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LOCALIZED TIGHT FRAMES ON SPHERES*

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Abstract. In this paper we wish to present a new class of tight frames on the sphere. These frames have excellent pointwise localization and approximation properties. These properties are based on pointwise localization of kernels arising in the spectral calculus for certain self-adjoint operators, and on a positive-weight quadrature formula for the sphere that the authors have recently developed. Improved bounds on the weights in this formula are another by-product of our analysis.

Key words. tight frames, n-sphere, localization

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1. Introduction.

Frames were introduced in the 1950s by Duffin and Schaeffer [4] to represent functions via over-complete sets. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. In that case, a set $\{\psi_j\}_{j\in\mathcal{J}}$ is a *frame* if there are constants $c, C > 0$ such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \le \sum_{j\in\mathcal{J}} |\langle f, \psi_j\rangle|^2 \le C\|f\|^2.$$

The smallest C and largest c are called upper and lower *frame bounds*. If $C = c$, we say the frame is *tight*. If $C = c = 1$, then the frame is *normalized*, and if in addition $\|\psi_j\| = 1$ for all j, then the frame is an orthonormal basis.

Frames, including tight ones, arise naturally in wavelet analysis on $\mathbb{R}^n$ when continuous wavelet transforms are discretized. They provide a redundancy that helps reduce the effect of noise in data, and they have been constructed, studied, and employed extensively in both theoretical and applied problems [1, 2, 6, 7, 10, 12].

Tight frames are similar in many respects to orthonormal wavelet bases; decomposing and synthesizing a signal or image from known data are tasks carried out with the same set of functions, the ones in the frame or in the basis. A feature that makes one frame preferable to another is simultaneous localization of the frame functions in both space and frequency. Frames with this feature have been successfully developed in $\mathbb{R}^n$ [1, 2].

On $\mathbb{S}^n$, the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, various types of both wavelets and frames have been constructed and used; see [8, 13, 16, 21] for references and more discussion. Tight, well-localized frames are another matter.

The purpose of this paper is to construct and study a class of well-localized, computationally implementable, tight frames on $\mathbb{S}^n$. Central to this construction is a key result of this paper, Theorem 3.5. This result concerns pointwise localization for a

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family of kernels for certain operators on $\mathbb{S}^n$; the family depends on a parameter and localization increases as the parameter becomes small. The frame functions, which are compactly supported in the frequency domain, are constructed from such kernels. This construction has an interesting connection to wavelet masks, which we will point out below. Another application of our localization result, one that is essential to turning the frame functions into a tight frame—that is, a hierarchical, multiresolution setting—is an improved positive-weight quadrature formula for $\mathbb{S}^n$, where the weights have known bounds. This quadrature formula is used for discretization purposes. In addition to Theorem 3.5, the main results of this paper are Proposition 5.1, Theorem 5.2, and Corollary 5.3. The first of these concerns the approximation power of the frames, the second shows that the frames are tight, and the third shows that the frame functions have excellent spatial localization.

The frame functions and quadrature formula are of interest in their own right. In particular, they can be used in the construction and characterization of many of the classical Banach spaces, including $L^p(\mathbb{S}^n)$, Besov spaces, and Triebel–Lizorkin spaces [18]. We mention also that the operator-theoretic approach that we use here may provide a foundation for extending our results to other Riemannian manifolds.

Strategy. The best way to view our method for constructing frames is to take an operator-theoretic approach. Let E_λ be the (right-continuous) spectral family for an unbounded, nonnegative, self-adjoint operator L defined on a Hilbert space $\mathcal{H}$. Thus, $\mathsf{L} = \int_{0-}^{\infty} \lambda d\mathsf{E}_\lambda$. On the sphere $\mathbb{S}^n$, this will be related to the square root of the Laplace–Beltrami operator shifted by a constant. For now, that connection isn't required.

We wish to decompose the spectral family in a way reminiscent of the decomposition of frequency space used by Meyer [10, 12] in connection with the construction of his wavelets. For this, we need a function $a \in C(\mathbb{R})$, with support in $[\frac{1}{2}, 2]$, and satisfying $|a(t)|^2 + |a(2t)|^2 \equiv 1$ on $[\frac{1}{2}, 1]$. Such a function can be easily constructed out of an *orthogonal wavelet mask* m_0 [2, section 8.3]. In fact, if $m_0(\xi) \in C^{k+1}$, then $a(t) := m_0(\pi \log_2(t))$ on $[\frac{1}{2}, 2]$, and 0 otherwise, is a C^k function that satisfies the appropriate criteria.

Define $b \in C(\mathbb{R})$ by

$$b(t) := \begin{cases} 1, & t \le 1, \\ a(t)\overline{\tilde{a}(t)}, & t > 1. \end{cases} \tag{1}$$

Using the properties of a we see that $\sum_{j=-\infty}^{J} |a(t/2^j)|^2 = b(t/2^J)$ if $t > 0$ and is 0 if $t \le 0$. Integrating both sides above with respect to $d\mathsf{E}_\lambda$ and using the spectral calculus for L, we obtain $\sum_{j=-\infty}^{J} a(\mathsf{L}/2^j)a(\mathsf{L}/2^j)^* = b(\mathsf{L}/2^J) - \mathsf{E}_0$. Define the operators,

$$\mathsf{A}_j = a(\mathsf{L}/2^j), \tag{2}$$

$$\mathsf{B}_J := b(\mathsf{L}/2^J), \tag{3}$$

and note that the relationship derived above becomes $\sum_{j=-\infty}^{J} \mathsf{A}_j \mathsf{A}_j^* = \mathsf{B}_J - \mathsf{E}_0$. Finally, it is easy to show that the strong limit of B_J as $J \to \infty$ is I, the identity. Taking limits above then yields $\sum_{j=-\infty}^{\infty} \mathsf{A}_j \mathsf{A}_j^* = \mathsf{I} - \mathsf{E}_0$.

We now can use this identity to define *decomposition* and *reconstruction* operators for $f \in \mathcal{H}$, which are, respectively,

$$f \to w_j = \mathsf{A}_j^* f \text{ and } f = \mathsf{E}_0 f + \sum_{j=-\infty}^{\infty} \mathsf{A}_j w_j.$$

PROPOSITION 1.1. *For any $a \in C(\mathbb{R})$ satisfying the conditions above, the operator frame that we have constructed is tight in the sense that*

$$\|f\|^2 = \|\mathsf{E}_0 f\|^2 + \sum_{j=-\infty}^{\infty} \|\mathsf{A}_j^* f\|^2.$$

In addition, we have that $\langle w_j, w_{j'} \rangle = 0$ for $|j - j'| \geq 2$, where $w_j = \mathsf{A}_j^* f$.

Proof. This follows immediately from the decomposition and reconstruction formulas above, the properties of a and of the spectral family. $\square$

Note that the decomposition arrived at above is *nearly* orthogonal. The level j decomposition w_j is *not* orthogonal to $w_{j\pm1}$, but it *is* orthogonal to the decomposition at all other levels.

As we pointed out above, when we deal with the sphere $\mathbb{S}^n$, we will take L proportional to $L_n := \sqrt{\lambda_n^2 - \Delta_{\mathbb{S}^n}}$, where $\lambda_n := \frac{n-1}{2}$. Notation and background pertinent to this operator, spherical harmonics, and related topics can be found in section 2.1.

In section 2.2, we show that with this choice of L the decomposition operator A_j^* is given in terms of a kernel $\overline{A_j}(\xi \cdot \eta)$, $\xi, \eta \in \mathbb{S}^n$, which is a polynomial in $\xi \cdot \eta$. Using the addition theorem for spherical harmonics, one can see that the level j decomposition $w_j(\eta) = \langle f(\xi), A_j(\xi \cdot \eta) \rangle_{\mathbb{S}^n}$ is a finite sum of spherical harmonics.

In the reconstruction phase, we need to find $\mathsf{A}_j w_j(\omega) = \langle w_j(\eta), \overline{A_j}(\omega \cdot \eta) \rangle_{\mathbb{S}^n}$. The integrand in this inner product is also a finite sum of spherical harmonics. At this point, the order of the spherical harmonics is such that we can compute the integral exactly using a quadrature formula introduced in [14, 15] and, in section 4, developed into the tool we need here. The point is that the frame functions have the form $\psi_{j,\xi}(\eta) = \sqrt{c_{j,\xi}} A_j(\eta \cdot \xi)$, where the $c_{j,\xi}$ and $\xi \in X_j$ are weights and nodes for the quadrature formula appropriate to level j. The details are given in section 5.

What makes these frame functions special is that they have excellent pointwise localization properties. These properties follow from the results on pointwise localization of certain kernels, given in section 3.

2. Near-orthogonal spectral decomposition for $\mathbb{S}^n$.

2.1. Background and notation for $\mathbb{S}^n$.

Centers and decompositions of $\mathbb{S}^n$*.* Let X be a finite set of distinct points in $\mathbb{S}^n$; we will call these the *centers*. There are several important quantities associated with this set: the *mesh norm*, $h_X = \sup_{y \in \mathbb{S}^n} \inf_{\xi \in X} d(\xi, y)$, where $d(\cdot,\cdot)$ is the geodesic distance between points on the sphere; the *separation radius*, $q_X = \frac{1}{2} \min_{\xi \neq \xi'} d(\xi, \xi')$; and the *mesh ratio*, $\rho_X := h_X / q_X \geq 1$.

For $\rho \geq 1$, let $\mathcal{F}_\rho = \mathcal{F}_\rho(\mathbb{S}^n)$ be the family of all sets of centers X with $\rho_X \leq \rho$; we will say that the family $\mathcal{F}_\rho$ is *ρ-uniform.* Unless confusion would arise, we will not indicate $\mathbb{S}^n$, and just use $\mathcal{F}_\rho$ to designate a family. The specific sphere $\mathbb{S}^n$ will be clear from the context. We will also say that a set of centers X is ρ-uniform if $X \in \mathcal{F}_\rho$. It is possible to show that for every $\rho \geq 2$ there exist nonempty ρ-uniform families for any $\mathbb{S}^n$ and that they contain sets of centers X for which h_X becomes arbitrarily small. The result is stated below. For a proof of the facts mentioned here as well as further discussion, see [19, section 2].

