditional in BMO , and $\sum_{Q \in \mathcal{Q}} |a_Q|^\tau < \infty$. The norm in $\dot{B}_\tau^{\alpha,k}(\Phi)$ is defined by

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)} := \inf \left\{ \left(\sum_{Q \in \mathcal{Q}} |a_Q|^\tau \right)^{1/\tau} : f - P = \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \right\}.$$

Observe that due to $\|\varphi_Q\|_q \sim |Q|^{1/q}$, $0 < q \leq \infty$, we have

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)} \sim \inf \left\{ \left(\sum_{Q \in \mathcal{Q}} (|Q|^{-1/q} \|a_Q \varphi_Q\|_q)^\tau \right)^{1/\tau} : f - P = \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \right\}.$$

3.6. Equivalent Besov Norms

Theorem 3.2. *The homogeneous Besov spaces $\dot{B}_\tau^{\alpha,k}(\Phi)$, $\dot{B}_\tau^{\alpha,k}(E, q)$ and $\dot{B}_\tau^{\alpha,k}(Q, q)$ for all $q \in [1, \infty)$ are the same with equivalent norms:*

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)} \sim \|f\|_{\dot{B}_\tau^{\alpha,k}(E, q)} \sim \|f\|_{\dot{B}_\tau^{\alpha,k}(Q, q)}$$

with constants of equivalence depending only on α , k , q , and the parameters of \mathcal{I} .

Proof. (a) We first show that if $f \in \dot{B}_\tau^{\alpha,k}(\Phi)$, then $f \in \dot{B}_\tau^{\alpha,k}(E,q)$ for $q \in [1, \infty)$ and

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)}. \quad (3.20)$$

Note that Hölder's inequality implies for any compact interval J

$$|J|^{-1/p} E_k(f, J)_p \leq |J|^{-1/q} E_k(f, J)_q, \quad 1 \leq p < q \leq \infty, \quad f \in L^q(J),$$

and hence

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(E,p)} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)}, \quad \text{if } 1 \leq p \leq q < \infty.$$

Therefore, it is sufficient to prove (3.20) in the (most unfavorable) case when $q > \max\{1, \tau\}$.

Assume $f \in \dot{B}_\tau^{\alpha,k}(\Phi)$. Then by Definition 3.3 f can be represented in the form

$$f - P = \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q$$

with unconditional convergence in BMO, where $P \in \Pi_k$, $f - P \in \text{VMO}$ and the coefficients $\{a_Q\}$ satisfy $(\sum_{Q \in \mathcal{Q}} |a_Q|^\tau)^{1/\tau} \leq 2 \|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)}$.

Denote by $\ell(I)$ the level of I ($\ell(I) = m$ if $I \in \mathcal{I}_m$) and, similarly, by $\ell(Q)$ the level of Q . Fix $I \in \mathcal{I}$. To estimate $\omega_k(f, \Omega_I)_q$ we split the representation of $f - P$ into two:

$$f - P = \sum_{j \geq \ell(I)} \sum_{Q \in \mathcal{Q}_j} a_Q \varphi_Q + \sum_{j < \ell(I)} \sum_{Q \in \mathcal{Q}_j} a_Q \varphi_Q =: G_I + H_I.$$

We use Proposition 2.1 to obtain

$$\begin{aligned} \omega_k(G_I, \Omega_I)_q &\leq c \|G_I\|_{L^q(\Omega_I)} \leq c \left(\sum_{\substack{\ell(Q) \geq \ell(I), \\ Q \cap \Omega_I \neq \emptyset}} \|a_Q \varphi_Q\|_q^\tau \right)^{1/\tau} \\ &\leq c \left(\sum_{\substack{\ell(Q) \geq \ell(I), \\ Q \cap \Omega_I \neq \emptyset}} |Q|^{\tau/q} |a_Q|^\tau \right)^{1/\tau}. \end{aligned} \quad (3.21)$$

To estimate $\omega_k(H_I, \Omega_I)_q$ we use that $\|\Delta_h^k \varphi_Q\|_\infty \leq c(|h|/|Q|)^k \leq c|h|/|Q|$ (for $|h| \leq |\Omega_I|/k$ and $Q \in \mathcal{Q}_j$ with $j < \ell(I)$) and $\Delta_h^k \varphi_Q(x) = 0$ if $[x, x+kh] \cap V_Q = \emptyset$, where V_Q is the set of all knots of φ_Q . We obtain

$$\begin{aligned} \omega_k(H_I, \Omega_I)_q &\leq \sum_{j < \ell(I)} \sum_{Q \in \mathcal{Q}_j} \omega_k(a_Q \varphi_Q, \Omega_I)_q \\ &\leq c \sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} \frac{|I|^{1+1/q}}{|Q|} |a_Q|. \end{aligned} \quad (3.22)$$

Using (3.5) we have

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)}^\tau \leq c \sum_{I \in \mathcal{I}} |I|^{-\tau/q} \omega_k(G_I, \Omega_I)_q^\tau + c \sum_{I \in \mathcal{I}} |I|^{-\tau/q} \omega_k(H_I, \Omega_I)_q^\tau =: S_1 + S_2. \quad (3.23)$$

To estimate S_1 we use (3.21) and obtain

$$\begin{aligned} S_1 &\leq \sum_{I \in \mathcal{I}} |I|^{-\tau/q} \sum_{\substack{\ell(Q) \geq \ell(I), \\ Q \cap \Omega_I \neq \emptyset}} |Q|^{\tau/q} |a_Q|^\tau \\ &= c \sum_{Q \in \mathcal{Q}} |a_Q|^\tau \sum_{\substack{I \in \mathcal{I}, \\ \ell(I) \leq \ell(Q), \\ \Omega_I \cap Q \neq \emptyset}} (|Q|/|I|)^{\tau/q} \leq c \sum_{Q \in \mathcal{Q}} |a_Q|^\tau \sum_{\nu \geq 0} \rho^{\nu\tau/q} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)}^\tau. \end{aligned}$$

Here $\rho < 1$ is from (2.4) and we used the nested structure of the partition \mathcal{I} ; we switched once the order of summation.

We now estimate S_2 . If $\tau \geq 1$ using (3.22) we get

$$\begin{aligned} S_2 &\leq c \sum_{I \in \mathcal{I}} |I|^{-\tau/q} \left(\sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} \frac{|I|^{1+1/q}}{|Q|} |a_Q| \right)^\tau \\ &= c \sum_{I \in \mathcal{I}} \left(\sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} (|I|/|Q|) |a_Q| \right)^\tau \\ &\leq c \sum_{I \in \mathcal{I}} \left(\sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} |I|/|Q| \right)^{\tau/\tau'} \sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} (|I|/|Q|) |a_Q|^\tau \\ &\leq c \sum_{I \in \mathcal{I}} \sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} (|I|/|Q|) |a_Q|^\tau. \end{aligned}$$

For the former inequality above we used Hölder's inequality. For the last inequality we used (2.4) as in the estimate of S_1 . Switching the order of summation in the last sums above we get

$$S_2 \leq c \sum_{Q \in \mathcal{Q}} |a_Q|^\tau \sum_{\substack{I \in \mathcal{I}, \\ \ell(I) > \ell(Q), \\ \Omega_I \cap V_Q \neq \emptyset}} |I|/|Q| \leq c \sum_{Q \in \mathcal{Q}} |a_Q|^\tau \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)}^\tau.$$

Here we used the simple fact that for any point $y \in \mathbb{R}$ and $J \in \mathcal{I}$ we have

$$\sum_{\substack{I \in \mathcal{I}, \\ \ell(I) > \ell(J), \\ I \ni y}} |I| \leq |J| \sum_{\nu > 0} \rho^\nu \leq c|J|,$$

where $0 < \rho < 1$ is from (2.4).

If $0 < \tau < 1$ we apply the same arguments in the estimate of S_2 but using the concavity of the function y^τ instead of Hölder's inequality. We have

$$\begin{aligned} S_2 &\leq c \sum_{I \in \mathcal{I}} \left(\sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} (|I|/|Q|)|a_Q| \right)^\tau \leq c \sum_{I \in \mathcal{I}} \sum_{j < \ell(I)} \sum_{\substack{Q \in \mathcal{Q}_j, \\ V_Q \cap \Omega_I \neq \emptyset}} (|I|/|Q|)^\tau |a_Q|^\tau \\ &\leq c \sum_{Q \in \mathcal{Q}} |a_Q|^\tau \sum_{\substack{I \in \mathcal{I}, \\ \ell(I) > \ell(Q), \\ \Omega_I \cap V_Q \neq \emptyset}} (|I|/|Q|)^\tau \leq c \sum_{Q \in \mathcal{Q}} |a_Q|^\tau \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)}^\tau. \end{aligned}$$

The above estimates for S_1 and S_2 and (3.23) yield (3.20).

(b) We now show that if $f \in \dot{B}_\tau^{\alpha,k}(E, q)$, $1 \leq q < \infty$, then $f \in \dot{B}_\tau^{\alpha,k}(\Phi)$ and

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(\Phi)} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(Q,q)} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)}. \quad (3.24)$$

Indeed, assume that $f \in \dot{B}_\tau^{\alpha,k}(E, q)$. Then by Theorem 3.1 there exists a polynomial $P \in \Pi_k$ such that with unconditional convergence in BMO

$$f - P = \sum_{Q \in \mathcal{Q}} b_{Q,q}(f) \varphi_Q,$$

and by (3.9) and (3.19)

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(Q,q)} \sim \left(\sum_{Q \in \mathcal{Q}} |b_{Q,q}(f)|^\tau \right)^{1/\tau} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E,q)}.$$

Now, using Definition 3.3 we conclude that $f \in \dot{B}_\tau^{\alpha,k}(\Phi)$ and (3.24) is valid.

(c) Parts (a), (b) of the proof establish the theorem for any fixed $1 \leq q < \infty$. Taking into account that the space $\dot{B}_\tau^{\alpha,k}(\Phi)$ does not depend on q this completes the proof. \square

Remark 3.1. Observe first that the space $\dot{B}_\tau^{\alpha,k}$ when $\tau \geq 1$ is the usual homogeneous Besov space $\dot{B}_{\tau\tau}^{\alpha,k}$. For simplicity of notation we suppress the second index τ . Theorem 3.2 shows that $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}(Q,q)}$ and $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}(E,q)}$ with $1 \leq q < \infty$ are other equivalent norms in the Besov space $\dot{B}_\tau^{\alpha,k}$, $\tau \geq 1$. These norms will be more useful for our purposes of spline approximation than the norm from (3.1).

