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B-spaces and their Characterization via Anisotropic Franklin Bases

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B-spaces (generalized Besov spaces) generated by multilevel nested triangulations of compact polygonal domains in $\mathbb{R}^2$ are explored. Mild conditions are imposed on the triangulations which prevent them from deterioration and at the same time allow for a lot of flexibility and, in particular, arbitrarily sharp angles. It is shown that the B-spaces can be characterized by the corresponding anisotropic Franklin bases. This result is applied to nonlinear n-term approximation from anisotropic Franklin bases.

1. Introduction

We consider general B-spaces generated by sequences of multilevel nested triangulations of compact polygonal domains in $\mathbb{R}^2$. For a given polygonal domain E in $\mathbb{R}^2$ we consider a sequence of nested triangulations $\mathcal{T}_0, \mathcal{T}_1, \dots$ of general nature. Mild conditions are imposed on the triangulations which prevent them from deterioration. At the same time these conditions allow for a great deal of flexibility and, in particular, arbitrarily sharp angles. Generalized Besov spaces B^{α}_{pq} (called B-spaces) are naturally associated with every such sequence of triangulations and provide a specific nonstandard means of measuring the smoothness of the functions. A particular class of B-spaces needed in nonlinear approximation were introduced in [7] and further developed and used in [2, 3, 8].

In this article, we show that the general B-spaces can be characterized via Franklin bases obtained by applying the Gram-Schmidt orthogonalization process to the corresponding Courant elements. Similar results for Besov spaces in regular setups are obtained in [6] (see also the references in [6]). Further, we show how the B-spaces can be used to characterize the approximation spaces of nonlinear n-term approximation from anisotropic Franklin systems in L_p $(1 < p < \infty)$. This is a follow up paper of [9], where among other things it is

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proved that the anisotropic Franklin bases are Schauder bases for C and L_1, and unconditional bases for L_p $(1 < p < \infty)$ and the corresponding Hardy space H_1.

The paper is organized as follows. In §2 we give all auxiliary results and introduce the anisotropic Franklin bases. In §3 we introduce the general B-spaces B^α_{pq} and show that the anisotropic Franklin bases characterize the B-spaces. In §4, we show that the approximation spaces of nonlinear n-term approximation from anisotropic Franklin bases can be characterized by certain B-spaces.

Notation. Throughout this article for a set $G \subset \mathbb{R}^2$, $|G|$ denotes the Lebesgue measure of G, while G° means the interior of G; $\mathbf{1}_G$ denotes the characteristic function of G, and $\tilde{\mathbf{1}}_G := |G|^{-1/2}\mathbf{1}_G$. For a finite set G, $\#G$ denotes the cardinality of G. Positive constants are denoted by $c, c_1, \dots$ (if not specified, they may vary at every occurrence), $A \approx B$ means $c_1 A \le B \le c_2 B$, and $A := B$ or $B =: A$ stands for "A is by definition equal to B". We set $\langle f, g\rangle := \int_E fg$.

2. Preliminaries

In this section we collect all prerequisites regarding triangulations, maximal operators, local approximation, etc., which will be needed in the development of the B-spaces and their characterization via Franklin bases.

2.1. Multilevel Triangulations

A set $E \subset \mathbb{R}^2$ is said to be a *bounded polygonal domain* if its interior E° is connected and E is the union of a finite set $\mathcal{T}_0$ of closed triangles with disjoint interiors: $E = \bigcup_{\triangle\in\mathcal{T}_0} \triangle$. We call $\mathcal{T} = \bigcup_{m=0}^\infty \mathcal{T}_m$ a *locally regular triangulation* of E or briefly an *LR-triangulation* with levels $(\mathcal{T}_m)_{m\ge 0}$ if the following conditions are fulfilled:

(a) Every level $\mathcal{T}_m$ is a partition of E, that is, $E = \bigcup_{\triangle\in\mathcal{T}_m} \triangle$ and $\mathcal{T}_m$ consists of closed triangles with disjoint interiors.

(b) The levels $(\mathcal{T}_m)$ of $\mathcal{T}$ are nested, i.e., $\mathcal{T}_{m+1}$ is a refinement of $\mathcal{T}_m$.

(c) Each triangle $\triangle \in \mathcal{T}_m$ has at least two and at most M_0 children (subtriangles) in $\mathcal{T}_{m+1}$, where $M_0 \ge 2$ is a constant.

(d) The valence N_v of each vertex v of any triangle $\triangle \in \mathcal{T}_m$ (the number of the triangles from $\mathcal{T}_m$ which share v as a vertex) is at most N_0, where N_0 is a constant.

(e) *No hanging vertices condition*: No vertex of any triangle $\triangle \in \mathcal{T}_m$ which belongs to the interior of E lies in the interior of an edge of another triangle from $\mathcal{T}_m$.

(f) There exist constants $0 < r < \rho < 1$ $(r \le \frac{1}{2})$ such that for each $\triangle \in \mathcal{T}_m$ $(m \ge 0)$ and any child $\triangle' \in \mathcal{T}_{m+1}$ of $\triangle$,

$$r|\triangle| \le |\triangle'| \le \rho|\triangle|. \tag{2.1}$$

(g) There exists a constant $0 < \delta \le 1$ such that for $\triangle', \triangle'' \in \mathcal{T}_m$ $(m \ge 0)$ with a common vertex,

$$\delta \le |\triangle'|/|\triangle| \le \delta^{-1}. \tag{2.2}$$

We call $\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$ a *regular triangulation* of a bounded polygonal domain $E \subset \mathbb{R}^2$ if $\mathcal{T}$ satisfies conditions (a)-(e) of LR-triangulations and also the *minimal angle condition*, that is, $\min \text{angle}\,(\triangle) \ge \beta$ for every triangle $\triangle \in \mathcal{T}$, where $\beta > 0$ is a constant. Evidently, every regular triangulation is locally regular but not the other way around. For other types of triangulations, see [7].