PROPOSITION 2.1 (see [19, Proposition 2.1]). *Let $\rho \geq 2$ and let $\mathcal{F}_\rho$ be the corresponding ρ-uniform family. Then, there exists a sequence of sets $X_k \in \mathcal{F}_\rho$, $k = 0, 1, \ldots,$ such that the sequence is nested, $X_k \subset X_{k+1}$, and such that at each step the mesh norms satisfy $\frac{1}{4} h_{X_k} < h_{X_{k+1}} \leq \frac{1}{2} h_{X_k}$.*

We will need to consider a decomposition of $\mathbb{S}^n$ into a finite number of nonoverlapping, connected regions R_ξ, each containing an interior point ξ that will serve for function evaluations as well as labeling. For example, if $\mathcal{X}$ is the Voronoi tessellation for a set of centers X, then we may take R_ξ to be the region associated with $\xi \in X$. In any case, we will let X be the set of the ξ's used for labels and $\mathcal{X} = \{R_\xi \subset \mathbb{S}^n \,|\, \xi \in X\}$. In addition, let $\|\mathcal{X}\| = \max_{\xi\in X}\{\mathrm{diam}(R_\xi)\}$. Du, Gunzburger, and Ju [3] construct a very interesting Voronoi tessellation in which $\xi \in X$ is the centroid of $R_\xi \in \mathcal{X}$.

Spherical harmonics. We turn to the situation in which the underlying Hilbert space is $\mathcal{H} = L^2(\mathbb{S}^n)$, with $d\mu$ being the usual measure on the n-sphere. Throughout the paper, we will let $\lambda_n := \frac{n-1}{2}$ and $\{Y_{\ell,m} : \ell = 0, 1, \ldots, m = 1 \ldots d_\ell^n\}$ be the usual orthonormal set of spherical harmonics [17, 24] associated with $\mathbb{S}^n$, where for $n \geq 2$,

$$d_\ell^n = \frac{\ell + \lambda_n}{\lambda_n}\binom{\ell + n - 2}{\ell} \overset{\ell\to\infty}{\sim} \frac{\ell^{n-1}}{\lambda_n(n-2)!}. \tag{4}$$

Denote by $\mathcal{H}_\ell$ the span of the spherical harmonics with fixed order ℓ, and let $\Pi_L = \bigoplus_{\ell=0}^{L}\mathcal{H}_\ell$ be the span of all spherical harmonics of order at most L. The orthogonal projection P_ℓ onto $\mathcal{H}_\ell$ is given by

$$\mathsf{P}_\ell f = \sum_{m=1}^{d_\ell^n}\langle f, Y_{\ell,m}\rangle Y_{\ell,m}. \tag{5}$$

Using the addition formula for spherical harmonics, one can write the kernel for this projection as

$$P_\ell(\xi,\eta) = \sum_{m=1}^{d_\ell^n} Y_{\ell,m}(\xi)\overline{Y_{\ell,m}(\eta)} = \frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\xi\cdot\eta), \tag{6}$$

where $\lambda_n = \frac{n-1}{2}$ and $P_\ell^{(\lambda_n)}$ is the ultraspherical polynomial of order λ_n and degree ℓ. We regard $\mathbb{S}^n$ as being the unit sphere in $\mathbb{R}^{n+1}$, and we let the quantity $\xi\cdot\eta$ denote the usual "dot" product for $\mathbb{R}^{n+1}$.

On the sphere, an operator K with a kernel of the form $K(\xi\cdot\eta)$ can be written as a convolution on $\mathbb{S}^n$; that is, $\mathsf{K}f = K * f$, where

$$K * f(\xi) = \int_{\mathbb{S}^n} K(\xi\cdot\eta)f(\eta)d\mu(\eta).$$

Because of the form of the convolution, these operators commute with rotations. Depending on the properties of the kernel, one may (and will!) apply these operators to spaces other than $L^2(\mathbb{S}^n)$.

The spherical harmonic $Y_{\ell,m}$ is an eigenfunction corresponding to the eigenvalue $-\ell(\ell+n-1) = \lambda_n^2 - (\ell+\lambda_n)^2$ for Laplace–Beltrami operator $\Delta_{\mathbb{S}^n}$ on $\mathbb{S}^n$. It follows that $\ell+\lambda_n$ is an eigenvalue corresponding to the eigenfunctions $Y_{\ell,m}$, $m = 1\ldots d_\ell^n$, of the pseudodifferential operator

$$\mathsf{L}_n := \sqrt{\lambda_n^2 - \Delta_{\mathbb{S}^n}} = \sum_{\ell=0}^{\infty}(\ell+\lambda_n)\mathsf{P}_\ell. \tag{7}$$

2.2. Operator frames and their kernels on $\mathbb{S}^n$. We now turn to the operators A_j defined in (2), when the underlying Hilbert space is $\mathcal{H} = L^2(\mathbb{S}^n)$ and L is proportional to the self-adjoint operator L_n given by (7). It is convenient to normalize the L_n's when $n \geq 2$ so that the lowest eigenvalue in the spectrum is in the interval $[1,2)$. To do that, let $j_n = \log_2\lfloor\lambda_n\rfloor$ for $n \geq 2$ and let $j_1 = 0$. We will work with $\mathsf{L} \to 2^{-j_n}\mathsf{L}_n$. Thus, $\mathsf{A}_j = a(2^{-j-j_n}\mathsf{L}_n)$, where the properties of $a \in C(\mathbb{R})$ are discussed in section 1. The spectral measure for $2^{-j_n}\mathsf{L}_n$ is $d\mathsf{E}_\lambda = \sum_{\ell=0}^\infty \mathsf{P}_\ell\delta\big(\lambda - 2^{-j_n}(\ell+\lambda_n)\big)$, where the P_ℓ's are the projections defined in (5) and have kernels given in (6). We can write the A_j's in kernel form:

$$A_j(\xi\cdot\eta) = \begin{cases} \frac{1}{\pi}\sum_{\ell=1}^\infty a(2^{-j}\ell)\cos(\ell\theta), & n=1,\ \xi\cdot\eta=\cos\theta, \\ \sum_{\ell=0}^\infty a\left(\frac{\ell+\lambda_n}{2^{j+j_n}}\right)\frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\xi\cdot\eta), & n\geq 2,\ j_n = \lfloor\log_2(\lambda_n)\rfloor. \end{cases} \tag{8}$$

The operator $\mathsf{B}_J = b(2^{-J-j_n}\mathsf{L}_n)$, with b defined in (1), has the kernel

$$B_J(\xi\cdot\eta) = \begin{cases} \frac{1}{2\pi}b(0) + \frac{1}{\pi}\sum_{\ell=1}^\infty b(2^{-J}\ell)\cos(\ell\theta), & n=1,\ \xi\cdot\eta=\cos\theta \\ \sum_{\ell=0}^\infty b\left(\frac{\ell+\lambda_n}{2^{J+j_n}}\right)\frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\xi\cdot\eta), & n\geq 2,\ j_n = \lfloor\log_2(\lambda_n)\rfloor. \end{cases} \tag{9}$$

Taking into account the support of a, when $n \geq 2$ in these operators it is easy to see that $\mathsf{B}_J = \sum_{j=0}^J \mathsf{A}_j\mathsf{A}_j^*$. For $n=1$, the projection P_0 enters and $\mathsf{B}_J = \mathsf{P}_0 + \sum_{j=0}^J \mathsf{A}_j\mathsf{A}_j^*$.

We will study and establish various properties of operator kernels similar to these in section 3. In section 5 we will discuss how these give rise to tight frames on $\mathbb{S}^n$ and discuss approximation properties of these frames.

3. Localization of kernels on $\mathbb{S}^n$. We want to study the localization properties of operator kernels related to the Laplace–Beltrami operator $\Delta_{\mathbb{S}^n}$ on the sphere. As we did earlier, let $\mathsf{L}_n := \sqrt{\lambda_n^2 - \Delta_{\mathbb{S}^n}}$ and let $\kappa(t) \in C^k(\mathbb{R})$, with $k \geq \max\{2, n-1\}$, be even and satisfy

$$|\kappa^{(r)}(t)| \leq C_\kappa(1+|t|)^{r-\alpha} \text{ for all } t\in\mathbb{R},\ r=0,\dots,k, \tag{10}$$

where $\alpha > n+k$ and $C_\kappa > 0$ are fixed constants. We remark that all compactly supported, even C^k functions satisfy (10), as do even functions in the Schwartz class $\mathcal{S}$. Even functions in $\mathcal{S}$ satisfy (10) for arbitrarily large k and α. Define the family of operators

$$\mathsf{K}_{\varepsilon,n} := \kappa(\varepsilon\mathsf{L}_n) = \sum_{\ell=0}^\infty \kappa(\varepsilon(\ell+\lambda_n))\mathsf{P}_\ell, \quad 0<\varepsilon\leq 1,$$

along with the associated family of kernels

$$K_{\varepsilon,n}(\underbrace{\xi\cdot\eta}_{\cos\theta}) := \begin{cases} \frac{1}{2\pi}\kappa(0) + \frac{1}{\pi}\sum_{\ell=1}^\infty \kappa(\varepsilon\ell)\cos\ell\theta, & n=1, \\ \sum_{\ell=0}^\infty \kappa(\varepsilon(\ell+\lambda_n))\frac{\ell+\lambda_n}{\lambda_n\omega_n}P_\ell^{(\lambda_n)}(\cos\theta), & n\geq 2, \end{cases} \tag{11}$$

where $\cos\theta = \xi\cdot\eta$ and $0<\varepsilon\leq 1$.

Our aim in this section is to obtain uniform bounds on the kernel $K_\varepsilon(\xi\cdot\eta)$ for small ε, with the bounds being explicitly dependent on ε.