An important difference occurs when $\tau < 1$. The norm $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}}$ from (3.1) that involves the modulus of smoothness $\omega_k(f, t)_\tau$ is hardly usable, while the norms $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}(Q,q)}$ and $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}(E,q)}$ with $1 \leq q < \infty$ work very well. We could have defined the Besov space $\dot{B}_\tau^{\alpha,k}$ for all $\tau > 0$ from the outset using the norm $\|\cdot\|_{\dot{B}_\tau^{\alpha,k}(Q,q)}$.

As is seen above we have introduced k as a parameter and consider α and k independent. The reason for this is that by allowing $\tau < 1$ the spaces $\dot{B}_\tau^{\alpha,k}(Q, q)$ are nontrivial and different for all $\alpha > 0$ and $k \geq 2$.

According to Theorem 3.2 all spaces $\dot{B}_\tau^{\alpha,k}(Q, q)$, $\dot{B}_\tau^{\alpha,k}(E, q)$, $1 \leq q < \infty$, and $\dot{B}_\tau^{\alpha,k}(\Phi)$ (also $\dot{B}_\tau^{\alpha,k}$ if $\tau \geq 1$, see (3.3)) are the same; from now on we shall use the notation $\dot{B}_\tau^{\alpha,k}$ for this Besov space.

4. Nonlinear Spline Approximation in BMO

Assume that \mathcal{I} is a regular multilevel partition of \mathbb{R} . We denote by $\Phi(\mathcal{I})$ the collection of all $k-1$ degree B-splines φ_Q generated by \mathcal{I} (see §2.3). Notice that $\Phi(\mathcal{I})$ is not a basis; $\Phi(\mathcal{I})$ is redundant. We consider the nonlinear n -term approximation in BMO from $\Phi(\mathcal{I})$.

Denote by $\Sigma_n(\mathcal{I})$ the set of all spline functions g of the form

$$g = \sum_{Q \in \Lambda_n} a_Q \varphi_Q,$$

where $a_Q \in \mathbb{C}$, $\Lambda_n \subset \mathcal{Q}(\mathcal{I})$, $\#\Lambda_n \leq n$, and Λ_n may vary with g . We denote by $\sigma_n(f, \mathcal{I})_{\text{BMO}}$ the error of BMO-approximation to $f \in \text{VMO}$ from $\Sigma_n(\mathcal{I})$:

$$\sigma_n(f)_{\text{BMO}} = \sigma_n(f, \mathcal{I})_{\text{BMO}} := \inf_{g \in \Sigma_n(\mathcal{I})} \|f - g\|_{\text{BMO}}.$$

Throughout this section we assume as before that

$$\alpha > 0, \quad 1/\tau := \alpha \quad \text{and} \quad k \geq 2,$$

and denote by $\dot{B}_\tau^{\alpha,k}$ the Besov space introduced in Section 3.

Convention. Clearly, if $f \in \dot{B}_\tau^{\alpha,k}$, then $\|f + P\|_{\dot{B}_\tau^{\alpha,k}} = \|f\|_{\dot{B}_\tau^{\alpha,k}}$ for all $P \in \Pi_k$. Hence, $\dot{B}_\tau^{\alpha,k}$ consists of equivalence classes modulo Π_k . In light of Theorem 3.1 for any function $f \in \dot{B}_\tau^{\alpha,k}$ there exists a polynomial $P \in \Pi_k$ such that $f - P \in \text{VMO}$. From now on we shall assume that each $f \in \dot{B}_\tau^{\alpha,k}$ is the *canonical representative* $f - P \in \text{VMO}$ of the equivalence class modulo Π_k generated by f . As before (see §2.1) we identify each $f \in \text{VMO}$ with $f + a$, a is an arbitrary constant.

The following pair of companion Jackson and Bernstein estimates are our main results in this article.

Theorem 4.1 (Jackson estimate). *If $f \in \dot{B}_\tau^{\alpha,k}$, then $f \in \text{VMO}$ and*

$$\sigma_n(f, \mathcal{I})_{\text{BMO}} \leq cn^{-\alpha} \|f\|_{\dot{B}_\tau^{\alpha,k}}, \quad n \in \mathbb{N}, \quad (4.1)$$

with $c > 0$ depending only on α, k , and the parameters of \mathcal{I} .

Theorem 4.2 (Bernstein estimate). *If $g \in \Sigma_n(\mathcal{I})$, $n \in \mathbb{N}$, then*

$$\|g\|_{\dot{B}_\tau^{\alpha,k}} \leq cn^\alpha \|g\|_{\text{BMO}} \quad (4.2)$$

with $c > 0$ depending only on α, k , and the parameters of \mathcal{I} .

Remark 4.1. As will be seen from the proof of the Bernstein estimate (4.2) (see §7) the assumption that g being in $\Sigma_n(\mathcal{I})$ belongs to C^{k-2} is not important; it is only used that $g \in C$. Therefore, estimate (4.2) is valid for B -splines that are only continuous.

As is well known the companion Jackson and Bernstein estimates (4.1), (4.2) imply complete characterization of the approximation spaces associated with spline approximation in BMO. We next describe this result.

Denote by $K(f, t)$ the K -functional associated with VMO and $\dot{B}_\tau^{\alpha,k}$, defined for $f \in \text{VMO}$ and $t > 0$ by (see e.g. [6, Chapter 6])

$$K(f, t) = K(f, t; \text{VMO}, \dot{B}_\tau^{\alpha,k}) := \inf_{g \in \dot{B}_\tau^{\alpha,k}} (\|f - g\|_{\text{BMO}} + t\|g\|_{\dot{B}_\tau^{\alpha,k}}).$$

The Jackson and Bernstein estimates (4.1), (4.2) imply the following direct and inverse estimates: If $f \in \text{VMO}$, then

$$\sigma_n(f)_{\text{BMO}} \leq cK(f, n^{-\alpha}), \quad n \geq 1, \tag{4.3}$$

and

$$K(f, n^{-\alpha}) \leq cn^{-\alpha} \left[\left(\sum_{\nu=1}^n \frac{1}{\nu} (\nu^\alpha \sigma_\nu(f, \mathcal{I})_{\text{BMO}})^\mu \right)^{1/\mu} + \|f\|_{\text{BMO}} \right], \quad n \geq 1, \tag{4.4}$$

where $\mu := \min\{\tau, 1\}$.

The proofs of (4.3), (4.4) are standard, see e.g. [6, Chapter 7, Theorem 5.1].

We define the approximation space $A_q^\gamma(\text{BMO}, \mathcal{I})$ generated by nonlinear n -term approximation in BMO from B -spliens to be the set of all functions $f \in \text{BMO}$ such that

$$\|f\|_{A_q^\gamma(\text{BMO}, \mathcal{I})} := \|f\|_{\text{BMO}} + \left(\sum_{n=1}^{\infty} (n^\gamma \sigma_n(f, \mathcal{I})_{\text{BMO}})^q \frac{1}{n} \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

The following characterization of the approximation spaces $A_q^\gamma(\text{BMO}, \mathcal{I})$ is immediate from inequalities (4.3), (4.4):

Theorem 4.3. *If $0 < \gamma < \alpha$ and $0 < q \leq \infty$, then*

$$A_q^\gamma(\text{BMO}, \mathcal{I}) = (\text{VMO}, \dot{B}_\tau^{\alpha,k})_{\frac{q}{\alpha}, q}.$$

In particular, if $f \in \text{VMO}$, then

$$K(f, t^\alpha) = O(t^\gamma) \quad \text{if and only if} \quad \sigma_n(f)_{\text{BMO}} = O(n^{-\gamma}).$$

Above $(X_0, X_1)_{\lambda, q}$ stands for the real interpolation space induced by two spaces X_0, X_1 , see e.g. [6, Chapter 6, §7].

5. Comparison with Spline Approximation in Other Spaces

5.1. Comparison between Spline Approximation in BMO and L^∞

Here we clarify the differences and similarities between nonlinear n -term approximation from B-splines in BMO and in the uniform norm.

Denote by $\sigma_n(f, \mathcal{I})_\infty$ the error of L^∞ -approximation to f from $\Sigma_n(\mathcal{I})$:

$$\sigma_n(f, \mathcal{I})_\infty := \inf_{S \in \Sigma_n(\mathcal{I})} \|f - S\|_\infty.$$

We denote by $\dot{B}_\tau^{\alpha, k}$ the Besov space introduced in Section 3. The following theorem follows from the Jackson estimates in [3, Theorem 4.1] or [17, Theorem 4.1].

Theorem 5.1 (Jackson estimate). *If $f \in \dot{B}_\tau^{\alpha, k}$, $\alpha \geq 1$, $1/\tau = \alpha$, $k \geq 2$, then f is continuous on \mathbb{R} , $\lim_{|x| \rightarrow \infty} f(x) = 0$, and*

$$\sigma_n(f, \mathcal{I})_\infty \leq cn^{-\alpha} \|f\|_{\dot{B}_\tau^{\alpha, k}}, \quad n \in \mathbb{N}, \quad (5.1)$$

with c depending only on α and the parameters of \mathcal{I} .

Observe that the B-spaces used for nonlinear spline approximation in L^∞ in [3, 17] are somewhat different because the approximation there takes place in dimension $d \geq 1$ or $d = 2$. However, from (3.5) in [3] or (2.15) in [17] it is clear that these spaces are the same as $\dot{B}_\tau^{\alpha, k}$ in dimension $d = 1$.

Theorem 5.2 (Bernstein estimate). *Let $\alpha > 0$ and $1/\tau = \alpha$. If $S \in \Sigma_n(\mathcal{I})$, $n \in \mathbb{N}$, then*

$$\|S\|_{\dot{B}_\tau^{\alpha, k}} \leq cn^\alpha \|S\|_\infty, \quad (5.2)$$

with c depending only on α and the parameters of \mathcal{I} .

This theorem follows by the Bernstein estimates in [3, Theorem 4.2], see also [17, Theorem 4.2].