We denote by $\mathcal{V}_m$ the set of all vertices of triangles from $\mathcal{T}_m$ and by $\mathcal{E}_m$ the set of all edges of triangles in $\mathcal{T}_m$. We also set $\mathcal{V} := \cup_{m\ge0}\mathcal{V}_m$ and $\mathcal{E} := \cup_{m\ge0}\mathcal{E}_m$.

We next give some basic facts concerning LR-triangulations. For more details about LR-triangulations, we refer the reader to [7] and [9].

The constants M_0, N_0, r, ρ, δ, and $\#\mathcal{T}_0$ associated with an LR-triangulation $\mathcal{T}$ are assumed fixed. We refer to them as *parameters* of $\mathcal{T}$.

The most important conditions (f)-(g) on LR-triangulations involve only areas of triangles but not angles. Consequently, if $\mathcal{T}$ is an LR-triangulation and $\triangle', \triangle'' \in \mathcal{T}_m$ have a common edge, then it may happen that $\triangle'$ is an equilateral triangle (or close to an equilateral triangle) but $\triangle''$ has a uncontrollably sharp angle.

In an LR-triangulation $\mathcal{T}$ there can be an equilateral (or close to such) triangle $\triangle^\diamond$ at any level T_m with descendants $\triangle_1 \supset \triangle_2 \supset \dots$ such that $\min \text{angle}(\triangle_j) \to 0$ as $j \to \infty$.

It is important to know how fast the area $|\triangle|$ of a triangle $\triangle \in \mathcal{T}_m$ may change when $\triangle$ moves away from a fixed triangle within the same level. Condition (f) suggests a geometric rate of change but in fact it is polynomial.

Lemma 1. *If $\triangle$, $\triangle' \in \mathcal{T}_m$ can be connected by n intermediate edges from $\mathcal{E}_m$, then*

$$c_1^{-1}(n+1)^{-s} \le |\triangle'|/|\triangle''| \le c_1(n+1)^s, \tag{2.3}$$

where $s, c_1 > 0$ depend only on the parameters of $\mathcal{T}$.

Graph distance. We next introduce the mth level graph distance between vertices, which will play an important role in our further development: For any two vertices $v', v'' \in \mathcal{T}_m$, $m \ge 0$, we define the *graph distance* $\rho_m(v', v'')$ as the minimum number of edges from $\mathcal{E}_m$ needed to connect v' and v''.

The following lemma will be needed later on (see [9]).

Lemma 2. *There exist constants $c > 0$ and $t > 0$ depending only on the parameters of $\mathcal{T}$ such that for any $v^\diamond \in \mathcal{V}_m$*

$$\#\{v \in \mathcal{V}_m : \rho_m(v, v^\diamond) \le n\} \le cn^t, \qquad n \ge 1. \tag{2.4}$$

Cells. For any vertex $v \in \mathcal{V}_m$ ($m \ge 0$), we denote by θ_v the union of all triangles from $\mathcal{T}_m$ which have v as a common vertex. We denote by Θ_m the set of all such cells θ_v with $v \in \mathcal{V}_m$ and set $\Theta = \bigcup_{m\ge 0} \Theta_m$. For a given cell $\theta \in \Theta$, we shall denote by v_θ the "central" vertex of θ.

For given $\theta', \theta'' \in \Theta_m$, we define the *graph distance* $\rho_m(\theta', \theta'')$ between θ' and θ'' by $\rho_m(\theta', \theta'') := \rho_m(v_{\theta'}, v_{\theta''})$, where $v_{\theta'}, v_{\theta''} \in \mathcal{V}_m$ are the "central vertices" of θ', θ''.

Definition of $\boldsymbol{\theta_x^m}$. We want to associate with each $x \in E$ a cell $\theta_x^m \in \Theta_m$, $m \ge 0$, which contains x. To this end we first associate with each triangle $\triangle \in \mathcal{T}_m$ a cell $\theta_\triangle^m \in \Theta_m$ such that $\triangle \subset \theta_\triangle^m$. Such a cell can be selected in three different ways. We choose one of them for each $\triangle \in \mathcal{T}_m$. Then for each $x \in E$ such that $x \in \triangle^\circ$ with $\triangle \in \mathcal{T}_m$, we define $\theta_x^m := \theta_\triangle^m$. If x lies on the edge of a triangle from $\mathcal{T}_m$, we define θ_x^m as any cell from θ_m such that x belongs to its interior, but if $x = v_\theta$ for some $\theta \in \Theta_m$, we set $\theta_x^m := \theta$.

Stars. In order to deal with graph distances and neighborhood relations it is convenient to employ the notion of the mth *level star of a set*: For any set $G \subset E$ and $m \ge 0$, we define the first mth level star of G by

$$\operatorname{Star}_m(G) := \operatorname{Star}_m^1(G) := \cup\{\theta \in \Theta_m : \theta^\circ \cap G \ne \emptyset\} \tag{2.5}$$

and inductively, $\operatorname{Star}_m^k(G) := \operatorname{Star}_m^1(\operatorname{Star}_m^{k-1}(G))$, $k > 1$.

Maximal operator. Every LR-triangulation $\mathcal{T}$ of E naturally generates a maximal operator $\mathcal{M}_\mathcal{T}^s$ defined by

$$(\mathcal{M}_\mathcal{T}^s f)(x) := \sup_{\theta:\, x\in\theta} \left(\frac{1}{|\theta|}\int_\theta |f(y)|^s dy\right)^{1/s} \tag{2.6}$$

where the supremum is over all cells $\theta \in \Theta$ containing x or $\theta = E$.

It is important that the Fefferman-Stein [5] vector valued maximal inequality holds for the maximal operator $\mathcal{M}_\mathcal{T}^s$ (for more details, see [9]):

Proposition 1. *Let $\mathcal{T}$ be an LR-triangulation of $E \subset \mathbb{R}^2$. If $0 < p < \infty$, $0 < q \le \infty$, and $0 < s < \min\{p, q\}$, then for any sequence of functions $(f_j)_{j=1}^\infty$ on E,*

$$\Big\|\Big(\sum_{j=1}^\infty |\mathcal{M}_\mathcal{T}^s f_j|^q\Big)^{1/q}\Big\|_p \le c\Big\|\Big(\sum_{j=1}^\infty |f_j|^q\Big)^{1/q}\Big\|_p, \tag{2.7}$$

where c depends only on p, q, s, and the parameters of $\mathcal{T}$.