The simple estimates given below in section 3.1 on the terms in the series used to define the kernels $K_{\varepsilon,n}$ confirm that, under mild conditions, these series are uniformly convergent. Let $n\geq 2$. Consider the ultraspherical identity [25, (4.7.14)] with $\lambda=\lambda_n$, $\frac{d}{dx}P_\ell^{(\lambda_n)}(x) = 2\lambda_n P_{\ell-1}^{(\lambda_n+1)}(x)$. Since $\lambda_n + 1 = \lambda_{n+2}$ and $\omega_n = \lambda_{n+2}\omega_{n+2}/\pi$, we have, for $\ell\geq 1$,

$$\frac{d}{dx}\left\{\left(\frac{\ell+\lambda_n}{\lambda_n\omega_n}\right)P_\ell^{(\lambda_n)}(x)\right\} = 2\pi\left(\frac{\ell-1+\lambda_{n+2}}{\lambda_{n+2}\omega_{n+2}}\right)P_{\ell-1}^{(\lambda_{n+2})}(x).$$

Multiply both sides by $\kappa(\varepsilon(\ell+\lambda_n))$ and sum on ℓ from 1 to ∞. Adjust the summation index on the right side and on the left use $\frac{d}{dx}P_0^{(\lambda_n)}(x) = 0$ to arrive at the identity below, which holds even when $n = 1$:

$$\frac{d}{dx}K_{\varepsilon,n}(x) = 2\pi K_{\varepsilon,n+2}(x). \tag{12}$$

3.1. Convergence issues and an L^∞ estimate on $K_{\varepsilon,n}$. The series defining the kernels are uniformly and absolutely convergent, by the M-test. This is easy to see for $n = 1$. For $n \geq 2$, start with the bound [25, (4.7.3) and (7.33.1)]

$$|P_\ell^{(\lambda_n)}(\cos\theta)| \leq \binom{\ell+n-2}{\ell} = P_\ell^{(\lambda_n)}(1), \tag{13}$$

and note that

$$\frac{\ell+\lambda_n}{\lambda_n}\binom{\ell+n-2}{\ell} \leq 2\binom{\ell+n-1}{\ell} \leq 2(1+\ell)^{n-1}.$$

From this and the assumptions on $\kappa(t)$, the terms in the series satisfy the bound

$$|\kappa(\varepsilon(\ell+\lambda_n))|\frac{\ell+\lambda_n}{\lambda_n\omega_n}\left|P_\ell^{(\lambda_n)}(\cos\theta)\right| \leq \frac{2\,C_\kappa(1+\ell)^{n-1}}{\omega_n(1+\varepsilon(\ell+\lambda_n))^\alpha} \leq \frac{2\,C_\kappa\varepsilon^{-(n-1)}}{\omega_n(1+\varepsilon\ell)^{\alpha-n+1}},$$

which suffices for the M-test, since $\alpha > n+k \geq n+2$ implies the series on the right above is convergent. Note that the estimate holds even when $n = 1$, provided the terms on the right are properly adjusted.

It is easy to take this a step further and obtain an estimate on $\|K_{\varepsilon,n}\|_\infty$, which we will need later on anyway.

Proposition 3.1. *If κ satisfies* (10)*, then*

$$\|K_{\varepsilon,n}\|_\infty \leq \frac{3\,C_\kappa}{\omega_n}\varepsilon^{-n}. \tag{14}$$

Proof. From the series definition of the kernel and the estimate on each term, we get this chain of inequalities:

$$\begin{aligned}\|K_{\varepsilon,n}\|_\infty &\leq \sum_{\ell=0}^{\infty}\frac{2\,C_\kappa\varepsilon^{-(n-1)}}{\omega_n(1+\varepsilon\ell)^{\alpha-n+1}}\\ &\leq \frac{2\,C_\kappa\varepsilon^{-(n-1)}}{\omega_n} + \int_0^\infty \frac{2\,C_\kappa\varepsilon^{-(n-1)}du}{\omega_n(1+\varepsilon u)^{\alpha-n+1}}\\ &\leq \frac{2\,C_\kappa\varepsilon^{-n}}{\omega_n}\left(\varepsilon+\frac{1}{\alpha-n}\right)\end{aligned}$$

Using $\varepsilon \leq 1$ and $\alpha - n > k \geq 2$ in the previous inequality and simplifying, we obtain (14). ∎

3.2. Integral representations. We now wish to obtain integral representations for the kernels $K_\varepsilon(\cos\theta)$. We begin with the Dirichlet–Mehler integral representation for the Gegenbauer polynomials [5, p. 177],

$$P_\ell^{(\lambda)}(\cos\theta) = \frac{2^\lambda\Gamma(\lambda+\frac{1}{2})\Gamma(\ell+2\lambda)}{\sqrt{\pi}\ell!\Gamma(\lambda)\Gamma(2\lambda)(\sin\theta)^{2\lambda-1}}\int_\theta^\pi \frac{\cos\left((\ell+\lambda)\varphi-\lambda\pi\right)}{(\cos\theta-\cos\varphi)^{1-\lambda}}d\varphi,$$

which holds for any real $\lambda > 0$. We will take $\lambda = \lambda_n = \frac{n-1}{2}$, with $n \geq 2$ throughout this section. Multiply both sides of the previous equation by $\frac{\ell+\lambda_n}{\lambda_n \omega_n}$ and then simplify to get

$$\frac{\ell+\lambda_n}{\lambda_n\omega_n} P_\ell^{(\lambda_n)}(\cos\theta) = \frac{\gamma_n(\ell+\lambda_n)(\ell+n-2)!}{\ell!(\sin\theta)^{n-2}} \int_\theta^\pi \frac{\cos\big((\ell+\lambda_n)\varphi - \lambda_n\pi\big)}{(\cos\theta-\cos\varphi)^{1-\lambda_n}} d\varphi, \tag{15}$$

where

$$\gamma_n := \frac{2^{\lambda_n}\Gamma(\lambda_n+\frac{1}{2})}{\sqrt{\pi}\lambda_n\omega_n\Gamma(\lambda_n)\Gamma(2\lambda_n)}. \tag{16}$$

Using the expression on the right in (15) in the series definition of $K_{\varepsilon,n}$, we get this representation:

$$K_{\varepsilon,n}(\cos\theta) = \frac{\gamma_n}{(\sin\theta)^{n-2}} \int_\theta^\pi \frac{C_{\varepsilon,n}(\varphi)}{(\cos\theta-\cos\varphi)^{1-\lambda_n}} d\varphi, \tag{17}$$

where $C_{\varepsilon,n}$ is given by the series

$$\begin{aligned} &C_{\varepsilon,n}(\varphi) \\ &:= \sum_{\ell=0}^\infty \kappa(\varepsilon(\ell+\lambda_n)) \frac{(\ell+\lambda_n)(\ell+n-2)!}{\ell!} \begin{cases} \sin(\lambda_n\pi)\sin(\ell+\lambda_n)\varphi, & n \text{ even}, \\ \cos(\lambda_n\pi)\cos(\ell+\lambda_n)\varphi, & n \text{ odd}. \end{cases} \end{aligned} \tag{18}$$

We want to put this series in a more convenient form. To begin, the factor $\frac{(\ell+\lambda_n)(\ell+n-2)!}{\ell!}$ is the product $(\ell+\lambda_n)(\ell+n-2)(\ell+n-3)\cdots(\ell+1)$, which can be rewritten as

$$\frac{(\ell+\lambda_n)(\ell+n-2)!}{\ell!} = \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \big((\ell+\lambda_n)^2 - (\lambda_n-r)^2\big) \times \begin{cases} \ell+\lambda_n, & \text{even}, \\ 1, & \text{odd}. \end{cases}$$

From this, we see that if we define the degree $n-1$ polynomial

$$Q_{n-1}(z) := \prod_{r=1}^{\lfloor\frac{n-1}{2}\rfloor} \big(z^2 - (\lambda_n-r)^2\big) \times \begin{cases} z\sin(\lambda_n\pi), & n \text{ even}, \\ \cos(\lambda_n\pi), & n \text{ odd}, \end{cases} \tag{19}$$

then we have that

$$C_{\varepsilon,n}(\varphi) = \sum_{\ell=0}^\infty \kappa(\varepsilon(\ell+\lambda_n)) Q_{n-1}(\ell+\lambda_n) \begin{cases} \sin(\ell+\lambda_n)\varphi, & n \text{ even}, \\ \cos(\ell+\lambda_n)\varphi, & n \text{ odd}. \end{cases} \tag{20}$$

We want to make a few observations about the polynomial Q_{n-1}. First, by direct calculation we have that $Q_{n-1}(-z) = (-1)^{n-1}Q_{n-1}(z)$, so that Q_{n-1} is an even function for odd n and an odd function for even n. Second, the zeros of Q_{n-1} are located at $\pm(\lambda_n - r)$ for $r = 1, \ldots, \lfloor\frac{n-1}{2}\rfloor$. This means that the function

$$g(t) := \kappa(\varepsilon t) Q_{n-1}(t) \begin{cases} \sin(t\varphi), & n \text{ even}, \\ \cos(t\varphi), & n \text{ odd}, \end{cases}$$

is even in t and has its zeros at $t = \pm(\lambda_n - r)$ for $r = 1 \ldots, \lfloor\lambda_n\rfloor$. In addition, we have defined g above so that from (20) we have $C_{\varepsilon,n}(\varphi) = \sum_{\ell=0}^\infty g(\ell+\lambda_n)$.

We want to apply the Poisson summation formula (PSF),

$$\sum_{\mu\in\mathbb{Z}} f(\mu) = \sum_{\nu\in\mathbb{Z}} \hat{f}(2\pi\nu), \quad \hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-i\omega t}dt,$$

which holds for "nice" f, to $f(t) = g(t+\lambda_n)$. Using the evenness of g and what we said about its zeros, we see that the left side of the PSF becomes

$$\sum_{\mu\in\mathbb{Z}} g(\mu+\lambda_n) = 2\sum_{\ell=0}^{\infty} g(\ell+\lambda_n) = 2C_{\varepsilon,n}(\varphi).$$

Employing elementary properties of the Fourier transform, we can show that

$$\hat{f}(\omega) = e^{i\lambda_n\omega}\hat{g}(\omega) = \varepsilon^{-1}e^{i\lambda_n\omega}Q_{n-1}(i\tfrac{d}{d\omega})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon}),$$

and so the right side of the PSF is

$$\begin{aligned}\sum_{\nu\in\mathbb{Z}} \hat{f}(2\pi\nu) &= \varepsilon^{-1}\sum_{\nu\in\mathbb{Z}} e^{2\pi\nu i\lambda_n}Q_{n-1}(i\tfrac{d}{d\omega})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})|_{\omega=2\pi\nu} \\ &= \varepsilon^{-1}\sum_{\nu\in\mathbb{Z}}(-1)^{(n-1)\nu}Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+2\pi\nu}{\varepsilon}).\end{aligned}$$

Equating the two sides of the PSF and dividing by 2, we arrive at the following result.

Proposition 3.2. *If κ satisfies* (10), *then for $n\geq 2$* (17) *holds with $C_{\varepsilon,n}$ given by*

$$C_{\varepsilon,n}(\varphi) = (2\varepsilon)^{-1}\sum_{\nu\in\mathbb{Z}}(-1)^{(n-1)\nu}Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+2\pi\nu}{\varepsilon}). \tag{21}$$

In addition, for the $n=1$ case we have

$$K_{\varepsilon,1}(\cos\theta) = (2\pi\varepsilon)^{-1}\sum_{\nu\in\mathbb{Z}}\hat{\kappa}(\tfrac{\theta+2\pi\nu}{\varepsilon}). \tag{22}$$

3.3. Estimates on $C_{\varepsilon,n}$. We need to obtain bounds on the kernels $C_{\varepsilon,n}$ from the previous section. The key to obtaining these bounds is this result.