Several clarifying remarks are in order:

(1) The Besov space $\dot{B}_\tau^{\alpha, k}$ is obviously not embedded in L^∞ when $\alpha < 1$. For this reason the Jackson estimate (5.1) is not valid when $\alpha < 1$. At the same time the Jackson estimate (4.1) holds for all $\alpha > 0$.

(2) Since $\|S\|_{BMO} \leq \|S\|_\infty$ the Bernstein estimate (5.2) is a consequence of the Bernstein estimate (4.2). Similarly, the Jackson estimate (4.1) follows by (5.1) in the case when $\alpha \geq 1$.

(3) It is interesting that in the case when $\alpha \geq 1$ the same Besov spaces $\dot{B}_\tau^{\alpha, k}$ work for spline approximation in both BMO and L^∞ .

(4) Algorithms for nonlinear n -term approximation from linear B-splines are developed in [4] and in more general settings in [3]. The results in [3, 17] use ideas that originate in earlier development of spline approximation in L^∞ in [7].

5.2. Comparison between Spline Approximation in BMO and L^p , $p < \infty$

There is a principle difference between nonlinear spline approximation in $\text{BMO}(\mathbb{R})$ (or $L^\infty(\mathbb{R})$) and in $L^p(\mathbb{R})$, $1 \leq p < \infty$, that we would like to clarify here.

To be specific, denote by $S(k, n)$ the set of all piecewise polynomials functions on \mathbb{R} of degree $k-1$ with $n+1$ free knots, that is, $S \in S(k, n)$ if there exist points $-\infty < x_0 < x_1 < \dots < x_n < \infty$ and polynomials $P_j \in \Pi_k$, $j = 1, \dots, n$, such that

$$S = \sum_{j=1}^n \mathbb{1}_{I_j} \cdot P_j, \quad I_j := [x_{j-1}, x_j]. \quad (5.3)$$

Here the knots $\{x_j\}$ are allowed to vary with S . Hence $S(k, n)$ is nonlinear. No multilevel partition is assumed. Given $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, define

$$S_n^k(f)_p := \inf_{S \in S(k, n)} \|f - S\|_p.$$

One is interested in characterizing the approximation spaces associate to this approximation process. The Besov spaces

$$\dot{B}_\tau^{\alpha, k}, \quad \alpha > 0, \quad 1/\tau := \alpha + 1/p, \quad k \geq 1,$$

naturally appear in this sort of problems. These spaces are standardly defined [19] by the following norm using moduli of smoothness as in (3.1) for $0 < \tau < \infty$:

$$\|f\|_{\dot{B}_\tau^{\alpha, k}} := \left(\int_0^\infty [t^{-\alpha} \omega_k(f, t)_\tau]^\tau \frac{dt}{t} \right)^{1/\tau}.$$

However, the definition of $\dot{B}_\tau^{\alpha, k}$ can be modified as in Definition 3.2 when $\tau < 1$, $q < p$, see [4, 16].

The following Jackson and Bernstein estimates have been established in [19]: If $f \in \dot{B}_\tau^{\alpha, k}$, then $f \in L^p$ and

$$S_n^k(f)_p \leq cn^{-\alpha} \|f\|_{\dot{B}_\tau^{\alpha, k}}$$

and for any $S \in S(k, n)$

$$\|S\|_{\dot{B}_\tau^{\alpha, k}} \leq cn^\alpha \|S\|_p.$$

Clearly, these two estimates allow to completely characterize the associated to $\{S_n^k(f)_p\}$ approximation spaces.

Discussion. As the following remarks show the nature of nonlinear spline (piecewise polynomials) approximation in $\text{BMO}(\mathbb{R})$ (or $L^\infty(\mathbb{R})$) and in $L^p(\mathbb{R})$, $p < \infty$, is totally different.

When approximating from piecewise polynomials in L^p , $1 \leq p < \infty$, there is no need to assume any smoothness or continuity, we simply work with discontinuous piecewise polynomials, see (5.3). The point is that smooth piecewise

polynomials and discontinuous piecewise polynomials produce the same rates of approximation for $p < \infty$. More importantly, if $S_I := \mathbb{1}_I \cdot P$ for some compact interval I and a polynomial $P \in \Pi_k$, $P \neq 0$, then

$$\|S_I\|_{\dot{B}_\tau^{\alpha,k}} < \infty, \quad \forall \alpha > 0, \quad 1/\tau = \alpha + 1/p.$$

In other words this piecewise polynomial function is infinitely smooth in the scale of the Besov spaces $\dot{B}_\tau^{\alpha,k}$.

In contrast, it is easy to see for $S_I \notin C(\mathbb{R})$ that

$$\|S_I\|_{\dot{B}_\tau^{\alpha,k}} = \infty \quad \text{for any } \alpha > 0 \text{ whenever } 1/\tau = \alpha, \text{ i.e. } p = \infty.$$

Furthermore, the Bernstein estimate (4.2) cannot be true for piecewise polynomials S of the form (5.3) even if $S \in C^{k-2}$. We next clarify this claim with the following simple example. Let $S(x) = 1$ for $x \in [0, 1]$, $S(x) = 0$ for $x \in (-\infty, -\varepsilon] \cup [1 + \varepsilon, \infty)$, and S is linear and continuous on $[-\varepsilon, 0]$ and $[1, 1 + \varepsilon]$, where $\varepsilon > 0$ is sufficiently small. It is readily seen that $\omega_2(S, t)_\tau^\tau \sim t$ for $\varepsilon \leq t \leq 1/2$ and hence

$$\|S\|_{\dot{B}_\tau^{\alpha,2}}^\tau \geq c \int_\varepsilon^{1/2} \frac{dt}{t} \geq c \ln(1/\varepsilon).$$

Therefore, the Bernstein estimate (4.2) is not valid for piecewise polynomials of the form (5.3) even if they are smooth. This is the reason for considering nonlinear approximation from splines generated by a hierarchy of B-splines.

6. Proof of Theorem 4.1

In this section we prove the Jackson estimate (4.1). We shall derive this estimate from an estimate for approximation in general sequence spaces.

6.1. Jackson Inequality for Nonlinear Approximations in \mathfrak{f}_∞^{0g}

Here we consider nonlinear n -term approximation from finitely supported sequences in the spaces \mathfrak{f}_∞^{0g} in a general setting developed in [13, §7.2].

Definition 6.1. Let $\mathcal{X} = \cup_{m=-\infty}^\infty \mathcal{X}_m$ be a countable multilevel index set. With every $\xi \in \mathcal{X}$ we associate an open set $U_\xi \subset \mathbb{R}$. We call $\{U_\xi : \xi \in \mathcal{X}\}$ a *nested structure associate with \mathcal{X}* if there exists constant $\lambda \geq 1$ such that:

- (a) $|\mathbb{R} \setminus \cup_{\xi \in \mathcal{X}_m} U_\xi| = 0 \quad \forall m \in \mathbb{Z}$;
- (b) If $\eta \in \mathcal{X}_m, \xi \in \mathcal{X}_\nu$ and $m \geq \nu$ then either $U_\eta \subset U_\xi$ or $U_\eta \cap U_\xi = \emptyset$;
- (c) For every $\xi \in \mathcal{X}_\nu, m < \nu$, there is unique $\eta \in \mathcal{X}_m$ such that $U_\xi \subset U_\eta$;

$$(d) |U_\eta| \leq \lambda |U_\zeta| \quad \forall \eta, \zeta \in \mathcal{X}_m, \forall m \in \mathbb{Z};$$

(e) Every U_ξ , $\xi \in \mathcal{X}_m$, has at least two children, i.e. there are $\eta, \zeta \in \mathcal{X}_{m+1}$ such that $U_\eta \subset U_\xi$, $U_\zeta \subset U_\xi$, $\eta \neq \zeta$.

We also assume that there exist one \mathfrak{D}^1 or two $\mathfrak{D}^1, \mathfrak{D}^2$ disjoint subsets of \mathbb{R} with the properties: (i) $|\mathbb{R} \setminus \cup_{j=1}^K \mathfrak{D}^j| = 0$; (ii) for every $\xi \in \mathcal{X}$ there is j , $1 \leq j \leq K$, such that $U_\xi \subset \mathfrak{D}^j$; (iii) if the sets U_η, U_ζ are contained in \mathfrak{D}^j for some $1 \leq j \leq K$, then there exists $U_\xi \subset \mathfrak{D}^j$ such that $U_\eta \subset U_\xi$, $U_\zeta \subset U_\xi$. Here $K = 1$ or $K = 2$.

Remark 6.1. Conditions (a)–(c) imply that for every $\xi \in \mathcal{X}_m$ we have

$$|U_\xi| = \sum_{\substack{\eta \in \mathcal{X}_{m+1} \\ U_\eta \subset U_\xi}} |U_\eta|, \quad \forall \xi \in \mathbb{N}.$$

Conditions (d)–(e) imply that there exists $\rho \in (0, 1)$, namely $\rho = \lambda/(\lambda + 1)$, such that

$$|U_\eta| \leq \rho |U_\xi| \quad \forall m \in \mathbb{Z}, \xi \in \mathcal{X}_m, \eta \in \mathcal{X}_{m+1}, U_\eta \subset U_\xi. \quad (6.1)$$

Recall that by definition the sequence spaces $\ell^\tau = \ell^\tau(\mathcal{X})$, $0 < \tau \leq \infty$, consists of all sequences $\{h_\xi : \xi \in \mathcal{X}\}$ such that

$$\|h\|_{\ell^\tau} := \left(\sum_{\xi \in \mathcal{X}} |h_\xi|^\tau \right)^{1/\tau} < \infty.$$

We define the sequence spaces $\mathfrak{g}^q = \mathfrak{g}^q(\mathcal{X})$, $0 < q \leq \infty$, as the set of all sequences $\{h_\xi : \xi \in \mathcal{X}\}$ such that

$$\|h\|_{\mathfrak{g}^q} := \sup_{\xi \in \mathcal{X}} \left(\sum_{U_\eta \subset U_\xi} |h_\eta|^q \frac{|U_\eta|}{|U_\xi|} \right)^{1/q} < \infty.$$

Note that the definition of $\ell^\tau(\mathcal{X})$ does not need a nested structure associate with \mathcal{X} , while the definition of $\mathfrak{g}^q(\mathcal{X})$, $q < \infty$, requires such kind of structure. Of course, for $q = \tau = \infty$ we have $\mathfrak{g}^\infty = \ell^\infty$.