2.2. Local Piecewise Linear Approximation and Courant Elements

The no-hanging-vertices condition (e) on LR-triangulations guarantees the existence of *Courant elements*, that is, for every cell $\theta \in \Theta_m$ there exists a unique continuous piecewise linear function φ_θ on E which is supported on θ and satisfies $\varphi_\theta(v_\theta) = 1$. This is the so called Courant element associated to θ. We denote $\Phi_m := \Phi_{m,\mathcal{T}} := (\varphi_\theta)_{\theta\in\Theta_m}$ and $\Phi := \Phi_{\mathcal{T}} := \cup_{m\geq 0}\Phi_m$.

We let $\mathcal{S}_m$ denote the space of all continuous piecewise linear functions over $\mathcal{T}_m$. Clearly, $S \in \mathcal{S}_m$ if and only if $S = \sum_{v\in\mathcal{V}_m} S(v)\varphi_{\theta_v}$. Evidently, $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ \dots$ and it is easy to see [7] that $\overline{\cup_{m\geq 0}\mathcal{S}_m} = L_p(E)$, $0 < p \leq \infty$.

We shall frequently use the obvious fact that all norms of a polynomial on a triangle are equivalent, namely, if P is a polynomial of degree $\leq k$ and $\triangle$ is a triangle in $\mathbb{R}^2$, then

$$\|P\|_{L_p(\triangle)} \approx |\triangle|^{1/p-1/q}\|P\|_{L_q(\triangle)}, \qquad 0 < p, q \leq \infty, \tag{2.8}$$

with constants of equivalence depending only on k, p, and q.

The L_p-stability of $\Phi_m = (\varphi_\theta)_{\theta\in\Theta_m}$ is immediate from (2.8): If $(a_\theta)_{\theta\in\Theta_m}$, is an arbitrary sequence of real numbers and $S := \sum_{\theta\in\Theta_m} a_\theta\varphi_\theta$, then

$$||S||_p \approx \Big(\sum_{\theta\in\Theta_m} \|a_\theta\varphi_\theta\|_p^p\Big)^{1/p} \approx \Big(\sum_{\theta\in\Theta_m} |\theta||a_\theta|^p\Big)^{1/p}, \qquad 0 < p \leq \infty. \tag{2.9}$$

The *local piecewise linear approximation* will play an important role in our further development. For $f \in L_q(E)$ and $\triangle \in \mathcal{T}_m$, we denote the error of L_q-approximation to f from $\mathcal{S}_m$ on $\operatorname{Star}_m(\triangle)$ by

$$\mathbb{S}_\triangle(f)_q := \mathbb{S}_\triangle(f,\mathcal{T})_q := \inf_{S\in\mathcal{S}_m} \|f - S\|_{L_q(\operatorname{Star}_m(\triangle))}. \tag{2.10}$$

The set Φ of all Courant elements is obviously redundant. A standard way of representing functions is by using the so called *quasi-interpolant operators* defined by

$$Q_m(f) := Q_{m,\mathcal{T}}(f) = \sum_{\theta\in\Theta_m} \langle f, \tilde\varphi_\theta\rangle\, \varphi_\theta, \qquad m \geq 0, \tag{2.11}$$

where the dual functions $\tilde\varphi_\theta$ are constructed so that they are supported in θ and $\langle\tilde\varphi_\theta, \varphi_{\theta'}\rangle = \delta_{\theta\theta'}$ for $\theta, \theta' \in \Theta_m$. In particular, the duals can be defined by

$$\tilde\varphi_\theta := \sum_{\triangle\in\mathcal{T}_m,\, \triangle\subset\theta} \mathbf{1}_\triangle \cdot \lambda_{\triangle,\theta}, \tag{2.12}$$

where $\lambda_{\triangle,\theta}$ is the linear polynomial which is equal to $\frac{9}{N_v|\triangle|}$ at v_θ (the "central vertex" of θ) and it takes the value $-\frac{3}{N_v|\triangle|}$ at the other two vertices of $\triangle$ (recall that N_v is the valence of v).

Evidently, Q_m is a linear projector, i.e., $Q_m(S) = S$ for $S \in \mathcal{S}_m$. It is important that Q_m is locally bounded and provides good local approximation: If $f \in L_q(E)$, $1 \le q \le \infty$, and $\triangle \in \mathcal{T}_m$, then

$$\|Q_m(f)\|_{L_q(\triangle)} \le c\|f\|_{L_q(\mathrm{Star}\,_m(\triangle))} \tag{2.13}$$

and

$$\|f - Q_m(f)\|_{L_q(\triangle)} \le c\,\mathbb{S}_\triangle(f)_q, \tag{2.14}$$

where the constants depend only on q and the parameters of $\mathcal{T}$.

We define

$$q_m := Q_m - Q_{m-1} \quad \text{for} \quad m \ge 0, \quad \text{where} \quad Q_{-1} := 0. \tag{2.15}$$

Clearly, $q_m(f) \in \mathcal{S}_m$. For a given function f we define the coefficients $(b_\theta(f))_{\theta\in\Theta_m}$ from the expression

$$q_m(f) =: \sum_{\theta\in\Theta_m} b_\theta(f)\varphi_\theta, \qquad m \ge 0. \tag{2.16}$$

It is readily seen that for $f \in L_p(E), 1 \le p \le \infty$,

$$f = \sum_{m=0}^{\infty}(Q_m(f) - Q_{m-1}(f)) = \sum_{m=0}^{\infty}\sum_{\theta\in\Theta_m} b_\theta(f)\varphi_\theta, \tag{2.17}$$

where $Q_{-1}(f) := 0$ and the convergence is in $L_p(E)$. For the proofs of all of the above and more details, we refer the reader to [7] (see also [8]).