Lemma 3.3. *Let κ satisfy* (10). *If $0\leq j\leq n-1$ and $0\leq r\leq k$ are integers, then $\frac{d^r}{dt^r}\{t^j\kappa\}\in L^1$ and $|\omega|^r|\hat{\kappa}^{(j)}(\omega)|\leq \|\frac{d^r}{dt^r}\{t^j\kappa\}\|_{L^1}$.*

Proof. Since $\kappa\in C^k$, the derivative $\frac{d^r}{dt^r}\{t^j\kappa\}$ is a linear combination of terms of the form $t^p\kappa^{(q)}$, each of which is bounded by a multiple of $(1+|t|)^{p+q-\alpha}$. This is in L^1 because $\alpha-p-q>\alpha-(n-1)-k>1$. This allows us to apply standard properties of the Fourier transform to obtain the formula $(-i)^{r+j}\omega^r\hat{\kappa}^{(j)}(\omega) = \widehat{\frac{d^r}{dt^r}\{t^j\kappa\}}$, which immediately implies the inequality. ∎

Consider the function $\left(\frac{\varphi+\omega}{\varepsilon}\right)^r Q_{n-1}(i\frac{d}{d\varphi})\hat{\kappa}(\frac{\varphi+\omega}{\varepsilon}) = \sum_{j=0}^{n-1}\varepsilon^{-j}q_{j,n}\left(\frac{\varphi+\omega}{\varepsilon}\right)^r\hat{\kappa}^{(j)}(\frac{\varphi+\omega}{\varepsilon})$, where $Q_{n-1}(z)=\sum_{j=0}^{n-1}q_{j,n}z^j$ is defined in (19). From Lemma 3.3, we have that

$$\begin{aligned}\left|\left(\tfrac{\varphi+\omega}{\varepsilon}\right)^r Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})\right| &\leq \textstyle\sum_{j=0}^{n-1}\varepsilon^{-j}|q_{j,n}|\,\|\frac{d^r}{dt^r}\{t^j\kappa\}\|_{L^1} \\ &\leq B_{n,k,\kappa}\varepsilon^{-(n-1)},\end{aligned}$$

where

$$B_{n,k,\kappa} := \left(\sum_{j=0}^{n-1} |q_{n,j}|\right) \max_{j<n,\, r\le k} \|\tfrac{d^r}{dt^r}\left\{t^j\kappa\right\}\|_{L^1}. \tag{23}$$

Adding the inequalities for $r = 0$ and $r = k$ and manipulating the result, we get that

$$\left|Q_{n-1}(i\tfrac{d}{d\varphi})\hat{\kappa}(\tfrac{\varphi+\omega}{\varepsilon})\right| \le \frac{2B_{n,k,\kappa}\varepsilon^{-(n-1)}}{1+\left|\frac{\varphi+\omega}{\varepsilon}\right|^k}.$$

We can use this inequality in conjunction with the series for $C_{\varepsilon,n}$ in (21) to arrive at the bound

$$|C_{\varepsilon,n}(\varphi)| \le (2\varepsilon)^{-1}\sum_{\nu\in\mathbb{Z}} \frac{2B_{n,k,\kappa}\varepsilon^{-(n-1)}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k} = \sum_{\nu\in\mathbb{Z}} \frac{B_{n,k,\kappa}\varepsilon^{-n}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k}, \tag{24}$$

which holds for all $\varphi \in \mathbb{R}$ and $0 < \varepsilon \le 1$. If we restrict φ to be in the interval $[0, \pi]$, then the dominant term in the series on the right comes from $\nu = 0$. The other terms are each bounded above by $B_{n,k,\kappa}\varepsilon^{k-n}((2|\nu| - 1)\pi)^{-k}$. Summing them and then estimating the resulting series by an integral gives us

$$\sum_{\nu\in\mathbb{Z},\nu\ne 0} \frac{B_{n,k,\kappa}\varepsilon^{-n}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k} \le B_{n,k,\kappa}\varepsilon^{k-n}\pi^{-k}\frac{2k-1}{k-1}.$$

Multiply top and bottom on the left above by $1 + \left(\frac{\varphi}{\varepsilon}\right)^k$ and use $0 \le \varphi \le \pi$ and $k \ge 2$ to get

$$\sum_{\nu\in\mathbb{Z},\nu\ne 0} \frac{B_{n,k,\kappa}\varepsilon^{-n}}{1+\left|\frac{\varphi+2\pi\nu}{\varepsilon}\right|^k} \le \frac{6B_{n,k,\kappa}\varepsilon^{-n}}{1+\left(\frac{\varphi}{\varepsilon}\right)^k}.$$

Combining this bound with that from (24) yields the result below.

PROPOSITION 3.4. *Let κ satisfy* (10)*, with $k \ge 2$ and $n \ge 2$. If $0 \le \varphi \le \pi$, then the kernel $C_{\varepsilon,n}$ defined in* (18) *satisfies the bound*

$$|C_{\varepsilon,n}(\varphi)| \le \frac{7B_{n,k,\kappa}\varepsilon^{-n}}{1+\left(\frac{\varphi}{\varepsilon}\right)^k}. \tag{25}$$

In addition, for the case $n = 1$, we have

$$|K_{\varepsilon,1}(\cos\theta)| \le \frac{7B_{1,k,\kappa}\varepsilon^{-1}}{1+\left(\frac{\theta}{\varepsilon}\right)^k}. \tag{26}$$

Proof. Only the second inequality requires comment. The proof we gave works for the $n = 1$ case because it has the form given in (22), which is essentially the same as that for the $C_{\varepsilon,n}$'s. □

3.4. Estimates on $K_{\varepsilon,n}$. We now turn to obtaining explicit bounds on the ΨDO kernels $K_{\varepsilon,n}$ similar to the bound on $K_{\varepsilon,1}$ in (26). From the integral representation in (17) and the bound on $C_{\varepsilon,n}$, we have that

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{7B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\theta)^{n-2}} \int_\theta^\pi \frac{(\cos\theta-\cos\varphi)^{\frac{n-3}{2}}d\varphi}{1+\left(\frac{\varphi}{\varepsilon}\right)^k}. \tag{27}$$

The two values of θ that present difficulties are $\theta = 0$ and $\theta = \pi$. The form of the inequality above is adequate for the $\theta = 0$ case, but needs to be reformulated for the $\theta = \pi$ case. To do that, we begin by denoting the angle supplementary to an angle α by $\tilde{\alpha}$, so throughout this section we will let $\tilde{\theta} = \pi - \theta$ and $\tilde{\varphi} = \pi - \varphi$. Changing variables in the integral on the right above and using $\sin\tilde{\alpha} = \sin\alpha$ and $\cos\tilde{\alpha} = -\cos\alpha$, we have the following reformulation of (27):

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{7B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\tilde{\theta})^{n-2}} \int_0^{\tilde{\theta}} \frac{(\cos\tilde{\varphi} - \cos\tilde{\theta})^{\frac{n-3}{2}} d\tilde{\varphi}}{1 + \left(\frac{\pi - \tilde{\varphi}}{\varepsilon}\right)^k}. \tag{28}$$

The next step is to bound both of these integrals. Recall the sum-to-product identity, $\cos\alpha - \cos\beta \equiv 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$, which holds for all α and β. Assuming that $\pi \ge \beta > \alpha \ge \pi/2$ and using the fact that $\frac{\sin t}{t}$ is decreasing for $0 \le t \le \pi$, we have that

$$6 < 8\frac{\sin(3\pi/4)}{3\pi/4}\frac{\sin(\pi/4)}{\pi/4} \le \frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2} = 8\frac{\sin\frac{\alpha+\beta}{2}}{\frac{\alpha+\beta}{2}}\frac{\sin\frac{\beta-\alpha}{2}}{\frac{\beta-\alpha}{2}} \le 8,$$

and so

$$\left(\frac{\cos\alpha - \cos\beta}{\beta^2 - \alpha^2}\right)^{\frac{n-3}{2}} \le 2^{\frac{3(n-3)}{2}} \times \begin{cases} \frac{2}{\sqrt{3}}, & n = 2, \\ 1, & n \ge 3 \end{cases} \le 2 \cdot 2^{\frac{3(n-3)}{2}}. \tag{29}$$

Assume that $\varepsilon \le \theta \le \pi/2$, and apply (29) to (27) to get the following chain of inequalities:

$$\begin{aligned} |K_{\varepsilon,n}(\cos\theta)| &\le \frac{14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\theta)^{n-2}} \int_\theta^\pi \frac{(\theta^2 - \varphi^2)^{\frac{n-3}{2}} d\varphi}{1 + \left(\frac{\varphi}{\varepsilon}\right)^k} \\ &\le 14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa}\gamma_n\varepsilon^{-n}\left(\frac{\theta}{\sin\theta}\right)^{n-2} \int_1^{\pi/\theta} \frac{(t^2-1)^{\frac{n-3}{2}} dt}{1 + (\theta/\varepsilon)^k t^k} \\ &\le \frac{14 \cdot 2^{\frac{3(n-3)}{2}} B_{n,k,\kappa}\gamma_n\varepsilon^{-n}(\pi/2)^{n-2}}{\left(\frac{\theta}{\varepsilon}\right)^k} \int_1^\infty \frac{(t^2-1)^{\frac{n-3}{2}} dt}{t^k}. \end{aligned}$$

Use $2(\theta/\varepsilon)^k \ge 1 + (\theta/\varepsilon)^k$, change variables of integration from $t \to 1/t$, and note that because $k \ge \max\{2, n-1\} \ge n-1$, the resulting integral on the right is bounded above by $\int_0^1 (1-t^2)^{\frac{n-3}{2}} dt = 2^{n-3}\Gamma(\lambda_n)^2/\Gamma(2\lambda_n)$ [26, p. 255]. After simplifying, we arrive at this estimate:

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{14 \cdot 2^{\frac{3(n-3)}{2}} \pi^{n-2} B_{n,k,\kappa}\gamma_n\Gamma(\lambda_n)^2/\Gamma(2\lambda_n)}{1 + \left(\frac{\theta}{\varepsilon}\right)^k}\varepsilon^{-n}.$$