The best nonlinear approximation of $h \in \mathfrak{g}^q$ from sequences with at most n non-zero elements is given by

$$\sigma_n(h)_{\mathfrak{g}^q} := \inf_{|\text{supp } \bar{h}| \leq n} \|h - \bar{h}\|_{\mathfrak{g}^q} = \inf_{\substack{\Lambda_n \subset \mathcal{X} \\ \#\Lambda_n \leq n}} \sup_{\xi \in \mathcal{X}} \left(\sum_{\substack{U_\eta \subset U_\xi \\ \eta \notin \Lambda_n}} |h_\eta|^q \frac{|U_\eta|}{|U_\xi|} \right)^{1/q}.$$

Theorem 6.1 (Jackson inequality). *Let $0 < \tau < \infty$ and $0 < q \leq \infty$. Assume $\{U_\xi : \xi \in \mathcal{X}\}$ is a nested structure associate with \mathcal{X} . There exists a constant $c = c(\tau, q, \lambda)$ such that for any $h \in \ell^\tau$ we have $h \in \mathfrak{g}^q$ and*

$$\sigma_n(h)_{\mathfrak{g}^q} \leq cn^{-1/\tau} \|h\|_{\ell^\tau}, \quad n \in \mathbb{N}.$$

6.2. Proof of Theorem 4.1

Let the support $Q \in \mathcal{Q}_m$ of φ be $Q = [x_{m,j}, x_{m,j+k}]$. We use Q in the place of ξ , \mathcal{Q}_m in the place of \mathcal{X}_m , $m \in \mathbb{Z}$, \mathcal{Q} in the place of \mathcal{X} and the open interval $I_Q := (x_{m,j}, x_{m,j+1})$ in the place of U_ξ from Definition 6.1. If \mathcal{I} is a regular multilevel partition of \mathbb{R} then the system of intervals $\{I_Q : Q \in \mathcal{Q}\}$ is a nested structure associate with $\mathcal{Q} = \cup_{m=-\infty}^{\infty} \mathcal{Q}_m$. It satisfies all conditions of Definition 6.1.

In order to determine K from Definition 6.1 we consider all knots $\{x_{m,j} : m, j \in \mathbb{Z}\}$. In view of (6.1) it is not possible to have two different points $y_1, y_2 \in \mathbb{R}$ that are common knots for all levels $m \in \mathbb{Z}$. If all levels have one common knot y_1 then $K = 2$, $\mathfrak{D}^1 = (-\infty, y_1)$, $\mathfrak{D}^2 = (y_1, \infty)$. If there is no common knot for all levels then $K = 1$, $\mathfrak{D}^1 = (-\infty, \infty)$.

Lemma 6.1. *If $\{a_Q\}_{Q \in \mathcal{Q}} \in \mathfrak{g}^1(\mathcal{Q})$ and $f = \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \in \text{VMO}(\mathbb{R})$ then*

$$\|f\|_{\text{BMO}} \leq c \|\{a_Q\}\|_{\mathfrak{g}^1}.$$

Proof. Denote

$$f_\nu := \sum_{j > \nu} \sum_{Q \in \mathcal{Q}_j} a_Q \varphi_Q, \quad \nu \in \mathbb{Z}.$$

We claim that

$$\|f_\nu\|_{\text{BMO}} \leq c \sup_{Q \in \mathcal{Q}_j, j > \nu} \sum_{I_{Q'} \subset I_Q} |a_{Q'}| \frac{|I_{Q'}|}{|I_Q|} =: \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)}. \quad (6.2)$$

Let J be an arbitrary compact interval in \mathbb{R} . Then there exist $m \in \mathbb{Z}$ such that if $Q \in \mathcal{Q}_m$ and $Q \cap J \neq \emptyset$, then $|Q| \sim |J|$ and $Q \subset 2J$. Denote $\tilde{J} := 2J$.

We may assume that $\nu < m$. We split f_ν into two: $f_\nu = f_m + (f_\nu - f_m)$. Using (2.10) we get

$$\begin{aligned} \frac{1}{|J|} \int_J |f_m(x)| dx &\leq \frac{1}{|J|} \int_J \sum_{Q' \subset \tilde{J}} |a_{Q'} \varphi_{Q'}(x)| dx \leq \frac{c}{|\tilde{J}|} \int_{\tilde{J}} \sum_{Q' \subset \tilde{J}} |a_{Q'} \varphi_{Q'}(x)| dx \\ &\leq c \sum_{Q \in \mathcal{Q}_m, Q \cap \tilde{J} \neq \emptyset} \sup_{Q \in \mathcal{Q}_j, j > m} \sum_{I_{Q'} \subset I_Q} |a_{Q'}| \frac{|I_{Q'}|}{|I_Q|} \leq c \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)}. \end{aligned} \quad (6.3)$$

Fix $y \in J$. Denote $F_{\nu m} := f_\nu - f_m$. We claim that

$$\frac{1}{|J|} \int_J |F_{\nu m}(x) - F_{\nu m}(y)| dx \leq c \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)}. \quad (6.4)$$

Indeed, let $Q \in \mathcal{Q}_j$, $j \leq m$, and assume $Q \cap J \neq \emptyset$. Then for $x \in Q$

$$|\varphi_Q(x) - \varphi_Q(y)| \leq |x - y| |\varphi'_Q(\xi)| \leq c |x - y| |Q|^{-1}, \quad \xi \in (x, y).$$

For every $x \in J$ using the above we get

$$\begin{aligned} |F_{\nu m}(x) - F_{\nu m}(y)| &\leq \sum_{j=\nu+1}^m \sum_{Q \in \mathcal{Q}_j, Q \ni x} |c_Q| |\varphi_Q(x) - \varphi_Q(y)| \\ &\leq c \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)} \sum_{j=\nu+1}^m \sum_{Q \in \mathcal{Q}_j, Q \ni x} |\varphi_Q(x) - \varphi_Q(y)| \\ &\leq c \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)} \sum_{j=\nu+1}^m \sum_{Q \in \mathcal{Q}_j, Q \ni x} |J||Q|^{-1} \leq c \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)}. \end{aligned}$$

In the last inequality we used (2.4) and that every point x is contained in k different Q 's from every level. Estimate (6.4) follows readily from the above inequalities.

From (6.3) and (6.4) it follows that

$$\frac{1}{|J|} \int_J |f_{\nu}(x) - F_{\nu m}(y)| dx \leq c \|\{a_Q\}\|_{\mathfrak{g}^1(\nu)},$$

which implies (6.2).

Now, condition $f \in \text{VMO}(\mathbb{R})$ implies $\lim_{\nu \rightarrow -\infty} \|f_{\nu} - f\|_{\text{BMO}} = 0$ and, hence, (6.2) proves the lemma. \square

Completion of the proof of Theorem 4.1. Let $h = \{a_Q\}_{Q \in \mathcal{Q}} \in \ell^{\tau}(\mathcal{Q})$. Then Theorem 6.1 with $n = 1$ implies $h \in \mathfrak{g}^1$ and Theorem 2.1 gives $f = \sum_{Q \in \mathcal{Q}} a_Q \varphi_Q \in \text{VMO}$. Using Lemma 6.1 and once more time Theorem 6.1 we obtain

$$\sigma_n(f)_{\text{BMO}(\mathbb{R})} \leq c \sigma_n(h)_{\mathfrak{g}^1(\mathcal{Q})} \leq cn^{-1/\tau} \|h\|_{\ell^{\tau}(\mathcal{Q})} \leq cn^{-\alpha} \|f\|_{\dot{B}_{\tau}^{\alpha, k}}, \quad n \in \mathbb{N}.$$

This proves Theorem 4.1. \square

7. Proof of Theorem 4.2

For the proof of the Bernstein estimate (4.2) we shall need Lemma 2.1 and the following

Lemma 7.1. *Let $f \in \text{BMO}$ and $f(x) = 0$ for $x \in J \setminus I$, where I, J are two intervals so that $I \subset J$ and $|J| = (1 + \delta)|I|$, $\delta > 0$. If $1 \leq \tau < \infty$, then*

$$\int_I |f(x)|^{\tau} dx \leq (1 + \delta^{-1})^{\tau} \int_J |f(x) - \text{Avg}_J f|^{\tau} dx \leq c|J| \|f\|_{\text{BMO}}^{\tau}, \quad (7.1)$$

where $c = c(\delta, \tau)$.

Proof. Using the hypothesis of the lemma we have

$$\begin{aligned}
& \left(\frac{1}{|I|} \int_I |f(x)|^\tau dx \right)^{1/\tau} \leq \left(\frac{1}{|I|} \int_I |f(x) - \text{Avg}_J f|^\tau dx \right)^{1/\tau} + |\text{Avg}_J f| \\
& \leq \left(\frac{1}{|I|} \int_I |f(x) - \text{Avg}_J f|^\tau dx \right)^{1/\tau} + \frac{1}{|J|} \int_J |f(x)| dx \\
& \leq \left(\frac{1}{|I|} \int_I |f(x) - \text{Avg}_J f|^\tau dx \right)^{1/\tau} + \frac{|I|}{|J|} \frac{1}{|I|} \int_I |f(x)| dx \\
& \leq \left(\frac{1}{|I|} \int_I |f(x) - \text{Avg}_J f|^\tau dx \right)^{1/\tau} + \frac{1}{1+\delta} \left(\frac{1}{|I|} \int_I |f(x)|^\tau dx \right)^{1/\tau}
\end{aligned}$$

with Hölder's inequality applied in the last inequality. This proves the first inequality in (7.1), while the second inequality follows from (2.2). \square

Proof of Theorem 4.2. Given a partition \mathcal{I} we set $t_m = \lambda \sup_{I \in \mathcal{I}_m} |\Omega_I|$, $m \in \mathbb{Z}$, with λ from (2.3). From (2.3) we infer that $t_m \lambda^{-2} \leq |\Omega_I| \leq t_m \lambda^{-1}$ for every $I \in \mathcal{I}_m$ and from (2.4) we obtain $rt_m \leq t_{m+1} \leq \rho t_m$, $m \in \mathbb{Z}$.