2.3. Anisotropic Franklin Bases

Here we define the Franklin system $\mathcal{F}_\mathcal{T}$ generated by Courant elements and present our main results on Franklin bases obtained in [9].

Let $\mathcal{T} := \bigcup_{m\ge0}\mathcal{T}_m$ be an LR-triangulation of E and recall that $\mathcal{V}_m$ denotes the set of all vertices of triangles from $\mathcal{T}_m$. We set $\mathcal{V}_0^* = \mathcal{V}_0$ and $\mathcal{V}_m^* = \mathcal{V}_m \setminus \mathcal{V}_{m-1}$ for $m \ge 1$ and write $\mathcal{V}^* = \bigcup_{m=0}^{\infty}\mathcal{V}_m^*$.

Let $\theta_0 := E$. Choose $\theta_{\max} \in \Theta_0$ to be of maximum area and denote $\Theta_0^* := \{\theta_0\} \cup \Theta_0 \setminus \{\theta_{\max}\}$, i.e., we replace $\theta_{\max}$ by $\theta_0 = E$. Moreover, we associate θ_0 with $v_{\theta_{\max}}$ and set $\varphi_{\theta_0} := \mathbf{1}_{\theta_0}$. For $m \ge 1$ denote by Θ_m^* the set of all cells $\theta \in \Theta_m$ with "central" vertices $v_\theta \in \mathcal{V}_m^*$ and set $\Theta^* := \bigcup_{m=0}^{\infty}\Theta_m^*$.

Note that for each m, the set $\{\varphi_\theta : \theta \in \bigcup_{i=0}^{m}\Theta_i^*\}$ is linearly independent. Also, $\mathcal{S}_m = \operatorname{span}\{\varphi_\theta : \theta \in \Theta_m\} = \operatorname{span}\{\varphi_\theta : \theta \in \bigcup_{i=0}^{m}\Theta_i^*\}$.

We consider an arbitrary (but fixed) *linear order* $\preceq$ on Θ^* satisfying the following conditions:

(i) If $\theta \in \Theta_m^*$ and $\theta' \in \Theta_n^*$ with $m < n$, then $\theta \preceq \theta'$ and (ii) $\theta_0 \preceq \theta$, $\forall\theta \in \Theta^*$.

We now define the *Franklin system* $\mathcal{F}_\mathcal{T}$ by applying the Gram-Schmidt orthogonalization process to $\{\varphi_\theta\}_{\theta\in\Theta^*}$ in $L_2(E)$ with respect to the order $\preceq$. We obtain an orthonormal system $\mathcal{F}_\mathcal{T} := \{f_\theta\}_{\theta\in\Theta^*}$ in $L_2(E)$ consisting of continuous piecewise linear functions. Each Franklin function f_θ is uniquely determined (up to a multiple ± 1) by the conditions:

(a) $f_\theta \in \operatorname{span}\{\varphi_{\theta'} : \theta' \preceq \theta\}$.

(b) $\langle f_\theta, \varphi_{\theta'}\rangle = 0$ for all $\theta' \prec \theta$,

(c) $\|f_\theta\|_2 = 1$.

Note that $f_{\theta_0} = \pm\tilde{\mathbf{1}}_{\theta_0} := \pm|E|^{-1/2}\,\mathbf{1}_E$.

Our main results on anisotropic Franklin systems from [9] read as follows: The Franklin system $\mathcal{F}_\mathcal{T} := \{f_\theta\}_{\theta\in\Theta^*}$ is a Schauder basis for $L_p(E)$, $1 \le p \le \infty$, with $L_\infty(E) := C(E)$ and a unconditional basis for $L_p(E)$, $1 < p < \infty$ and the corresponding Hardy space $H_1(E,\mathcal{T})$. Also, $H_1(E,\mathcal{T})$ is exactly the space of all functions in L_1 for which the Franklin system expansion converge unconditionally in L_1. Finally, the Franklin bases characterize the corresponding anisotropic BMO spaces.

For the purposes of this article, we need the localization properties of the Franklin functions [9] (we use the notation from §2.1).

Proposition 2. *There exist constants* $0 < q_1 < 1$ *and* $c > 0$ *depending only on the parameters of* $\mathcal{T}$ *such that for any* $\theta \in \Theta_m^*$ $(m \ge 0)$,

$$|f_\theta(x)| \le c|\theta|^{-1/2} q_1^{\rho_m(\theta,\theta_x^m)}, \qquad x \in E, \tag{2.18}$$

and for any $s > 0$ *there exists a constant* c_s *such that*

$$|f_\theta(x)| \le c_s|\theta|^{-1/2}(\mathcal{M}_\mathcal{T}^s \mathbf{1}_\theta)(x), \qquad x \in E, \tag{2.19}$$

where $\mathcal{M}_\mathcal{T}^s$ *is the maximal operator defined in* (2.6). *Furthermore,*

$$c_p^{-1}|\theta|^{1/p-1/2} \le \|f_\theta\|_{L_p(\theta)} \le \|f_\theta\|_p \le c_p|\theta|^{1/p-1/2}, \qquad 0 < p \le \infty. \tag{2.20}$$

3. B-spaces

In this section we define the general B-spaces B_{pq}^α and show that they can be characterized by the corresponding Franklin bases.

3.1. Definition of B-spaces and Basic Properties

We begin by introducing the B-space $B_{pq}^\alpha := B_{pq}^\alpha(T)$ induced by an arbitrary LR-triangulation $\mathcal{T}$ of a compact polygonal domain E in $\mathbb{R}^2$. Since our primary goal here is to relate them to the corresponding Franklin bases, we consider only B-spaces which are imbedded in L_1. We say that the indices α, p, and q are *admissible* if one of the following holds:

(a) $0 < p, q \le \infty$ and $\alpha > (1/p-1)_+$ or

(b) $0 < p < 1$, $0 < q \le 1$, and $\alpha = 1/p - 1$.

As will be shown these conditions guarantee the desired embedding.