The messy quantity in the numerator can be simplified considerably. This requires employing the definition of γ_n in (16), the formula for ω_n, the familiar properties of the Γ-function, along with the less familiar duplication formula [26, p. 240], $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$, and manipulating the expressions involved. The result is that

$$2^{\frac{3(n-3)}{2}}\pi^{n-2}\gamma_n\Gamma(\lambda_n)^2/\Gamma(2\lambda_n) = \frac{\omega_{n-1}}{4\sqrt{\pi}}\,,\ \omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

Thus we can rewrite the previous inequality, which holds for $\varepsilon \le \theta \le \pi/2$, as

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{7\omega_{n-1}B_{n,k,\kappa}}{2\sqrt{\pi}(1+(\frac{\theta}{\varepsilon})^k)}\varepsilon^{-n}.$$

If we now apply (29) to (28), with $0 \le \tilde\theta \le \pi/2$ (or, equivalently, $\pi/2 \le \theta \le \pi$), then

$$\begin{aligned}|K_{\varepsilon,n}(\cos\theta)| &\le \frac{7B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(\sin\tilde\theta)^{n-2}}\int_0^{\tilde\theta}\frac{(\tilde\theta^2-\tilde\varphi^2)^{\frac{n-3}{2}}d\tilde\varphi}{1+\left(\frac{\pi-\tilde\varphi}{\varepsilon}\right)^k}\\ &\le \frac{14\cdot 2^{\frac{3(n-3)}{2}}B_{n,k,\kappa}\gamma_n\varepsilon^{-n}}{(1+\left(\frac{\theta}{\varepsilon}\right)^k)(\sin\tilde\theta)^{n-2}}\int_0^{\tilde\theta}(\tilde\theta^2-\tilde\varphi^2)^{\frac{n-3}{2}}d\tilde\varphi.\end{aligned}$$

Carrying out manipulations analogous to those for the previous case, we obtain

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{7\omega_{n-1}B_{n,k,\kappa}}{4\sqrt{\pi}(1+(\frac{\theta}{\varepsilon})^k)}\varepsilon^{-n}.$$

The final case concerns $0 \le \theta \le \varepsilon$. For such θ, we have, from the L^∞ bound in (14), that

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{3\,C_\kappa}{\omega_n}\varepsilon^{-n} \le \frac{3\,C_\kappa}{\omega_n}\left(\frac{1+(\frac{\theta}{\varepsilon})^k}{1+(\frac{\theta}{\varepsilon})^k}\right)\varepsilon^{-n} \le \frac{6\,C_\kappa}{\omega_n(1+(\frac{\theta}{\varepsilon})^k)}\varepsilon^{-n},$$

which, when combined with (22) for $n=1$, gives us the main result of this section.

THEOREM 3.5. *Let κ satisfy* (10), *with $k \ge \max\{2, n-1\}$. If $0 \le \theta \le \pi$, then the kernel $K_{\varepsilon,n}$ satisfies the bound*

$$|K_{\varepsilon,n}(\cos\theta)| \le \frac{\beta_{n,k,\kappa}}{1+(\frac{\theta}{\varepsilon})^k}\varepsilon^{-n}, \tag{30}$$

where

$$\beta_{n,k,\kappa} := \begin{cases} 7B_{1,k,\kappa} & \text{if } n=1,\\ \max\left\{\frac{6\,C_\kappa}{\omega_n}, \frac{7\omega_{n-1}B_{n,k,\kappa}}{2\sqrt{\pi}}\right\} & \text{if } n \ge 2.\end{cases} \tag{31}$$

We conclude this section with an application of this theorem to obtaining a bound on the L^1 norm of $K_{\varepsilon,n}(\xi\cdot\eta)$, with η fixed. By the Funk–Hecke formula [17, Theorem 6], this norm is given by

$$\int_{\mathbb{S}^n}|K_{\varepsilon,n}(\xi\cdot\eta)|d\mu(\xi) = \omega_{n-1}\int_0^\pi |K_{\varepsilon,n}(\cos\theta)|\sin^{n-1}\theta\,d\theta,$$

which is of course independent of η. For that reason we will drop any reference to η and denote the norm by $\|K_{\varepsilon,n}\|_1$. Here is the bound we want.

COROLLARY 3.6. *Let $n \ge 1$. If κ satisfies* (10), *with $k > \max\{2, n\}$, then*

$$\|K_{\varepsilon,n}\|_1 \le 2\omega_{n-1}\beta_{n,k,\kappa}.$$

Proof. By Theorem 3.5 and the remarks above, we have

$$\|K_{\varepsilon,n}\|_1 \le \omega_{n-1}\int_0^\pi |K_{\varepsilon,n}(\cos\theta)|\sin^{n-1}\theta\,d\theta \le \beta_{n,k,\kappa}\omega_{n-1}\varepsilon^{-n}\int_0^\pi \frac{\sin^{n-1}\theta\,d\theta}{1+(\frac{\theta}{\varepsilon})^k}.$$

The integral on the right side above can be estimated this way:

$$\begin{aligned}\int_0^\pi \frac{\sin^{n-1}\theta\,d\theta}{1+(\frac{\theta}{\varepsilon})^k} &< \varepsilon^n\int_0^{\pi/\varepsilon}\frac{t^{n-1}dt}{1+t^k}\\ &< \varepsilon^n\left\{\int_0^1 t^{n-1}dt+\int_1^\infty \frac{dt}{t^{k+1-n}}\right\} < 2\varepsilon^n.\end{aligned}$$

The corollary then follows immediately from the estimate. ☐

3.5. Operator properties of $\mathsf{K}_{\varepsilon,n}$. We now turn to the operator properties of $\mathsf{K}_{\varepsilon,n}$. Our first result is calculating the norm of the map of $\mathsf{K}_{\varepsilon,n}: L^p \to L^q$. After that we will prove a lemma showing that for certain κ the operator $\mathsf{K}_{\varepsilon,n}$ will be a reproducing kernel on Π_L. We will close the section with a result showing that for such κ and $\varepsilon \le (L+\lambda_n)^{-1}$ the norm of $f - \mathsf{K}_{\varepsilon,n}f$ is comparable to the distance from f to Π_L in appropriate norms.

THEOREM 3.7. *If κ satisfies* (10)*, with $k > \max\{2, n\}$, then, for all $1 \le p \le \infty$ and $1 \le q \le \infty$, the operator $\mathsf{K}_{\varepsilon,n}: L^p(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$ is bounded and its norm satisfies*

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \le 2\omega_{n-1}\beta_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-(\frac{1}{p}-\frac{1}{q})_+},$$

where $\beta_{n,k,\kappa}$ is defined in (31) *and $(x)_+ = x$ for $x > 0$ and $(x)_+ = 0$ otherwise.*

Proof. The operators are all of the form $K_{\varepsilon,n} * f$ and so, for the (p, q) pairs $(1, 1)$, (∞,∞), $(\infty, 1)$, all satisfy $\|K_{\varepsilon,n} * f\|_q \le \|K_{\varepsilon,n}\|_1\|f\|_p$. By the Riesz–Thorin theorem [28, p. 95] and Corollary 3.6, we then have for $1 \le q \le p \le \infty$

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \le \|K_{\varepsilon,n}\|_1 \le 2\omega_{n-1}\beta_{n,k,\kappa}.$$

For the pair $(1,\infty)$, we have $\|K_{\varepsilon,n} * f\|_\infty \le \|K_{\varepsilon,n}\|_\infty\|f\|_1$. By (14) and (31), we have $\|K_{\varepsilon,n}\|_\infty \le \frac{1}{2}\beta_{n,k,\kappa}\varepsilon^{-n}$, and so $\|K_{\varepsilon,n} * f\|_\infty \le \frac{1}{2}\beta_{n,k,\kappa}\varepsilon^{-n}\|f\|_1$. Apply the Riesz–Thorin theorem to the pairs (p, q), where $\frac{1}{p} = (1-t)\alpha + t$ and $\frac{1}{q} = (1-t)\alpha$, where $0 < t < 1$ and $0 < \alpha < 1$, $(\frac{1}{\alpha},\frac{1}{\alpha})$ and $(1,\infty)$ to get

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \le (2\beta_{n,k,\kappa}\omega_{n-1})^{1-t}\left(\frac{1}{2}\beta_{n,k,\kappa}\varepsilon^{-n}\right)^t = 2\omega_{n-1}\beta_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-t}.$$

Since $\frac{1}{p} = (1-t)\alpha + t = \frac{1}{q} + t$, $t = \frac{1}{p} - \frac{1}{q}$. Thus, for $q > p$, we have

$$\|\mathsf{K}_{\varepsilon,n}\|_{p,q} \le 2\omega_{n-1}\beta_{n,k,\kappa}(4\omega_{n-1}\varepsilon^n)^{-(\frac{1}{p}-\frac{1}{q})}.$$

Putting the last inequality together with that for $q \le p$ yields the result. ☐

The following lemma is obvious.

LEMMA 3.8. *Let $L > 0$ be an integer and let $0 < \varepsilon \le (L + \lambda_n)^{-1}$. If κ satisfies* (10)*, with $k \ge \max\{2, n-1\}$, and if $\kappa(t) \equiv 1$ on $[0, 1]$, then $K_{\varepsilon,n}(\xi\cdot\eta)$ is a reproducing kernel on Π_L, the space of spherical harmonics having degree at most L.*

Remark 3.9. Let $L > 0$ be an integer. If we choose ε so that $L = \lfloor\varepsilon^{-1} - \lambda_n\rfloor$, then by combining the previous theorem and lemma we get a familiar result about harmonic polynomials: *If $S \in \Pi_L$, then $\|S\|_q \le C_nL^{n(\frac{1}{p}-\frac{1}{q})_+}\|S\|_p$.*

We let $E_L(f)_p$ denote the distance of $f \in L^p(\mathbb{S}^n)$ to Π_L, i.e.,

$$E_L(f)_p := \inf_{S\in\Pi_L} \|f - S\|_p. \tag{32}$$

Corollary 3.10. *Let κ satisfy (10), with $k > \max\{2, n\}$, and in addition suppose $\kappa(t) \equiv 1$ on $[0, 1]$. If $f \in L^p(\mathbb{S}^n)$, $1 \le p \le \infty$, and $\varepsilon \le (L + \lambda_n)^{-1}$, then*

$$\|f - K_{\varepsilon,n} * f\|_p \le (1 + 2\omega_{n-1}\beta_{n,k,\kappa})E_L(f)_p. \tag{33}$$

Also, for $1 \le p < \infty$ or, if $p = \infty$, for $f \in C(\mathbb{S}^n)$, we have $\lim_{\varepsilon\downarrow 0} K_{\varepsilon,n} * f = f$.