Let $g = \sum_{Q \in \Lambda_n} a_Q \varphi_Q \in \Sigma_n$. Denote by $x_0 < x_1 < \dots < x_N$, $N \leq (k+1)n$, the knots of φ_Q , $Q \in \Lambda_n$, in increasing order (if two B-splines have a common knot then it appears only once in this sequence). There exist polynomials $P_j \in \Pi_{k-1}$, $j = 1, \dots, N$, such that

$$g = \sum_{j=1}^N \mathbb{1}_{J_j} \cdot P_j, \quad J_j := [x_{j-1}, x_j], \quad g \in C(\mathbb{R}). \quad (7.2)$$

In fact $g \in C^{k-2}(\mathbb{R})$. Also, denote $J_0 := (-\infty, x_0]$ and $J_{N+1} := [x_N, \infty)$. Note that the points $x_0 < x_1 < \dots < x_N$ are among the knots of partition \mathcal{I} and two consecutive x_{j-1}, x_j may belong to different levels of \mathcal{I} .

(a) Let $1 \leq \tau < \infty$. For every $m \in \mathbb{Z}$ and every $I \in \mathcal{I}_m$ we shall establish the estimate

$$E_k(g, \Omega_I)_\tau \leq c \left(\sum_{J_\nu \subset \Omega_I} |J_\nu| + \sum_{\substack{0 < |J_\mu \cap \Omega_I| < |\Omega_I| \\ J_\mu \setminus \Omega_I \neq \emptyset}} \min \left\{ |J_\mu|, \frac{t_m^{1+\tau}}{|J_\mu|^\tau} \right\} \right) \|g\|_{BMO}^\tau. \quad (7.3)$$

The second sum in (7.3), where the summation is on J_μ , contains 0, 1 or 2 terms. Every such term represents an interval J_μ which *partially* covers Ω_I and contains *in its interior* one of the end points of Ω_I . Note that J_0 and J_{N+1} have length ∞ and if they are among J_μ 's then the corresponding term is 0. We shall term the intervals $J_j, j = 1, \dots, N$, with $|J_j| > t_m$ *big intervals* for the level m .

For the proof of (7.3) we consider five cases depending on the position of the interval Ω_I , $I \in \mathcal{I}_m$, relative to the intervals $J_j, j = 0, 1, \dots, N+1$, of g .

Case 1: Ω_I is a subset to one of the intervals $J_j, j = 0, 1, \dots, N+1$.

In this case (7.3) is trivially satisfied because $E_k(g, \Omega_I)_\tau = 0$.

Case 2: Both end points of Ω_I are either among the knots $x_j, j = 1, \dots, N$, or are in the interior of intervals J_μ with $|J_\mu| \leq t_m$.

Note that $\min\{|J_\mu|, t_m^{1+\tau}/|J_\mu|^\tau\} = |J_\mu|$ if J_μ belongs to the second sum in (7.3). Set $J^* = \cup_{|J_\nu \cap \Omega_I| > 0} J_\nu$. Then (7.3) follows from

$$E_k(g, \Omega_I)_\tau^\tau \leq E_1(g, J^*)_\tau^\tau \leq |J^*| \|g\|_{\text{BMO}}^\tau = \sum_{|J_\nu \cap \Omega_I| > 0} |J_\nu| \|g\|_{\text{BMO}}^\tau.$$

Case 3: Both end points of Ω_I are in the interior of two intervals J_μ with $|J_\mu| > t_m$ and there is only one knot among $x_j, j = 1, \dots, N$, in the interior of Ω_I .

Let x_j be the knot of g in the interior of Ω_I . Then the two big intervals covering the ends of Ω_I are J_j and J_{j+1} . We have

$$\begin{aligned} E_k(g, \Omega_I)_\tau^\tau &\leq E_1(g, \Omega_I)_\tau^\tau \leq c|\Omega_I| \left(|\Omega_I| \|g'\|_{L^\infty(J_j \cup J_{j+1})} \right)^\tau \\ &\leq c|\Omega_I|^{1+\tau} \left(\|g'\|_{L^\infty(J_j)}^\tau + \|g'\|_{L^\infty(J_{j+1})}^\tau \right). \end{aligned}$$

Further, using Lemma 2.1 we get for $J = J_j$ or $J = J_{j+1}$

$$\begin{aligned} \|g'\|_{L^\infty(J)} &= \|(g - \text{Avg}_J g)'\|_{L^\infty(J)} \\ &\leq \frac{c}{|J|} \|g - \text{Avg}_J g\|_{L^\infty(J)} \leq \frac{c}{|J|^2} \|g - \text{Avg}_J g\|_{L^1(J)} \\ &= \frac{c}{|J|} \cdot \frac{1}{|J|} \int_J |g(x) - \text{Avg}_J g| dx \leq \frac{c}{|J|} \|g\|_{\text{BMO}}. \end{aligned} \quad (7.4)$$

Therefore

$$E_k(g, \Omega_I)_\tau^\tau \leq ct_m^{1+\tau} \sum_{\substack{0 < |J_\mu \cap \Omega_I| < |\Omega_I| \\ J_\mu \setminus \Omega_I \neq \emptyset}} \frac{1}{|J_\mu|^\tau} \|g\|_{\text{BMO}}^\tau. \quad (7.5)$$

Note that (7.5) also holds if one of J_j and J_{j+1} is unbounded because $g = 0$ on this interval. Inequality (7.3) reduces to (7.5) in this case.

Case 4: Both end points of Ω_I are in the interior of two intervals J_μ with $|J_\mu| > t_m$ and there are at least two knots among $x_j, j = 1, \dots, N$, inside Ω_I .

Let $x_{j_1}, \dots, x_{j_2-1}, x_{j_1} < x_{j_2-1}$, be the knots of g in the interior of Ω_I . Then the intervals J_μ from the second sum in (7.3) are J_{j_1} are J_{j_2} . Denote $J = [x_{j_1}, x_{j_2-1}]$, $J^* = J_{j_1} \cup J \cup J_{j_2}$, and

$$V = \sum_{Q \in \Lambda_n, \#V_Q \cap J \geq 2} a_Q \varphi_Q.$$

Recall V_Q denotes the knots of φ_Q . Note that $|J| < |\Omega_I|$ and $|J_1|, |J_2| > \lambda|\Omega_I|$. Hence $\#V_Q \cap J \geq 2$ implies $\#V_Q \cap J = k + 1$ (for $Q \in \Lambda_n$), i.e. if a B-spline is involved in the sum of V , then it has all knots in J . The same argument shows that if g has two B-splines from *different* levels with only one knot in J , then this is a common knot. (Assuming the contrary, from the refinement property

of \mathcal{I} we get that the higher level contains an interval of length $< |\Omega_I|$ and hence it cannot contain an interval of length $> \lambda|\Omega_I|$. This contradicts the assumption that a B-spline from this level has a single knot in J .) Denote by y the common knot of all B-splines of g with only one knot in the interior of J (if there are such splines). Hence we can write

$$g(x) = U(x) + V(x), \quad x \in J^*,$$

where

$$U = \mathbb{1}_{[x_{j_1-1}, y]} P_{j_1} + \mathbb{1}_{[y, x_{j_2}]} P_{j_2}, \quad P_{j_1}, P_{j_2} \in \Pi_{k-1}, \quad U \in C(J).$$

Here P_{j_1}, P_{j_2} are from (7.2). In case there are no B-splines of g with only one knot in the interior of J the above representation holds with $P_{j_1} = P_{j_2}$ and y being any of $x_{j_1}, \dots, x_{j_2-1}$.

We have

$$E_k(g, \Omega_I)_\tau^\tau \leq cE_k(U, \Omega_I)_\tau^\tau + cE_k(V, \Omega_I)_\tau^\tau. \quad (7.6)$$

To estimate the best approximation of V we use Lemma 7.1 and $3J \subset J^*$ to obtain

$$\begin{aligned} E_k(V, \Omega_I)_\tau^\tau &\leq c\|V\|_\tau^\tau = c \int_J |V(x)|^\tau dx \leq c \int_{3J} |V(x) - \text{Avg}_{3J} V|^\tau dx \\ &\leq c \int_{3J} |g(x) - \text{Avg}_{3J} g|^\tau dx + c \int_{3J} |U(x) - \text{Avg}_{3J} U|^\tau dx \quad (7.7) \\ &\leq c|J|\|g\|_{BMO}^\tau + c|J|(|J|\|U'\|_{L^\infty(J^*)})^\tau. \end{aligned}$$

To estimate the best approximation of U we write

$$E_k(U, \Omega_I)_\tau^\tau \leq E_1(U, \Omega_I)_\tau^\tau \leq c|\Omega_I|(|\Omega_I|\|U'\|_{L^\infty(J^*)})^\tau \quad (7.8)$$

Inserting (7.7) and (7.8) in (7.6) we obtain

$$E_k(g, \Omega_I)_\tau^\tau \leq c \sum_{J_\nu \subset \Omega_I} |J_\nu| \|g\|_{BMO}^\tau + ct_m^{1+\tau} \|U'\|_{L^\infty(J^*)}^\tau. \quad (7.9)$$

For the estimate of U' we use Lemma 2.1 and (7.4) with $J = J_{j_1}$ to obtain

$$\|U'\|_{L^\infty([x_{j_1-1}, y])} \leq c\|U'\|_{L^\infty(J_{j_1})} = c\|g'\|_{L^\infty(J_{j_1})} \leq \frac{c}{|J_{j_1}|} \|g\|_{BMO}. \quad (7.10)$$

Similarly for the interval $[y, x_{j_2}] \supset J_{j_2}$.

Combining (7.9) and (7.10) we obtain (7.3). Estimate (7.3) is also valid in the case when one or both intervals J_{j_1}, J_{j_2} are unbounded because $g = U = 0$ here.

Case 5: One of the end points of Ω_I is among the knots $x_j, j = 1, \dots, N$, or is in the interior of an interval J_μ with $|J_\mu| \leq t_m$ and the other end point is in the interior of an interval J_μ with $|J_\mu| > t_m$.