For a given LR-triangulation of E, we define $B^{\alpha}_{pq}(\mathcal{T})$ as the set of all functions $f \in L_p(E)$ such that

$$|f|_{B^{\alpha}_{pq}(\mathcal{T})} := \Big(\sum_{m=0}^{\infty} \Big(\sum_{\triangle \in \mathcal{T}_m} (|\triangle|^{-\alpha}\, \mathbb{S}_{\triangle}(f)_p)^p \Big)^{q/p} \Big)^{1/q} < \infty, \tag{3.1}$$

where $\mathbb{S}_{\Delta}(f)_p$ is the error of L_p-approximation to f from $\mathcal{S}_m$ on $\mathrm{Star}_{\,m}(\triangle)$ (see (2.10)). We set

$$\|f\|_{B^{\alpha}_{pq}(\mathcal{T})} := |E|^{-\alpha} \|f\|_p + |f|_{B^{\alpha}_{pq}(\mathcal{T})}. \tag{3.2}$$

Evidently, $\|\cdot\|_{B^{\alpha}_{pq}(\mathcal{T})}$ is a norm if $p, q \geq 1$ and quasi-norm otherwise.

The B-space B^{α}_{pq} has an atomic decomposition. We define

$$\|f\|^{A}_{B^{\alpha}_{pq}(\mathcal{T})} := \inf_{f = \sum_{\theta \in \Theta} a_\theta \varphi_\theta} \Big(\sum_{m=0}^{\infty} \Big(\sum_{\theta \in \Theta_m} (|\theta|^{-\alpha} \|a_\theta \varphi_\theta\|_p)^p \Big)^{q/p} \Big)^{1/q}, \tag{3.3}$$

where the infimum is taken over all representations $f = \sum_{\theta \in \Theta} a_\theta \varphi_\theta$ in $L_p(E)$.

A third approach to the B-spaces B^{α}_{pq} is by using the decomposition via quasi-interpolants from (2.17). We define

$$\|f\|^{Q}_{B^{\alpha}_{pq}(\mathcal{T})} := \Big(\sum_{m=0}^{\infty} \Big(\sum_{\theta \in \Theta_m} (|\theta|^{-\alpha} \|b_\theta(f) \varphi_\theta\|_p)^p \Big)^{q/p} \Big)^{1/q}. \tag{3.4}$$

The following lemma serves as a justification of our definition of admissible indices.

Lemma 3. *If α, p, and q are admissible indices and $\|f\|^{A}_{B^{\alpha}_{pq}(\mathcal{T})} < \infty$, then*

$$\|f\|_{L_1} \leq c \|f\|^{A}_{B^{\alpha}_{pq}(\mathcal{T})}. \tag{3.5}$$

Proof. We consider only the case when $p < 1$, $q > 1$, and $\alpha > 1/p - 1$, since the other cases are similar. Let $f = \sum_{\theta \in \Theta} a_\theta \varphi_\theta$ in L_p. Then using (2.8) and (2.9) we infer

$$\begin{aligned} \|f\|_1 &\leq \sum_{m=0}^{\infty} \Big\| \sum_{\theta \in \Theta_m} a_\theta \varphi_\theta \Big\|_1 \leq c \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} \|a_\theta \varphi_\theta\|_1 \\ &\leq c \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} |\theta|^{1-1/p} \|a_\theta \varphi_\theta\|_p \\ &= c|E|^{\varepsilon} \sum_{m=0}^{\infty} \sum_{\theta \in \Theta_m} (|\theta|/|E|)^{\varepsilon} |\theta|^{-\alpha} \|a_\theta \varphi_\theta\|_p, \end{aligned}$$

where $\varepsilon := \alpha - (1/p - 1) > 0$. By (2.1)-(2.2) if $\theta \in \Theta_m$, then $|\theta|/|E| \le c\rho^m$. We use this, the fact that $p < 1$, and Hölder's inequality to obtain

$$\begin{aligned}\|f\|_1 &\le c|E|^\varepsilon \sum_{m=0}^{\infty} \rho^{\varepsilon m} \Big(\sum_{\theta\in\Theta_m} (|\theta|^{-\alpha} \|a_\theta \varphi_\theta\|_p)^p \Big)^{1/p} \\ &\le c|E|^\varepsilon \Big(\sum_{m=0}^{\infty} \Big(\sum_{\theta\in\Theta_m} (|\theta|^{-\alpha} \|a_\theta \varphi_\theta\|_p)^p \Big)^{q/p} \Big)^{1/q} \Big(\sum_{m=0}^{\infty} \rho^{\varepsilon m q'} \Big)^{1/q'},\end{aligned}$$

where $1/q + 1/q' = 1$. Since $0 < \rho < 1$, the above yields (3.5). □

Theorem 1. *For a given LR-triangulation $\mathcal{T}$ of E and admissible indices α, p, and q the the norms $\|\cdot\|_{B^\alpha_{pq}(\mathcal{T})}$, $\|\cdot\|^A_{B^\alpha_{pq}(\mathcal{T})}$, and $\|\cdot\|^Q_{B^\alpha_{pq}(\mathcal{T})}$, defined in* (3.2)-(3.4)*, are equivalent with constants of equivalence depending only on α, p, q, and the parameters of $\mathcal{T}$.*

The proof of this theorem is fairly simple and will be omitted (see the proofs of the corresponding results in [2, 3, 7]; see also the more complicated proof of Theorem 2 below).

Remark 1. As was shown above the B-spaces B^α_{pq} are in essence sequence spaces and hence they can be interpolated by utilizing standard techniques. We do not present such result here. For some interpolation results on B-spaces, see [3].

Remark 2. In general the B-spaces are different from Besov spaces. However, if $\mathcal{T}$ is a regular triangulation of a compact polygonal domain E in $\mathbb{R}^2$, then the B-space $B^\alpha_{pq}(\mathcal{T})$ coincides with the Besov space $B^{2\alpha}_q(L_p(E))$ for sufficiently small $\alpha > 0$. For more details, see [7].