Proof. By Lemma 3.8, then $K_{\varepsilon,n} * S = S$ if $S \in \Pi_L$. It follows that $f - K_{\varepsilon,n} * f = (I + \mathsf{K}_\varepsilon)(f - S)$. From this and Theorem 3.7, we have that $\|f - K_{\varepsilon,n} * f\|_p \le (1 + 2\omega_{n-1}\beta_{n,k,\kappa})\|f - S\|_p$. Taking the infimum over all $S \in \Pi_L$ yields (33). That $\lim_{\varepsilon\downarrow 0} K_{\varepsilon,n} * f = f$ follows from (33) together with the fact that the spherical harmonics are dense in L^p for $1 \le p < \infty$ and in $C(\mathbb{S}^n)$ in the usual L^∞ norm [24, section IV.2]. ◻

The estimate in (33) is useful for obtaining rates of approximation, simply because rates of approximation by spherical harmonics are well known for many classes of functions; see, for example, Rustamov [23]. For further discussion, see the remarks following Proposition 5.1.

4. Quadrature on $\mathbb{S}^n$.

To do the discretizations required to construct tight spherical frames in section 5, we need a strengthened version of the quadrature formula given in [14, 15]. There are two reasons for this. First, the earlier quadrature formula applies to a partition of $\mathbb{S}^n$ that is restricted. Second, it utilizes a set of centers that is not a general set of scattered points, but rather a set that has been "culled" from one. Our aim is to use the results obtained in section 3 to produce an improved positive-weight quadrature formula that avoids these restrictions. Indeed, out of this will also come strengthened versions of the inequalities derived in [14].

4.1. Marcinkiewicz–Zygmund inequalities.

In this section we wish to give Marcinkiewicz–Zygmund type inequalities. These inequalities provide equivalences between norms defined through integrals and discrete norms stemming from sampled points and certain weights. Here, instead of polynomials, we will work with functions of the form $K_{\varepsilon,n} * f$ for $f \in L^1(\mathbb{S}^n)$.

The place to start is with a decomposition of the sphere into a finite number of nonoverlapping, connected regions R_ξ, each containing an interior point ξ that will serve for function evaluations as well as labeling. For example, given a set of centers X, one can form the corresponding Voronoi tessellation, and then take R_ξ to be the region associated with $\xi \in X$. In any case, we will let X be the set of the ξ's used for labels and $\mathcal{X} = \{R_\xi \subset \mathbb{S}^n \mid \xi \in X\}$. In addition, let $\|\mathcal{X}\| = \max_{\xi\in X}\{\operatorname{diam}(R_\xi)\}$.

The quantity that we wish to estimate first is the magnitude of the difference between the continuous and discrete norms for $g = K_{\varepsilon,n} * f$,

$$E_{\mathcal{X}} := \Big| \|g\|_1 - \sum_{\xi\in X} |g(\xi)|\mu(R_\xi)\Big|,$$

where we assume that $f \in L^1(\mathbb{S}^n)$. It is straightforward to show that

$$E_{\mathcal{X}} \le \sum_{\xi\in X} \int_{R_\xi} |g(\eta) - g(\xi)|d\mu(\eta) \le \sup_{\zeta\in\mathbb{S}^n} F_{\varepsilon,\mathcal{X}}(\zeta)\|f\|_1\,,$$

where $F_{\varepsilon,\mathcal{X}}(\zeta) := \sum_{\xi\in X}\int_{R_\xi}\left|K_{\varepsilon,n}(\eta\cdot\zeta)-K_{\varepsilon,n}(\xi\cdot\zeta)\right|d\mu(\eta)$, which is the quantity we need to estimate.

Choose ζ to be the north pole of $\mathbb{S}^n$ and let θ be the colatitude in spherical coordinates; set $\theta_\eta = \cos^{-1}(\eta\cdot\zeta)$ and $\theta_\xi = \cos^{-1}(\xi\cdot\zeta)$. Denote by θ_ξ^+ and θ_ξ^-, respectively, the high and low values for θ over R_ξ. Using (12) for the derivative of $K_{\varepsilon,n}$, we can write $F_{\varepsilon,\mathcal{X}}(\zeta)$ as

$$\begin{aligned}F_{\varepsilon,\mathcal{X}}(\zeta) &= 2\pi\sum_{\xi\in X}\int_{R_\xi}\left|\int_{\theta_\zeta}^{\theta_\eta}K_{\varepsilon,n+2}(\cos t)\sin t dt\right|d\mu(\eta)\\ &\le 2\pi\sum_{\xi\in X}\mu(R_\xi)\int_{\theta_\xi^-}^{\theta_\xi^+}|K_{\varepsilon,n+2}(\cos t)|\sin t\,dt.\end{aligned}$$

Divide $\mathbb{S}^n$ into $M=\lfloor\pi/\|\mathcal{X}\|\rfloor$ equal bands in which $(m-1)\pi/M\le\theta\le m\pi/M$, $m=1,\dots,M$. To avoid trivial situations and simplify later inequalities, we will assume that $M\ge 3$. Call these bands $B_1,\dots,B_M$. Each R_ξ can have nontrivial intersection with at most two adjacent bands, because $\operatorname{diam}(R_\xi)\le\|\mathcal{X}\|\le\pi/M$. So if $R_\xi\subset B_m\cup B_{m+1}$, then $(m-1)\pi/M\le\theta_\xi^-\le\theta_\xi^+\le(m+1)\pi/M$. In addition, the sum of the contributions from all $R_\xi\subset B_m\cup B_{m+1}$ is bounded above by the quantity

$$I_m := 2\pi\mu(B_m\cup B_{m+1})\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}|K_{\varepsilon,n+2}(\cos t)|\sin t\,dt,$$

where $\mu(B_m\cup B_{m+1}) = \omega_{n-1}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\sin^{n-1}t\,dt$. It follows that $F_{\varepsilon,\mathcal{X}}(\zeta)\le\sum_{m=1}^{M-1}I_m$. From Theorem 3.5, if we assume $k\ge n+2>\max\{2,n+1\}$ and if we use various linear approximations to the sine, we have

$$I_m \le 2\pi\omega_{n-1}\beta_{n+2,k,\kappa}\varepsilon^{-n-2}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}t^{n-1}dt\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\frac{t}{1+(\frac{t}{\varepsilon})^k}dt. \tag{34}$$

For $2\le m\le M-1$, we can bound the first integral by $\frac{2\pi}{M}\left(\frac{m+1}{M}\pi\right)^{n-1}$. In the second integral, we divide and multiply the integrand by t^{n-1}, and replace the t^{n-1} in the denominator by its lowest value. The result is that

$$\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\frac{t}{1+(\frac{t}{\varepsilon})^k}dt \le \left(\frac{M}{(m-1)\pi}\right)^{n-1}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\frac{t^n}{1+(\frac{t}{\varepsilon})^k}dt.$$

Putting these two bounds together yields

$$I_m \le \frac{4\pi^2}{M}\omega_{n-1}\beta_{n+2,k,\kappa}\varepsilon^{-n-2}\underbrace{\left(\frac{m+1}{m-1}\right)^{n-1}}_{\le 3^{n-1}}\int_{\frac{m-1}{M}\pi}^{\frac{m+1}{M}\pi}\frac{t^n}{1+(\frac{t}{\varepsilon})^k}dt.$$

Summing both sides from $m=2$ to $M-1$, taking account of intervals appearing twice in the sum, and doing some obvious manipulations, we obtain

$$\begin{aligned}\sum_{m=2}^{M-1}I_m &< \frac{8\pi^2 3^{n-1}\omega_{n-1}}{M\varepsilon}\beta_{n+2,k,\kappa}\int_{\frac{\pi}{M\varepsilon}}^{\frac{\pi}{\varepsilon}}\frac{t^n}{1+t^k}dt\\ &< \frac{8\pi^2 3^{n-1}\omega_{n-1}}{M\varepsilon}\beta_{n+2,k,\kappa}\underbrace{\int_0^\infty\frac{t^n}{1+t^k}dt}_{\le 3/2} < \frac{4\pi^2 3^n\omega_{n-1}}{M\varepsilon}\beta_{n+2,k,\kappa}.\end{aligned}$$

We now need to estimate I_1. From (34) we have

$$I_1 \le n^{-1}\omega_{n-1}(2\pi/M)^n \int_0^{\frac{2\pi}{M}} \frac{\beta_{n+2,k,\kappa}\varepsilon^{-n-2}}{1+(\frac{t}{\varepsilon})^k} t dt < \frac{\omega_{n-1}}{2n}\left(\frac{2\pi}{M\varepsilon}\right)^{n+2}\beta_{n+2,k,\kappa}.$$

We arrive at the estimate

$$F_{\varepsilon,\mathcal{X}}(\zeta) \le 2\pi\omega_{n-1}\beta_{n+2,k,\kappa}\frac{2\pi}{M\varepsilon}\left\{\frac{1}{2n}\left(\frac{2\pi}{M\varepsilon}\right)^{n+1} + 3^n\right\}.$$

To finish up, we want to put our inequalities in terms of the ratio $\|\mathcal{X}\|/\varepsilon$. Since we have assumed that $M \ge 3$, we have that $\pi/M \le \frac{4}{3}\|\mathcal{X}\|$. Using this in the previous inequality and simplifying, we arrive at

$$F_{\varepsilon,\mathcal{X}}(\zeta) < 16\pi \cdot 3^{n-1}\omega_{n-1}\beta_{n+2,k,\kappa}\frac{\|\mathcal{X}\|}{\varepsilon}\left\{1 + \frac{3}{2n}\left(\frac{8\|\mathcal{X}\|}{9\varepsilon}\right)^{n+1}\right\}.$$

We remark that if $\|\mathcal{X}\| \le \varepsilon \le 1$, then the assumption that $M \ge 3$ is automatically fulfilled. In addition, the right side of the inequality above is independent of ζ, so it holds for the left replaced by $\sup_{\zeta\in\mathbb{S}^n} F_{\varepsilon,\mathcal{X}}(\zeta)$. Finally, the inequality itself simplifies considerably. We collect all these observations in the result below.