This is a simplified version of Case 4. Let $x_{j_1}, \dots, x_{j_2-1}, x_{j_1} \leq x_{j_2-1}$, be the knots of g in the interior of Ω_I . Without loss of generality, let $|J_{j_1}| > t_m$ be the big interval containing the left end of I . Then for the right end of I belongs to $J_{j_2} = [x_{j_2-1}, x_{j_2}]$ and $|J_{j_2}| \leq t_m$. Set $J = [x_{j_1}, x_{j_2}]$, $J^* = J_{j_1} \cup J$, and

$$U = \mathbb{1}_{J^*} P_{j_1}, \quad P_{j_1} \in \Pi_k; \quad V(x) = g(x) - U(x), \quad x \in J^*.$$

Thus, the polynomial U coincides with g on J_{j_1} and the spline V is zero on J_{j_1} . We have

$$E_k(g, \Omega_I)_\tau^\tau = E_k(V, \Omega_I)_\tau^\tau \leq c \|V\|_\tau^\tau \leq c |J| \|g\|_{\text{BMO}}^\tau + c |J| (\|J\| \|U'\|_{L^\infty(J^*)})^\tau \tag{7.11}$$

as in (7.7) with the interval $3J$ replaced with $[x_{j_1} - |J|/2, x_{j_1}] \cup J \subset J^*$.

For the estimation of U' we use (7.10), which together with (7.11) gives (7.3). Estimate (7.3) is also valid in the case when J_{j_1} is unbounded because $g = U = 0$ here. Thus, (7.3) is proved.

Using estimates (7.3) we obtain

$$\begin{aligned} \sum_{I \in \mathcal{I}_m} |I|^{-1} E_k(g, \Omega_I)_\tau^\tau &\leq c \sum_{I \in \mathcal{I}_m} |I|^{-1} \left(\sum_{J_\nu \subset \Omega_I} |J_\nu| + \sum_{\substack{0 < |J_\mu \cap \Omega_I| < |\Omega_I| \\ J_\mu \setminus \Omega_I \neq \emptyset}} \min \left\{ |J_\mu|, \frac{t_m^{1+\tau}}{|J_\mu|^\tau} \right\} \right) \|g\|_{\text{BMO}}^\tau \\ &\leq c \left(\sum_{|J_\nu| \leq t_m} \frac{|J_\nu|}{t_m} + \sum_{|J_\nu| > t_m} \frac{t_m^\tau}{|J_\nu|^\tau} \right) \|g\|_{\text{BMO}}^\tau. \end{aligned} \tag{7.12}$$

In the last inequality we use that every J_ν with $|J_\nu| \leq t_m$ may belong to at most $2k - 1$ different intervals Ω_I , $I \in \mathcal{I}_m$, and that every J_μ (independently of $|J_\nu| \leq t_m$ or $t_m < |J_\nu| < \infty$) may partially cover at most $4k - 2$ different intervals Ω_I , $I \in \mathcal{I}_m$.

Finally, taking a sum on m in (7.12) and using $t_{m+1} \leq \rho t_m$ and $N \leq (k+1)n$ we obtain

$$\begin{aligned} \sum_{I \in \mathcal{I}} |I|^{-1} E_k(g, \Omega_I)_\tau^\tau &\leq c \sum_{m \in \mathbb{Z}} \left(\sum_{|J_\nu| \leq t_m} \frac{|J_\nu|}{t_m} + \sum_{|J_\nu| > t_m} \frac{t_m^\tau}{|J_\nu|^\tau} \right) \|g\|_{\text{BMO}}^\tau \\ &= c \sum_{\nu=1}^N \left(\sum_{\substack{m \in \mathbb{Z} \\ |J_\nu| \leq t_m}} \frac{|J_\nu|}{t_m} + \sum_{\substack{m \in \mathbb{Z} \\ |J_\nu| > t_m}} \frac{t_m^\tau}{|J_\nu|^\tau} \right) \|g\|_{\text{BMO}}^\tau \leq cn \|g\|_{\text{BMO}}^\tau. \end{aligned} \tag{7.13}$$

In view of (3.3) this completes the proof of the theorem in the case $1 \leq \tau < \infty$.

(b) Let $0 < \tau < 1$. We shall use the identification $\dot{B}_\tau^{\alpha,k} = \dot{B}_\tau^{\alpha,k}(E, 1)$, see (3.5). From (7.3) with $\tau = 1$ and the concavity of y^τ for $0 < \tau < 1$ we obtain

for every $I \in \mathcal{I}_m$, $m \in \mathbb{Z}$, the inequality

$$\begin{aligned} & (|I|^{-1}E_k(g, \Omega_I)_1)^\tau \\ & \leq c \left(\sum_{J_\nu \subset \Omega_I} \frac{|J_\nu|^\tau}{t_m^\tau} + \sum_{\substack{0 < |J_\mu \cap \Omega_I| < |\Omega_I| \\ J_\mu \setminus \Omega_I \neq \emptyset}} \min \left\{ \frac{|J_\mu|^\tau}{t_m^\tau}, \frac{t_m^\tau}{|J_\mu|^\tau} \right\} \right) \|g\|_{BMO}^\tau. \end{aligned} \quad (7.14)$$

Now, proceeding as in the proof of (7.12) and (7.13) we obtain from (7.14)

$$\begin{aligned} & \sum_{I \in \mathcal{I}} (|I|^{-1}E_k(g, \Omega_I)_1)^\tau \\ & \leq c \sum_{\nu=1}^N \left(\sum_{\substack{m \in \mathbb{Z} \\ |J_\nu| \leq t_m}} \frac{|J_\nu|^\tau}{t_m^\tau} + \sum_{\substack{m \in \mathbb{Z} \\ |J_\nu| > t_m}} \frac{t_m^\tau}{|J_\nu|^\tau} \right) \|g\|_{BMO}^\tau \leq cn \|g\|_{BMO}^\tau. \end{aligned}$$

In view of (3.5) this completes the proof of the theorem in the case $0 < \tau < 1$. \square

8. Appendix

8.1. Proofs of Lemma 2.5 and Theorem 2.1

Proof of Lemma 2.5. Let $\{a_Q\}_{Q \in \mathcal{Q}}$ be a sequence of complex numbers and fix a compact interval $J \subset \mathbb{R}$. Consider first the case when $0 < \tau < p$. By [16, Theorem 3.3] we have

$$\left\| \sum_{Q \in \mathcal{Q}, Q \subset J} |a_Q \varphi_Q| \right\|_p \leq c \left(\sum_{Q \in \mathcal{Q}, Q \subset J} \|a_Q \varphi_Q\|_p^\tau \right)^{1/\tau}. \quad (8.1)$$

Clearly, $\|a_Q \varphi_Q\|_p \leq c|a_Q||Q|^{1/p} \leq c|a_Q||J|^{1/p}$, which along with (8.1) implies (2.21).

In the case $\tau \geq p$ we choose $q > \tau$ and use Hölder's inequality and (2.21) in the proven case from above to obtain

$$\left\| \frac{1}{|J|^{1/p}} \sum_{\substack{Q \in \mathcal{Q} \\ Q \subset J}} |a_Q \varphi_Q| \right\|_p \leq \left\| \frac{1}{|J|^{1/q}} \sum_{\substack{Q \in \mathcal{Q} \\ Q \subset J}} |a_Q \varphi_Q| \right\|_q \leq c \left(\sum_{\substack{Q \in \mathcal{Q} \\ Q \subset J}} |a_Q|^\tau \right)^{1/\tau}.$$

The proof is complete. \square

Proof of Theorem 2.1. Part (b) is trivial. For the proof of part (a) assume $\tau > 1$. Denote

$$f_\nu := \sum_{j > \nu} \sum_{Q \in \mathcal{Q}_j} a_Q \varphi_Q, \quad \nu \in \mathbb{Z}.$$

We claim that

$$\|f_\nu\|_{\text{BMO}} \leq c \left(\sum_{j>\nu} \sum_{Q \in \mathcal{Q}_j} |a_Q|^\tau \right)^{1/\tau} =: \|\{a_Q\}\|_{\ell^\tau(\nu)}. \quad (8.2)$$

Let J be an arbitrary compact interval in \mathbb{R} . Then there exist $m \in \mathbb{Z}$ such that if $Q \in \mathcal{Q}_m$ and $Q \cap J \neq \emptyset$, then $|Q| \sim |J|$ and $Q \subset 2J$. Denote $\tilde{J} := 2J$.

First, we consider the less favorable case $\nu < m$. We split f_ν into two: $f_\nu = f_m + (f_\nu - f_m)$. Using Lemma 2.5 we get

$$\begin{aligned} \frac{1}{|J|} \int_J |f_m(x)| dx &\leq \frac{c}{|\tilde{J}|} \int_{\tilde{J}} \sum_{j>m} \sum_{\substack{Q \in \mathcal{Q}_j, \\ Q \subset \tilde{J}}} |a_Q \varphi_Q(x)| dx \\ &\leq c \left(\sum_{j>m} \sum_{Q \in \mathcal{Q}_j} |a_Q|^\tau \right)^{1/\tau} \leq c \|\{a_Q\}\|_{\ell^\tau(\nu)}. \end{aligned} \quad (8.3)$$

Denote $F_{\nu m} := f_\nu - f_m$ and fix $y \in J$. We claim that

$$\frac{1}{|J|} \int_J |F_{\nu m}(x) - F_{\nu m}(y)| dx \leq c \|\{a_Q\}\|_{\ell^\tau(\nu)}. \quad (8.4)$$

Indeed, let $Q \in \mathcal{Q}_j$, $j \leq m$, and assume $Q \cap J \neq \emptyset$. Then for $x \in Q$

$$|\varphi_Q(x) - \varphi_Q(y)| \leq |x - y| \|\varphi'_Q\|_\infty \leq c|x - y||Q|^{-1}.$$

Fix $x \in J$ and assume that x belongs to the interior of some $Q^* \in \mathcal{Q}_m$. Using the above we get

$$\begin{aligned} |F_{\nu m}(x) - F_{\nu m}(y)| &\leq \sum_{j=\nu+1}^m \sum_{\substack{Q \in \mathcal{Q}_j, \\ Q \ni x}} |a_Q| |\varphi_Q(x) - \varphi_Q(y)| \\ &\leq c \|\{a_Q\}\|_{\ell^\tau(\nu)} \sum_{j=\nu+1}^m \sum_{\substack{Q \in \mathcal{Q}_j, \\ Q \ni x}} |x - y| |Q|^{-1} \\ &\leq c \|\{a_Q\}\|_{\ell^\tau(\nu)} |J| \sum_{j \leq m} \sum_{\substack{Q \in \mathcal{Q}_j, \\ Q \ni x}} |Q|^{-1} \\ &\leq c \|\{a_Q\}\|_{\ell^\tau(\nu)} |J| |Q^*|^{-1} \leq c \|\{a_Q\}\|_{\ell^\tau(\nu)}. \end{aligned}$$

Here we used that $\sum_{j \leq m} \sum_{Q \in \mathcal{Q}_j, Q \ni x} |Q|^{-1} \leq c|Q^*|^{-1}$, which follows from the conditions on the underlying regular multilevel partition \mathcal{I} . Estimate (8.4) follows readily from the above inequalities.