3.2. Franklin Basis Decomposition of B-spaces

Our main goal in this section is to show that the B-spaces $B^\alpha_{pq}(\mathcal{T})$ can be characterized via representations using Franklin bases. We define

$$\|f\|^F_{B^\alpha_{pq}(\mathcal{T})} := \Big(\sum_{m=0}^{\infty} \Big(\sum_{\theta\in\Theta^*_m} (|\theta|^{-\alpha} \|c_\theta(f) f_\theta\|_p)^p \Big)^{q/p} \Big)^{1/q}, \tag{3.6}$$

where $c_\theta(f) := \langle f, f_\theta \rangle$.

Theorem 2. *Suppose α, p, and q are admissible indices and let $\mathcal{T}$ be an LR-triangulation of a bounded polygonal domain $E \subset \mathbb{R}^2$. Then $f \in B^\alpha_{pq}(\mathcal{T})$ if and only if $\|f\|^F_{B^\alpha_{pq}(\mathcal{T})} < \infty$ and*

$$\|f\|^F_{B^\alpha_{pq}(\mathcal{T})} \approx \|f\|_{B^\alpha_{pq}(\mathcal{T})}. \tag{3.7}$$

Proof. (a) We first show that if $\|f\|^F_{B^\alpha_{pq}} < \infty$, then

$$\|f\|_{B^\alpha_{pq}} \le c\|f\|^F_{B^\alpha_{pq}}. \tag{3.8}$$

We consider only the case when $1 < p < q < \infty$, since the other cases are the same or easier.

Let $\triangle \in \mathcal{T}_j$ $(j \ge 0)$ and denote briefly $\triangle^* := \operatorname{Star}_j(\triangle)$ (see (2.5)). Evidently, $\mathbb{S}_\triangle(g)_p = 0$ if $g \in \mathcal{S}_{j-1}$. Therefore, using that $f = \sum_{\theta\in\Theta^*} c_\theta f_\theta$ in L_p with $c_\theta := c_\theta(f) := \langle f, f_\theta\rangle$, we have

$$\mathbb{S}_\triangle(f)_p \le \sum_{m=j}^{\infty} \Big\| \sum_{\theta\in\Theta^*_m} c_\theta f_\theta \Big\|_{L_p(\triangle^*)}. \tag{3.9}$$

For $\theta \in \Theta^*_m$, we denote $g_\theta(x) := |\theta|^{-1/2} q_1^{\rho_m(\theta,\theta^m_x)/2}$. Then by (2.18), we have $|f_\theta(x)| \le c g_\theta(x) q_1^{\rho_m(\theta,\theta^m_x)/2}$, $x \in E$. Applying Hölder's inequality, we obtain

$$\begin{aligned}\Big|\sum_{\theta\in\Theta^*_m} c_\theta f_\theta(x)\Big|^p &\le c\Big(\sum_{\theta\in\Theta^*_m} |c_\theta| g_\theta(x) q_1^{\rho_m(\theta,\theta^m_x)/2}\Big)^p \\ &\le c \sum_{\theta\in\Theta^*_m} (|c_\theta||g_\theta(x))^p \Big(\sum_{\theta\in\Theta^*_m} q_1^{\rho_m(\theta,\theta^m_x)p'/2}\Big)^{p/p'}, \end{aligned} \tag{3.10}$$

where as usual $1/p + 1/p' = 1$.

Fix $\omega \in \Theta_m$ and denote

$$\Theta^\nu_m := \{\theta \in \Theta_m : \rho_m(\omega,\theta) = \nu\}, \qquad \nu \ge 0.$$

Note that by Lemma 2, $\#\Theta^\nu_m \le c(\nu+1)^t$. Therefore, for an arbitrary $\beta > 0$,

$$\begin{aligned}\sum_{\theta\in\Theta_m} q_1^{\beta\rho_m(\omega,\theta)} &\le \sum_{\nu=0}^{\infty}\sum_{\theta\in\Theta^\nu_m} q_1^{\beta\nu} \le \sum_{\nu=0}^{\infty} \#\Theta^\nu_m q_1^{\beta\nu} \\ &\le c\sum_{\nu=0}^{\infty} (\nu+1)^t q_1^{\beta\nu} \le c < \infty. \end{aligned} \tag{3.11}$$

We use (3.11) in (3.10) with $\beta = p'/2$ and integrate to obtain

$$\Big\|\sum_{\theta\in\Theta^*_m} c_\theta f_\theta\Big\|_{L_p(\triangle^*)} \le c\Big(\sum_{\theta\in\Theta^*_m} \|c_\theta g_\theta\|^p_{L_p(\triangle^*)}\Big)^{1/p}. \tag{3.12}$$

We need estimate $\|g_\theta\|_{L_p(\triangle^*)}$. To this end we define, for $\theta \in \Theta_m$ and $\triangle \in \Theta_j$ $(m \ge j)$,

$$\rho_m(\theta,\triangle^*) := \inf\{\rho_m(\theta,\theta^m_x) : x \in (\triangle^*)^\circ\} \tag{3.13}$$

and

$$\rho_j(\triangle,\theta) := \inf\{\rho_j(\theta^j_x,\theta^j_y) : x \in \triangle^\circ,\ y \in \theta^\circ\}.$$

We next show that

$$\|g_\theta\|^p_{L_p(\triangle^*)} \le c\|f_\theta\|_p^p\, q_2^{\rho_j(\triangle,\theta)}, \qquad \text{where } 0 < q_2 < 1. \tag{3.14}$$

Denote briefly $r := \rho_m(\theta, \triangle^*)$ and let $E_r := \{x \in E : \rho_m(\theta, \theta^m_x) \ge r\}$. Also, let $\Theta^\nu_m := \{\eta \in \Theta_m : \rho_m(\theta, \eta) = \nu\}$. Then $E_r = \cup_{\nu=r}^\infty \cup_{\eta\in\Theta^\nu_m} \eta$. Evidently, $\triangle^* \subset E_r$ and hence

$$\|g_\theta\|^p_{L_p(\triangle^*)} \le \|g_\theta\|^p_{L_p(E_r)} \le \sum_{\nu=r}^\infty \sum_{\eta\in\Theta^\nu_m} \|g_\theta\|^p_{L_p(\eta)}.$$