PROPOSITION 4.1. *Let κ satisfy* (10) *with $k \ge n+2$, and for $f \in L^1(\mathbb{S}^n)$ let $g = K_{\varepsilon,n} * f$. If $\mathcal{X}$ is the decomposition of $\mathbb{S}^n$ described above and if $\|\mathcal{X}\| \le \varepsilon \le 1$, then*

$$\left| \|g\|_1 - \sum_{\xi\in X} |g(\xi)|\mu(R_\xi)\right| \le 16\pi \cdot 3^n\omega_{n-1}\beta_{n+2,k,\kappa}\frac{\|\mathcal{X}\|}{\varepsilon}\|f\|_1. \tag{35}$$

This result leads immediately to a version of the Marcinkiewicz–Zygmund inequalities for $\mathbb{S}^n$. This result extends an earlier result proved in [14, Theorem 3.1]. As we noted at the start of the section, the earlier result held only for restricted classes of decompositions.

THEOREM 4.2. *Let $L > 0$ be an integer and let $\delta \in (0,1)$. If $\mathcal{X}$ is the decomposition of $\mathbb{S}^n$ described above and $S \in \Pi_L$, then there exists a constant $s_n \ge 1$, which depends only on n, such that*

$$(1-\delta)\|S\|_1 \le \sum_{\xi\in X} |S(\xi)|\mu(R_\xi) \le (1+\delta)\|S\|_1 \tag{36}$$

holds whenever $\|\mathcal{X}\| \le \delta s_n^{-1}(L+\lambda_n)^{-1}$.

Proof. Let κ satisfy (10), with $k \ge n+2$. In addition, require $\kappa(t) \equiv 1$ for $t \in [0,1]$. Choose $\varepsilon = (L+\lambda_n)^{-1}$. By Lemma 3.8, $S = K_{\varepsilon,n} * S$, and so if we take $f = S$ and $\|\mathcal{X}\| \le \varepsilon = (L+\lambda_n)^{-1} \le 1$ in Proposition 4.1, then $g = K_{\varepsilon,n} * S = S$ there. Manipulating the resulting expression in (35) then gives us

$$\tilde{s}_n := \sup \frac{\left|\|S\|_1 - \sum_{\xi\in X}|S(\xi)|\mu(R_\xi)\right|}{(L+\lambda_n)\|\mathcal{X}\|\|S\|_1} \le 16\pi \cdot 3^n\omega_{n-1}\beta_{n+2,k,\kappa}\,,$$

where the supremum is over all $\mathcal{X}$ and $L > 0$ such that $\|\mathcal{X}\| \le (L+\lambda_n)^{-1}$ and clearly depends *only* on n. Now, let

$$s_n := \max\{1, \tilde{s}_n\} \le \max\{1, 16\pi \cdot 3^n\omega_{n-1}\beta_{n+2,k,\kappa}\}. \tag{37}$$

If we further restrict $\|\mathcal{X}\|$ so that $\|\mathcal{X}\| \le \delta s_n^{-1}(L+\lambda_n)^{-1}$, then (36) follows. ∎

4.2. Positive-weight quadrature for $\mathbb{S}^n$. Our aim is to extend the quadrature formula in [14, Theorem 4.1] to more general sets of centers and decompositions than the restricted class covered there. Even more important for us here is obtaining upper and *lower* bounds on the positive weights. For the restricted case covered in [14], upper bounds were given in [15], but nothing was said about lower bounds, which we need for constructing tight-frames on $\mathbb{S}^n$.

There is an important map associated with Π_L and the decomposition $\mathcal{X}$ and the corresponding finite set X. Let $|X|$ be the cardinality of X. We define the sampling map, $T_X : \Pi_L \to \mathbb{R}^{|X|}$, by $T_X S := (S(\xi))_{\xi\in X}$. From Theorem 4.2, it follows that if $\|\mathcal{X}\| \le \delta s_n^{-1}(L+\lambda_n)^{-1}$ holds and if $T_X S = 0$, we have that $\|S\|_1 = 0$ and, hence, $S \equiv 0$. The sampling map, which is linear, is therefore injective. Moreover, if we let the subspace $V_L = T_X \Pi_L \subset \mathbb{R}^{|X|}$, then the inverse map $T_X^{-1} : V_L \to \Pi_L$ is of course linear. Also, we will let $S_X = (S(\xi))_{\xi\in X}$.

Since our interest here is in weights for quadrature, we start with the linear functional $\Phi : \Pi_L \to \mathbb{R}$ given by

$$\Phi(S) := \int_{\mathbb{S}^n} S(\eta)d\mu(\eta),\ S \in \Pi_L.$$

Let $\Phi_X(S_X) = \Phi(T_X^{-1}(S_X)) = \Phi(S)$. If $S_X \ge 0$, then $|S(\xi)| = S(\xi)$ for $\xi \in X$, and so from (36) we have that

$$\Big|\Phi(S) - \sum_{\xi\in X} S(\xi)\mu(R_\xi)\Big| \le \Big|\|S\|_1 - \sum_{\xi\in X} S(\xi)\mu(R_\xi)\Big| \le \frac{\delta}{1-\delta}\sum_{\xi\in X} S(\xi)\mu(R_\xi),$$

provided only that $\|\mathcal{X}\| \le \delta s_n^{-1}(L+\lambda_n)^{-1}$. For any $\delta < \frac{1}{2}$, this implies that

$$\frac{1-2\delta}{1-\delta}\sum_{\xi\in X} S(\xi)\mu(R_\xi) \le \Phi(S) \le \frac{1}{1-\delta}\sum_{\xi\in X} S(\xi)\mu(R_\xi).$$

From this, we see that the linear functional

$$\Psi_X(S_X) := \Phi_X(S_X) - \frac{1-2\delta}{1-\delta}\sum_{\xi\in X} S(\xi)\mu(R_\xi) \tag{38}$$

is positive on the cone $0 \le S_X \in V_L$, which itself is contained in the positive cone of $\mathbb{R}^{|X|}$.

There are two facts we will take account of. The first is that the positive cone of V_L is contained in the positive cone of $\mathbb{R}^{|X|}$. The second is that the vector $(1)_{\xi\in X}$, which is in both cones, is an interior point of the positive cone of $\mathbb{R}^{|X|}$. By the Krein–Rutman theorem [9], there exists a positive linear functional $\widetilde{\Psi}_X$ that extends Ψ_X to all $\mathbb{R}^{|X|}$. Consequently, there exist weights $\alpha_\xi \ge 0$ such that $\widetilde{\Psi}_X(x) = \sum_{\xi\in X}\alpha_\xi x_\xi$. Using this and $\Phi_X(S_X) = \Phi(S)$ in (38), we obtain

$$\Phi(S) = \sum_{\xi\in X} c_\xi S(\xi), \quad c_\xi := a_\xi + \frac{1-2\delta}{1-\delta}\mu(R_\xi),\ a_\xi \ge 0. \tag{39}$$

This is of course a positive-weight quadrature formula on $\mathbb{S}^n$, with weights bounded below by $\frac{1-2\delta}{1-\delta}\mu(R_\xi)$.

We want to get upper bounds as well. To do that, we let $L' = \lfloor \frac{L}{2} \rfloor$ and fix $\xi_0 \in X$. If $S \in \Pi_{L'}$, then S^2 is in Π_L. The quadrature formula (39) then implies that

$$\|S\|_2^2 = \Phi(S^2) = \sum_{\xi \in X} c_\xi (S(\xi))^2 \geq c_{\xi_0}(S(\xi_0))^2.$$

Choose $S(\eta) = \sum_{\ell=0}^{L'} \sum_{m=1}^{d_\ell^n} Y_{\ell,m}(\eta)\overline{Y_{\ell,m}(\xi_0)} = \sum_{\ell=0}^{L'} \frac{\ell+\lambda_n}{\omega_n \lambda_n} P_\ell^{(\lambda_n)}(\xi_0 \cdot \eta)$, which is real valued. Using the orthogonality of the $Y_{\ell,m}$'s, one can show that $\|S\|_2^2 = S(\xi_0) = \sum_{\ell=0}^{L'} \frac{\ell+\lambda_n}{\omega_n \lambda_n} \binom{\ell+n-2}{\ell}$. From the previous inequality, (13) and (4), and the fact that $\dim \Pi_{L'} = d_{L'}^{n+1}$ [17, p. 4], we get $c_{\xi_0} \leq \omega_n / d_{L'}^{n+1}$, where $L' := \lfloor L/2 \rfloor$. We summarize these results below.

THEOREM 4.3. *Adopt the notation of Theorem* 4.2. *In particular, s_n is given by* (37) *and depends only on n. For any $0 < \delta < \frac{1}{2}$ and any integer $L > 0$, if $\|\mathcal{X}\| \leq \delta s_n^{-1}(L+\lambda_n)^{-1}$, then there exist positive weights c_ξ, $\xi \in X$, such that the quadrature formula*

$$\int_{\mathbb{S}^n} f(\eta) d\mu(\eta) \doteq \sum_{\xi \in X} c_\xi f(\xi) \tag{40}$$

is exact for spherical harmonics in Π_L. Also, the weights satisfy the bounds

$$\frac{1-2\delta}{1-\delta} \mu(R_\xi) \leq c_\xi \leq \frac{\omega_n}{d_{L'}^{n+1}},\ L' = \lfloor L/2 \rfloor. \tag{41}$$

The theorem just proved starts with L and puts conditions on the decomposition $\mathcal{X}$. The centers in X play a secondary role, serving as labels for regions in $\mathcal{X}$ and as evaluation points in the quadrature formula.

It's useful to turn this around and have the centers X play the primary role. To do that, we need to make the assumption that we are considering only ρ-uniform X; that is, for some fixed ρ we assume that the mesh ratio $h_X/q_X = \rho_X \leq \rho$. We will take the $\mathcal{X} = \mathcal{X}_V$ to be the Voronoi decomposition associated with X. For this decomposition, we have $h_X \leq \|\mathcal{X}_V\|$. Also, since the smallest distance between two points in X is $2q_X$, every $R_\xi \in \mathcal{X}_V$ contains a spherical cap with center ξ and radius $q_X \geq h_X/\rho$; hence, $\mu(R_\xi) \geq \omega_{n-1}(2/\pi)^{n-1}\rho^{-n}h_X^n/n$. Applying Theorem 4.3, we arrive at this result.