From (8.3) and (8.4) it follows that

$$\frac{1}{|J|} \int_J |f_\nu(x) - F_{\nu m}(y)| dx \leq c \|\{a_Q\}\|_{\ell^\tau(\nu)},$$

which implies (8.2).

In the easier case $\nu \geq m$ (8.2) will follow directly from an estimate similar to (8.3). In turn, (8.2) implies that for any $\nu, \mu \in \mathbb{Z}$, $\mu > \nu$,

$$\|f_\nu - f_\mu\|_{BMO} \leq c \left(\sum_{j=\nu+1}^{\mu} \sum_{Q \in \mathcal{Q}_j} |a_Q|^\tau \right)^{1/\tau} \rightarrow 0 \quad \text{as } \nu, \mu \rightarrow -\infty.$$

Since BMO is complete, it follows that $\lim_{\nu \rightarrow -\infty} f_\nu = f$ for some $f \in BMO$, where the convergence is in the BMO-norm. It also follows that $\|f\|_{BMO} \leq c \|\{a_Q\}\|_{\ell^\tau}$, which confirms (2.20).

Finally, because the norm $\|\{a_Q\}\|_{\ell^\tau}$ does not change when reshuffling the terms in its definition, it readily follows from the above proof that the convergence in $\sum_{Q \in \mathcal{Q}} a_Q \varphi_Q$ is unconditional in BMO. \square

8.2. Proof of Equivalence (3.3)

Denote

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(E)} := \left(\sum_{I \in \mathcal{I}} (|I|^{-\alpha} \omega_k(f, \Omega_I)_\tau)^\tau \right)^{1/\tau}.$$

Denote by \mathcal{D}_m the m th level dyadic intervals ($|J| = 2^{-m}$ if $J \in \mathcal{D}_m$) and set $\mathcal{D} := \cup_{m \in \mathbb{Z}} \mathcal{D}_m$. Clearly, see (2.12),

$$\omega_k(f, 2^{-m})_\tau^\tau \leq \sum_{J \in \mathcal{D}_m} \omega_k(f, (2k+1)J)_\tau^\tau.$$

From the conditions on \mathcal{I} it follows that for each $J \in \mathcal{D}$ the interval $(2k+1)J$ is contained in some interval Ω_I , $I \in \mathcal{I}$, of minimum length (hence, $|\Omega_I| \sim |J|$), and each Ω_I , $I \in \mathcal{I}$, contains a uniformly bounded number of such intervals $J \in \mathcal{D}$. Therefore,

$$\begin{aligned} \|f\|_{\dot{B}_\tau^{\alpha,k}}^\tau &\sim \sum_{m \in \mathbb{Z}} 2^m \omega_k(f, 2^{-m})_\tau^\tau \\ &\leq c \sum_{J \in \mathcal{D}} |J|^{-1} \omega_k(f, (2k+1)J)_\tau^\tau \leq c \sum_{I \in \mathcal{I}} (|I|^{-\frac{1}{\tau}} \omega_k(f, \Omega_I)_\tau)^\tau \end{aligned}$$

and hence $\|f\|_{\dot{B}_\tau^{\alpha,k}} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E)}$.

For the estimate in the other direction we use (2.15). We obtain

$$\begin{aligned} &\sum_{I \in \mathcal{I}, \frac{1}{2} < 2^m |I| \leq 2} (|I|^{-\frac{1}{\tau}} \omega_k(f, \Omega_I)_\tau)^\tau \\ &\leq c \sum_{I \in \mathcal{I}, \frac{1}{2} < 2^m |I| \leq 2} |I|^{-2} \int_0^{|\Omega_I|} \int_{\Omega_I} |\Delta_h^k(f, x, \Omega_I)|^\tau dx dh \\ &\leq c 2^{2m} \int_0^{c2^{-m}} \int_{\mathbb{R}} |\Delta_h^k f(x)|^\tau dx dh \\ &\leq c 2^m \omega_k(f, c2^{-m})_\tau^\tau \leq c 2^m \omega_k(f, 2^{-m})_\tau^\tau. \end{aligned}$$

Here we used that only finitely many of the intervals $\{\Omega_I : I \in \mathcal{I}, \frac{1}{2} < 2^m |I| \leq 2\}$ may overlap at any point $x \in \mathbb{R}$, and $\omega_k(f, c2^{-m})_\tau \leq c' \omega_k(f, 2^{-m})_\tau$. From above and (3.2) we get

$$\|f\|_{\dot{B}_\tau^{\alpha,k}(E)} = \sum_{I \in \mathcal{I}} (|I|^{-\frac{1}{\tau}} \omega_k(f, \Omega_I)_\tau)^\tau \leq c \sum_{m \in \mathbb{Z}} 2^m \omega_k(f, 2^{-m})_\tau^\tau \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}}^\tau.$$

This and the estimate in the other direction from above yield the equivalence $\|f\|_{\dot{B}_\tau^{\alpha,k}} \sim \|f\|_{\dot{B}_\tau^{\alpha,k}(E)}$. \square

8.3. Proof of Theorem 3.1

Let $f \in \dot{B}_\tau^{\alpha,k}(E, q)$. In light of Proposition 3.2 there exists a polynomial $P \in \Pi_k$ such that $\|f - P\|_{\text{BMO}} \leq c \|f\|_{\dot{B}_\tau^{\alpha,k}(E, q)}$. Let $T_{m,q}$ be the quasi-interpolant from (2.19)

(a) First we show that

$$\lim_{m \rightarrow \infty} \|f - P - T_{m,q}(f - P)\|_{\text{BMO}} = 0. \quad (8.5)$$

Fix $\varepsilon > 0$. In light of (3.5) there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{j=m_0}^{\infty} \sum_{I \in \mathcal{I}_j} |I|^{-\tau/q} E_k(f, \Omega_I)_q^\tau < \varepsilon^\tau. \quad (8.6)$$

Fix $m \geq m_0$. Let J be an arbitrary compact interval and let ν be its level (see §2.2). We next consider two cases depending on the size of $|J|$.

Case 1: $\nu > m$. There exist two adjacent intervals I_1, I_2 in \mathcal{I}_ν such that $J \subset I_1 \cup I_2$, $|J| \sim |I_1| \sim |I_2|$. For an appropriate constant c^\diamond (to be selected) we have

$$\begin{aligned} & \frac{1}{|J|} \int_J |f(x) - P(x) - T_{m,q}(f - P)(x) - c^\diamond|^q dx \\ &= \frac{1}{|J|} \int_J |f(x) - T_{m,q}(f)(x) - c^\diamond|^q dx \leq \frac{c}{|J|} \int_J |f(x) - T_{\nu,q}(f)(x)|^q dx \\ &+ c \|T_{\nu,q}(f) - T_{m,q}(f) - c^\diamond\|_{L^\infty(J)}^q =: S_1 + S_2. \end{aligned} \quad (8.7)$$

To estimate S_1 we use Lemma 2.4 and (8.6) to obtain

$$\begin{aligned} S_1 &\leq \frac{1}{|J|} \int_{I_1 \cup I_2} |f(x) - T_{\nu,q}(f)(x)|^q dx \\ &\leq c |I_1|^{-1} E_k(f, \Omega_{I_1})_q^q + c |I_2|^{-1} E_k(f, \Omega_{I_2})_q^q < c \varepsilon^q. \end{aligned} \quad (8.8)$$

To estimate S_2 we shall use the abbreviated notation $\mathbf{q}_j := T_{j,q}(f) - T_{j-1,q}(f)$ (see (3.6)). We fix $y \in J$ and select the constant $c^\diamond := T_{\nu,q}(f)(y) - T_{m,q}(f)(y)$.

Then for any $x \in J$ we have

$$|T_{\nu,q}(f)(x) - T_{m,q}(f)(x) - c^\diamond| = \left| \sum_{j=m+1}^{\nu} (\mathbf{q}_j(x) - \mathbf{q}_j(y)) \right| \leq |J| \sum_{j=m+1}^{\nu} \|\mathbf{q}'_j\|_{L^\infty(J)}.$$

The choice of ν implies that for any $j = m+1, \dots, \nu$ there exist two adjacent intervals I'_j, I''_j in \mathcal{I}_j such that $J \subset I'_j \cup I''_j$. Using that \mathbf{q}_j is a polynomial of degree $k-1$ on I'_j and on I''_j we obtain from Lemma 2.1 and (2.3)

$$\begin{aligned} |J| \sum_{j=m+1}^{\nu} \|\mathbf{q}'_j\|_{L^\infty(J)} &\leq |J| \sum_{j=m+1}^{\nu} \left(\|\mathbf{q}'_j\|_{L^\infty(I'_j)} + \|\mathbf{q}'_j\|_{L^\infty(I''_j)} \right) \\ &\leq c|J| \sum_{j=m+1}^{\nu} \left(|I'_j|^{-1} \|\mathbf{q}_j\|_{L^\infty(I'_j)} + |I''_j|^{-1} \|\mathbf{q}_j\|_{L^\infty(I''_j)} \right) \\ &\leq c \sum_{j=m+1}^{\nu} \frac{|J|}{|I'_j|} \left(|I'_j|^{-1/q} \|\mathbf{q}_j\|_{L^q(I'_j)} + |I''_j|^{-1/q} \|\mathbf{q}_j\|_{L^q(I''_j)} \right). \end{aligned}$$

Using (3.11), (8.6) and (2.4) in the above, we obtain

$$S_2 = c \|T_{\nu,q}(f) - T_{m,q}(f) - c^\diamond\|_{L^\infty(J)}^q < c\varepsilon^q.$$

This together with (8.7) and (8.8) implies

$$\frac{1}{|J|} \int_J |f(x) - P(x) - T_{m,q}(f - P)(x) - c^\diamond|^q dx \leq c\varepsilon^q. \quad (8.9)$$

Case 2: $\nu \leq m$. Hence $|J| \geq c|I|$ for all $I \in \mathcal{I}_m$ and $\sum_{I \in \mathcal{I}_m, I \cap J \neq \emptyset} |I| \leq c|J|$. Using Lemma 2.4 and (8.6) we obtain

$$\begin{aligned} \frac{1}{|J|} \int_J |f(x) - T_{m,q}(f)(x)|^q dx &\leq \frac{1}{|J|} \sum_{I \in \mathcal{I}_m, I \cap J \neq \emptyset} \int_I |f(x) - T_{m,q}(f)(x)|^q dx \\ &\leq \frac{c}{|J|} \sum_{I \in \mathcal{I}_m, I \cap J \neq \emptyset} E_k(f, \Omega_I)_q^q \leq \frac{c}{|J|} \sum_{I \in \mathcal{I}_m, I \cap J \neq \emptyset} |I| \varepsilon^q \leq c\varepsilon^q. \end{aligned}$$

In turn, this and estimate (8.9) yield

$$\|f - P - T_{m,q}(f - P)\|_{BMO} \leq c\varepsilon, \quad \forall m \geq m_0,$$

which implies (8.5).