Further, we use the definition of g_θ to obtain

$$\|g_\theta\|^p_{L_p(\triangle^*)} \le c|\theta|^{-p/2} \sum_{\nu=r}^\infty \sum_{\eta\in\Theta^\nu_m} |\eta| q_1^{\nu p/2} \le c|\theta|^{1-p/2} \sum_{\nu=r}^\infty \sum_{\eta\in\Theta^\nu_m} (|\eta|/|\theta|) q_1^{\nu p/2}.$$

By (2.3), $|\eta|/|\theta| \le c(\nu+1)^s$ and by Lemma 2, $\#\Theta^\nu_m \le c(\nu+1)^t$. Consequently,

$$\begin{aligned} \|g_\theta\|^p_{L_p(\triangle^*)} &\le c|\theta|^{1-p/2} \sum_{\nu=r}^\infty \#\Theta^\nu_m (\nu+1)^s q_1^{\nu p/2} \\ &\le c|\theta|^{1-p/2} \sum_{\nu=r}^\infty (\nu+1)^{t+s} q_1^{\nu p/2} \\ &\le c|\theta|^{1-p/2} q_2^r = c|\theta|^{1-p/2} q_2^{\rho_m(\theta,\triangle^*)}, \end{aligned} \tag{3.15}$$

for some $0 < q_2 < 1$. Now taking into account that $\|f_\theta\|_p \approx |\theta|^{1/p-1/2}$ by (2.20) and $\rho_m(\theta, \triangle^*) \ge \rho_j(\triangle, \theta) - 1$, since $m \ge j$, we conclude that (3.15) yields (3.14).

Combining (3.9) with (3.12) and (3.14), we obtain

$$|\triangle|^{-\alpha}\, \mathbb{S}_\triangle(f)_p \le c \sum_{m=j}^\infty \Big(\sum_{\theta\in\Theta^*_m} (|\theta|/|\triangle|)^{\alpha p} |\theta|^{-\alpha p} \|c_\theta f_\theta\|_p^p q_2^{\rho_j(\triangle,\theta)} \Big)^{1/p}.$$

Let $\omega \in \Theta_j$ be such that $\theta \subset \omega$. Then by (2.1)-(2.2), $|\theta|/|\omega| \le c\rho^{m-j}$ and using (2.3), $|\omega|/|\triangle| \le c(\rho_j(\triangle, \omega) + 1)^s \le c(\rho_j(\triangle, \theta) + 1)^s$. Therefore,

$$(|\theta|/|\triangle|)^{\alpha p} q_2^{\rho_j(\triangle,\theta)} \le c_1(\rho_j(\triangle,\theta)+1)^{s\alpha p} q_2^{\rho_j(\triangle,\theta)} \rho^{\alpha p(m-j)} \le c q_3^{\rho_j(\triangle,\theta)} \rho^{\alpha p(m-j)},$$

for some $0 < q_3 < 1$. Thus

$$|\triangle|^{-\alpha}\, \mathbb{S}_\triangle(f)_p \le c \sum_{m=j}^\infty \rho^{\alpha(m-j)} \Big(\sum_{\theta\in\Theta^*_m} A^p_\theta q_3^{\rho_j(\triangle,\theta)} \Big)^{1/p},$$

where $A_\theta := |\theta|^{-\alpha}\|c_\theta f_\theta\|_p$. Finally, applying Hölder's inequality, we get

$$
\begin{aligned}
(|\triangle|^{-\alpha}\,\mathbb{S}_\triangle(f)_p)^p &\le c\Big(\sum_{m=j}^{\infty}\rho^{\alpha p(m-j)/2}\sum_{\theta\in\Theta_m^*}A_\theta^p q_3^{\rho_j(\triangle,\theta)}\Big)\Big(\sum_{m=j}^{\infty}\rho^{\alpha p'(m-j)/2}\Big)^{p/p'} \\
&\le c\sum_{m=j}^{\infty}\rho^{\alpha p(m-j)/2}\sum_{\theta\in\Theta_m^*}A_\theta^p q_3^{\rho_j(\triangle,\theta)}, \qquad (3.16)
\end{aligned}
$$

since $0<\rho<1$.

We are now prepared to prove (3.8). Using (3.16) in the definition of $|f|_{B^\alpha_{pq}}$, we have

$$
\begin{aligned}
|f|^q_{B^\alpha_{pq}} &= \sum_{j=0}^{\infty}\Big(\sum_{\triangle\in\mathcal{T}_j}(|\triangle|^{-\alpha}\mathbb{S}_\triangle(f)_p)^p\Big)^{q/p} \\
&\le c\sum_{j=0}^{\infty}\Big(\sum_{\triangle\in\mathcal{T}_j}\sum_{m=j}^{\infty}\rho^{\alpha p(m-j)/2}\sum_{\theta\in\Theta_m^*}A_\theta^p q_3^{\rho_j(\triangle,\theta)}\Big)^{q/p} \\
&\le c\sum_{j=0}^{\infty}\Big(\sum_{m=j}^{\infty}\rho^{\alpha p(m-j)/2}\sum_{\theta\in\Theta_m^*}A_\theta^p\sum_{\triangle\in\mathcal{T}_j}q_3^{\rho_j(\triangle,\theta)}\Big)^{q/p},
\end{aligned}
$$

where we once switched the order of summation. Similarly as in (3.11), we have $\sum_{\triangle\in\mathcal{T}_j}q_3^{\rho_m(\triangle,\theta)}\le c<\infty$. On the other hand, using Hölder's inequality,