COROLLARY 4.4. *Adopt the notation of Theorem* 4.3 *and let X be a ρ-uniform set of centers. If $h_X \leq \delta s_n^{-1}(L+\lambda_n)^{-1}$, then the quadrature formula* (40) *holds with weights satisfying*

$$\omega_{n-1}(2/\pi)^{n-1}\left(\frac{1-2\delta}{1-\delta}\right)\rho^{-n}h_X^n \leq c_\xi \leq \frac{\omega_n}{d_{L'}^{n+1}}, \qquad L' = \lfloor L/2 \rfloor. \tag{42}$$

Set $\delta = 1/4$. To get a better idea of how the weights are bounded in terms of $h = h_X$ or L, note that by (4) we have $d_{L'}^{n+1} \sim \frac{(L/2)^n}{\lambda_{n+1}(n-1)!}$. In addition, if we take L as large as possible, but still consistent with the condition that $h_X \leq \delta s_n^{-1}(L+\lambda_n)^{-1}$, then $L \sim h^{-1}$. In that case, we see that

$$c_\xi = \mathcal{O}\{h^n\} = \mathcal{O}\{L^{-n}\}, \tag{43}$$

where the constants hidden by $\mathcal{O}$ are dependent only on the dimension n.

So far we have only addressed the *existence* of positive weights, along with bounds on them. In fact, the existence of such weights implies the *feasibility* of solving a quadratic programming problem that produces weights minimizing $\sum_{\xi\in X} c_\xi^2$, subject to constraints. Thus it is possible to numerically compute the weights. For more details, see [14, section 4.3].

5. Tight frames on $\mathbb{S}^n$. In this section, we discuss three important features of the operator frames on $\mathbb{S}^n$ introduced earlier in section 2.2. The first is the approximation power of these frames in various spaces. The second is how to turn them into *tight* frames for $\mathbb{S}^n$. This requires discretizing them using the quadrature results from the previous section. The third and final feature is their excellent localization properties.

We will turn to discussing the approximation power of these operator frames, after a brief word about notation. Throughout this section, the operators A_j and B_J are their kernels A_j and B_J, which are defined in section 2.2. The function $b(t)$ is defined in (1). We assume that the function $a(t)$, whose properties are discussed in section 1, is in $C^k(\mathbb{R})$.

PROPOSITION 5.1. *Let $k > \max\{n, 2\}$, and let b be defined by* (1)*, with $a \in C^k(\mathbb{R})$. If $f \in L^p(\mathbb{S}^n)$, $1 \le p \le \infty$, and if $L > 0$ is an integer such that $2^{-J-j_n} \le (L+\lambda_n)^{-1}$, then*

$$\|f - \mathsf{B}_J f\|_p \le C_{b,k,n} E_L(f)_p, \qquad E_L(f)_p := \operatorname{dist}_{L^p}(f, \Pi_L). \tag{44}$$

Also, for $1 \le p < \infty$ or, if $p = \infty$, for $f \in C(\mathbb{S}^n)$, we have $\lim_{J\to\infty} \mathsf{B}_J f = f$.

Proof. Apply Corollary 3.10 with $\kappa = b$, k as above, and $\varepsilon = 2^{-J-j_n}$. ∎

The proposition implies that $\mathsf{B}_J f$ approximates f to within an error comparable to $E_L(f)_p$, which is that for the best approximation to f from Π_L in L^p. Much work [11, 20, 22, 23, 27] has been done on estimating this error for various smoothness classes and spaces. This work allows us to obtain rates of approximation when f has additional smoothness requirements. A typical result [11] is this: If $f \in L^p(\mathbb{S}^n)$, with $\|f\|_p = 1$, belongs to a smoothness class $W_p^\alpha(\mathbb{S}^n)$, which is analogous to a Sobolev space, then $E_L(f)_p \sim L^{-\alpha}$. Choosing f similarly and taking $L \sim 2^J$, we get a corresponding result for our case: $\|f - \mathsf{B}_J f\|_p \sim 2^{-\alpha J}$.

We now turn to constructing tight frames on $\mathbb{S}^n$. The quadrature formulas from section 4.2 will play a pivotal role in their construction; we will also require a sequence of sets of centers to use in conjunction with them. Let $\rho \ge 2$ be fixed. By Proposition 2.1, we can find a sequence of sets of centers $\{X_j \in \mathcal{F}_\rho\}_{j=0}^\infty$ such that X_j is nested and such that the mesh norm $h_j := h_{X_j}$ halves going from j to $j+1$; that is, $h_{j+1} \le h_j/2$. In what follows, assume that the X_j's form such a sequence.

Recall that on $\mathbb{S}^n$, the frame transform $f \to w_j = \widetilde{\mathsf{A}}_j^* f$ takes the form $w_j(\eta) = \mathsf{A}_j^* f(\eta) = \langle f(\zeta), A_j(\zeta\cdot\eta)\rangle_{L^2(\mathbb{S}^n)}$. Because $A_j(\zeta\cdot\eta)$ is a spherical polynomial with degree less than 2^{j+j_n+1}, the function $w_j(\eta)$ is a spherical polynomial of degree less than 2^{j+j_n+1}. In the reconstruction formula this then contributes the term

$$\mathsf{A}_j w_j(\omega) = \int_{\mathbb{S}^n} A_j(\omega\cdot\eta) w_j(\eta) d\mu(\eta).$$

The product $A_j(\omega\cdot\eta)w_j(\eta)$ is a spherical polynomial having degree less than $2^{j+j_n+1} + 2^{j+j_n+1} = 2^{j+j_n+2}$.

We can integrate this *exactly* using the quadrature formula (40), with $L = 2^{j+j_n+2}$. First of all, the condition on the mesh norm h in both Theorem 4.3

and Corollary 4.4 is that $h \le \delta s_n^{-1}(L+\lambda_n)^{-1}$, where $\delta \in (0,1/2)$ is arbitrary. Choose $\delta = 1/4$ to be definite. For $n=1$ (the circle), we have $\lambda_1 = 0$ and $j_1 = 0$, and the condition is $h \le \delta s_1^{-1} 2^{-j-2} = s_1^{-1} 2^{-j-4}$. For $n \ge 2$, note that $2^{j+j_n+2} + \lambda_n \le 2^{j+2}\lfloor \lambda_n \rfloor + \lambda_n < 2^{j+3}\lambda_n$. The condition for $n \ge 2$ is then fulfilled if $h \le \delta(\lambda_n s_n)^{-1} 2^{-j-3} = (\lambda_n s_n)^{-1} 2^{-j-5}$. It is clear that these conditions can be met by using the sets X_j.

Let the quadrature weight corresponding to the center $\xi \in X_j$ be denoted by $c_{j,\xi}$, so that

$$\mathsf{A}_j w_j(\omega) = \sum_{\xi \in X_j} c_{j,\xi} A_j(\xi \cdot \omega) w_j(\omega) = \sum_{\xi \in X_j} \langle f, \psi_{j,\xi}\rangle \psi_{j,\xi}, \tag{45}$$

where

$$\psi_{j,\xi}(\eta) := \sqrt{c_{j,\xi}}\, A_j(\eta \cdot \xi),\ \xi \in X_j, \tag{46}$$

is the analysis frame function at level j. The frame function $\psi_{j,\xi}$ is computable: A_j is known and, as we noted at the end of section 4.2, the weights can be found numerically. We can now prove this result.

THEOREM 5.2. *Let $k > \max\{n,2\}$, and let A_j be the kernel in* (8)*, with $a \in C^k(\mathbb{R})$. If $f \in C(\mathbb{S}^n)$ or, for $1 \le p < \infty$, if $f \in L^p(\mathbb{S}^n)$, then*

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in X_j} \langle f, \psi_{j,\xi}\rangle \psi_{j,\xi},$$

with convergence being in the appropriate space. In addition, if $f \in L^2(\mathbb{S}^n)$, the frame $\{\psi_{j,\xi}\}_{j \in \mathbb{Z}_+, \xi \in X_j}$ is tight:

$$\|f\|^2 = \begin{cases} \frac{1}{2\pi}|\langle f, 1\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\xi \in X_j} |\langle f, \psi_{j,\xi}\rangle|^2, & n = 1, \\ \sum_{j=0}^{\infty}\sum_{\xi \in X_j} |\langle f, \psi_{j,\xi}\rangle|^2, & n \ge 2. \end{cases}$$

Finally, the frame functions have vanishing moments that increase with j, and are orthogonal on nonadjacent levels.

Proof. From (9) and (45), for $n \ge 2$ we get $\mathsf{B}_J f = \sum_{j=0}^{J}\sum_{\xi \in X_j} \langle f, \psi_{j,\xi}\rangle \psi_{j,\xi}$. By Proposition 5.1 this converges to f in all of the spaces mentioned. To prove that the frame is tight, just observe that for $f \in L^2(\mathbb{S}^n)$, we have $\langle \mathsf{B}_J f, f\rangle = \sum_{j=0}^{J}\sum_{\xi \in X_j} \left|\langle f, \psi_{j,\xi}\rangle\right|^2$. Taking the limit as $J \to \infty$ then yields the equation for $\|f\|^2$. The statement concerning vanishing moments follows from the structure of the A_j's, and the orthogonality between nonadjacent levels is proved in Proposition 1.1. The $n=1$ case has a projection P_0 in B_J, where P_0 projects onto the constants. The effect of this is to add a term to the series for $\|f\|^2$. ∎

Our last result concerns the localization properties of the frame function defined by (46).

COROLLARY 5.3. *Let $k > \max\{n,2\}$ and let $\psi_{j,\xi}$ be given by* (46)*. If $\theta := \cos^{-1}(\eta \cdot \xi)$, then for all $\theta \in [0,\pi]$ there are constants C and C', which depend on k, n, and a, such that these hold:*

$$|\psi_{j,\xi}(\eta)| \le \frac{2^{n(j+j_n)/2}\, C}{1 + (2^{j+j_n}\,\theta)^k} \ \text{and}\ |B_J(\eta \cdot \xi)| \le \frac{2^{n(J+j_n)}\, C'}{1+(2^{J+j_n}\,\theta)^k}.$$

Proof. Use Theorem 3.5, with $\kappa = b$ and $\varepsilon = 2^{-J-j_n}$, to bound $B_J(\xi \cdot \eta)$, and again, with $\kappa = a$ and $\varepsilon = 2^{-j-j_n}$, to bound $A_j(\eta \cdot \xi)$. Next, use $L = 2^{j+j_n+2}$ in (43) to see that $c_\xi = \mathcal{O}\{2^{-(j+j_n)n}\}$, where the constants depend only on n. To bound $\psi_{j,\xi}$, use the bounds on A_j and c_ξ in (46). ∎

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