(b) We next prove that

$$\lim_{m \rightarrow -\infty} \|T_{m,q}(f - P)\|_{BMO} = 0. \quad (8.10)$$

Let $\varepsilon > 0$. By (3.5) it follows that there exists $m_1 \in \mathbb{Z}$ such that

$$\sum_{j=-\infty}^{m_1} \sum_{I \in \mathcal{I}_j} |I|^{-\tau/q} E_k(f, \Omega_I)_q^\tau < \varepsilon^\tau. \quad (8.11)$$

Fix $m < m_1$. Let J be an arbitrary compact interval and let $\nu - 1$ be its level (see §2.2). Then J contains some interval $I \in \mathcal{I}_\nu$ and $|J| \sim |I|$.

As in part (a) we shall use the abbreviated notation $\mathbf{q}_j := T_{j,q}(f) - T_{j-1,q}(f)$. Observe that $T_{m,q}(f - P) = T_{m,q}(f) - P$. Using this we write

$$T_{m,q}(f - P) = \sum_{j=N+1}^m \mathbf{q}_j + T_{N,q}(f) - P = \sum_{j=N+1}^m \sum_{Q \in \mathcal{Q}_j} b_{Q,q}(f) \varphi_Q + T_{N,q}(f) - P,$$

where $N < m$, $N < \nu$ and $\nu - N$ is sufficiently large (to be determined). Clearly, for any constant c^* (to be selected) there exists a constant c^{**} such that

$$\begin{aligned} \frac{1}{|J|} \int_J |T_{m,q}(f - P) - c^{**}|^q dx &\leq c \left\| \sum_{j=N+1}^m \sum_{Q \in \mathcal{Q}_j} b_{Q,q}(f) \varphi_Q \right\|_{\text{BMO}}^q \\ &+ \frac{c}{|J|} \int_J |T_{N,q}(f)(x) - P(x) - c^*|^q dx =: S_1 + S_2. \end{aligned} \quad (8.12)$$

To estimate S_1 we invoke Theorem 2.1, (3.9), (3.8), (3.11), (8.11) and obtain

$$\begin{aligned} S_1 &\leq c \left(\sum_{j=-\infty}^m \sum_{Q \in \mathcal{Q}_j} |b_{Q,q}(f)|^\tau \right)^{q/\tau} \leq c \left(\sum_{j=-\infty}^m \sum_{I \in \mathcal{I}_j} |I|^{-\tau/q} \|\mathbf{q}_j\|_{L^q(I)}^\tau \right)^{q/\tau} \\ &\leq c \left(\sum_{j=-\infty}^m \sum_{I \in \mathcal{I}_j} |I|^{-\tau/q} E_k(f, \Omega_I)_q^\tau \right)^{q/\tau} < c\varepsilon^q. \end{aligned} \quad (8.13)$$

To estimate S_2 we recall that $N < \nu$ and hence there are two adjacent intervals I_1, I_2 in \mathcal{I}_N such that $J \subset I_1 \cup I_2$. Let $I^\circ \in \mathcal{I}_{N-1}$ be the only parent of I_1 ($I_1 \subset I^\circ$). Clearly, $\Omega_{I_1} \cup \Omega_{I_2} \subset \Omega_{I^\circ}$. Let $R \in \Pi_k$ be a polynomial such that

$$\|f - R\|_{L^q(\Omega_{I^\circ})} \leq c E_k(f, \Omega_{I^\circ})_q. \quad (8.14)$$

We now choose the constant c^* to be $c^* := R(y) - P(y)$, where $y \in J$ is fixed. We have

$$\begin{aligned} S_2 &\leq \frac{c}{|J|} \int_J |T_{N,q}(f)(x) - P(x) - c^*|^q dx \\ &\leq \frac{c}{|J|} \int_J |T_N(\mathcal{P}_{N,q} - R)(x)|^q dx + \frac{c}{|J|} \int_J |R(x) - P(x) - c^*|^q dx =: U_1 + U_2. \end{aligned} \quad (8.15)$$

Using (2.17) and (2.10) we get

$$\begin{aligned} U_1 &= \frac{c}{|J|} \int_J |T_N(\mathcal{P}_{N,q} - R)(x)|^q dx \leq c \|T_N(\mathcal{P}_{N,q} - R)\|_{L^\infty(I_1 \cup I_2)}^q \\ &\leq c \|\mathcal{P}_{N,q} - R\|_{L^\infty(\Omega_{I_1} \cup \Omega_{I_2})}^q \leq c \max_{I \in \mathcal{I}_N, I \subset \Omega_{I_1} \cup \Omega_{I_2}} \|\mathcal{P}_{N,q} - R\|_{L^\infty(I)}^q \end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{I \in \mathcal{I}_N, I \subset \Omega_{I_1} \cup \Omega_{I_2}} |I|^{-1} \|\mathcal{P}_{N,q} - R\|_{L^q(I)}^q \\
&\leq c \sum_{I \in \mathcal{I}_N, I \subset \Omega_{I_1} \cup \Omega_{I_2}} (|I|^{-1} \|f - \mathcal{P}_{N,q}\|_{L^q(I)}^q + |I|^{-1} \|f - R\|_{L^q(I)}^q).
\end{aligned}$$

By the definition of $\mathcal{P}_{N,q}$ we have $\|f - \mathcal{P}_{N,q}\|_{L^q(I)} \leq cE_k(f, I)_q \leq cE_k(f, \Omega_I)_q$ (see (2.18) and (2.16)). We use this, (8.14), (8.11) and the above estimates for U_1 to obtain

$$U_1 \leq c \sum_{j=N-1}^N \sum_{I \in \mathcal{I}_j, I \cap (\Omega_{I_1} \cup \Omega_{I_2}) \neq \emptyset} |I|^{-1} E_k(f, \Omega_I)_q^q < c\varepsilon^q. \quad (8.16)$$

Here we used that in the double sum above there is a constant number (depending only on k) of terms.

In estimating U_2 we shall use the abbreviated notation $I_\star := I_1 \cup I_2$, where $I_1, I_2 \in \mathcal{I}_N$ are determined above. Using that R and P are polynomials on I_\star and Lemma 2.1 we get

$$\begin{aligned}
U_2 &= \frac{c}{|J|} \int_J |R(x) - P(x) - (R(y) - P(y))|^q dx \leq c(|J| \| (R - P)' \|_{L^\infty(I_\star)})^q \\
&= c(|J| \| (R - P - \tilde{c})' \|_{L^\infty(I_\star)})^q \leq c(|J|/|I_\star|)^q \|R - P - \tilde{c}\|_{L^\infty(I_\star)}^q \\
&\leq c(|J|/|I_\star|)^q |I_\star|^{-1} \|R - P - \tilde{c}\|_{L^q(I_\star)}^q,
\end{aligned}$$

where the constant \tilde{c} is defined by $\tilde{c} := \text{Avg}_{I_\star}(f - P)$. We now use (8.14), (2.2), (8.11), and obtain

$$\begin{aligned}
&|I_\star|^{-1} \|R - P - \tilde{c}\|_{L^q(I_\star)}^q \\
&\leq c|I_\star|^{-1} \|f - R\|_{L^q(I_\star)}^q + c|I_\star|^{-1} \|f - P - \text{Avg}_{I_\star}(f - P)\|_{L^q(I_\star)}^q \\
&\leq c|I^\diamond|^{-1} E_k(f, \Omega_{I^\diamond})_q^q + c\|f - P\|_{\text{BMO}}^q \leq c\varepsilon^q + c\|f - P\|_{\text{BMO}}^q.
\end{aligned}$$

On the other hand, because $|J| \sim |I|$ with $I \in \mathcal{I}_\nu$ and $I_\star = I_1 \cup I_2$ with $I_1, I_2 \in \mathcal{I}_N$, we infer from (2.4) that $|J|/|I_\star| \leq c\rho^{\nu-N}$. Putting all of the above together we obtain

$$U_2 \leq c\rho^{(\nu-N)q} (\varepsilon^q + \|f - P\|_{\text{BMO}}^q).$$

Combining this with (8.15) and (8.16) we get

$$S_2 \leq c\varepsilon^q + c\rho^{(\nu-N)q} \|f - P\|_{\text{BMO}}^q.$$

In turn, this along with (8.12) and (8.13) yield

$$\frac{1}{|J|} \int_J |T_{m,q}(f - P) - c^\star|^q dx \leq c\varepsilon^q + c\rho^{(\nu-N)q} \|f - P\|_{\text{BMO}}^q, \quad \forall m < m_1.$$

Since the constant c in this estimate is independent of N and $f - P \in \text{BMO}$, by letting $N \rightarrow -\infty$ we arrive at

$$\frac{1}{|J|} \int_J |T_{m,q}(f - P) - c^*|^q dx \leq c\varepsilon^q, \quad \forall m < m_1.$$

This estimate implies $\|T_{m,q}(f - P)\|_{\text{BMO}} \leq c\varepsilon$ for all $m < m_1$, which yields (8.10).

Clearly, decomposition (3.18) follows at once by (8.5) and (8.10). Inequality (3.19) follows by Lemma 3.2. The unconditional convergence in (3.18) is a consequence of Theorem 2.1. Finally, the unconditional convergence in BMO of the series in (3.18) and the fact that each φ_Q is in $C_0(\mathbb{R})$ leads to the conclusion that $f - P$ is in VMO. \square

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