$$
\begin{aligned}
&\Big(\sum_{m=j}^{\infty}\rho^{\alpha p(m-j)/2}\sum_{\theta\in\Theta_m^*}A_\theta^p\Big)^{q/p} \\
&\qquad\le\sum_{m=j}^{\infty}\Big(\rho^{\alpha p(m-j)/4}\sum_{\theta\in\Theta_m^*}A_\theta^p\Big)^{q/p}\Big(\sum_{m=j}^{\infty}\Big(\rho^{\alpha p(m-j)/4}\Big)^{\gamma}\Big)^{1/\gamma} \\
&\qquad\le c\sum_{m=j}^{\infty}\rho^{\alpha q(m-j)/4}\Big(\sum_{\theta\in\Theta_m^*}A_\theta^p\Big)^{q/p},
\end{aligned}
$$

where $\gamma>1$ is determined from $p/q+1/\gamma=1$. Consequently,

$$
\begin{aligned}
|f|^q_{B^\alpha_{pq}} &\le c\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\rho^{\alpha q(m-j)/4}\Big(\sum_{\theta\in\Theta_m^*}A_\theta^p\Big)^{q/p} \\
&\le c\sum_{m=0}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}A_\theta^p\Big)^{q/p}\sum_{j=0}^{m}\rho^{\alpha q(m-j)/4} \\
&\le c\sum_{m=0}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}A_\theta^p\Big)^{q/p},
\end{aligned}
$$

where we once switched the order of summation and used that $0<\rho<1$. Therefore, $|f|_{B^\alpha_{pq}} \le c\|f\|^F_{B^\alpha_{pq}}$.

It remains to show that $\|f\|_p \le c|E|^\alpha\|f\|^F_{B^\alpha_{pq}}$. Exactly as in (3.12), we obtain

$$\|f\|_p \le \sum_{j=0}^\infty \Big\| \sum_{\theta\in\Theta^*_m} c_\theta f_\theta\Big\|_p \le c\sum_{j=0}^\infty \Big(\sum_{\theta\in\Theta^*_m} \|c_\theta f_\theta\|_p^p\Big)^{1/p}$$

and continuing as in the proof of Lemma 3 we arrive at $\|f\|_p \le c|E|^\alpha\|f\|^F_{B^\alpha_{pq}}$. Thus (3.8) is established.

(b) We next show that

$$\|f\|^F_{B^\alpha_{pq}} \le c\|f\|_{B^\alpha_{pq}} \tag{3.17}$$

provided $\|f\|_{B^\alpha_{pq}} < \infty$. We again consider only the most complicated case when $1<p<q<\infty$. We first estimate $|c_\theta(f)|$, where $c_\theta(f) := \langle f, f_\theta\rangle$. Since $f\in L_p(E)$, then by (2.17)

$$f = Q_0 f + \sum_{j=1}^\infty (Q_j(f) - Q_{j-1}(f)) =: \sum_{j=0}^\infty q_j. \tag{3.18}$$

Fix $\theta\in\Theta^*_m$ $(m\ge 1)$. Then since $f_\theta \perp \mathcal{S}_{m-1}$ and $q_j\in\mathcal{S}_j$,

$$|c_\theta(f)| \le \int_E \Big| f_\theta(x)\sum_{j=m}^\infty q_j(x)\Big| dx \le \sum_{j=m}^\infty \int_E |f_\theta(x) q_j(x)| dx.$$

Denote $g_\theta(x) := q_1^{\rho_m(\theta,\theta^m_x)/2}$ and $h_\theta(x) := |\theta|^{-1/2} q_1^{\rho_m(\theta,\theta^m_x)/2}$. Exactly as in the estimate of $\|g_\theta\|_{L_p(\triangle^*)}$ above (see (3.15) and also the estimate of $\|f_\theta\|_p$ in [9]) we have $\|h_\theta\|_\tau \approx |\theta|^{1/\tau-1/2}$ for $0<\tau\le\infty$. By (2.18), $|f_\theta(x)| \le c g_\theta(x) h_\theta(x)$, $x\in E$. Using the above and Hölder's inequality, we obtain

$$\begin{aligned}|c_\theta(f)| &\le c\sum_{j=m}^\infty \|q_j g_\theta\|_p \|h_\theta\|_{p'} \le c|\theta|^{1/p'-1/2} \sum_{j=m}^\infty \Big(\sum_{\triangle\in\mathcal{T}_j} \|q_j g_\theta\|^p_{L_p(\triangle)}\Big)^{1/p}\\ &\le c|\theta|^{1/2-1/p} \sum_{j=m}^\infty \Big(\sum_{\triangle\in\mathcal{T}_j} \|q_j\|^p_{L_p(\triangle)} q_1^{\rho_m(\theta,\triangle)/2}\Big)^{1/p},\end{aligned} \tag{3.19}$$

where $\rho_m(\theta,\triangle)$ is defined as in (3.13).

For $\triangle\in\mathcal{T}_j$, we denote by $\triangle'$ the only triangle in $\mathcal{T}_{j-1}$ such that $\triangle\subset\triangle'$. Then by (2.14),

$$\|q_j\|_{L_p(\triangle)} \le \|f-Q_j(f)\|_{L_p(\triangle)} + \|f - Q_{j-1}(f)\|_{L_p(\triangle')} \le c(\mathbb{S}_\triangle(f)_p + \mathbb{S}_{\triangle'}(f)_p)$$

and evidently $\rho_m(\theta,\triangle') \le \rho_m(\theta,\triangle)$. Using this in (3.19), we obtain

$$|c_\theta(f)| \le c|\theta|^{1/2-1/p} \sum_{j=m-1}^\infty \Big(\sum_{\triangle\in\mathcal{T}_j} \mathbb{S}_\triangle(f)_p^p \cdot q_1^{\rho_m(\theta,\triangle)/2}\Big)^{1/p},$$

and since $\|f_\theta\|_p \approx |\theta|^{1/p-1/2}$, we have

$$|\theta|^{-\alpha p}\|c_\theta(f)f_\theta\|_p^p \le c|\theta|^{-\alpha p}\Big(\sum_{j=m-1}^{\infty}\Big(\sum_{\triangle\in\mathcal{T}_j}\mathbb{S}_\triangle(f)_p^p\cdot q_1^{\rho_m(\theta,\triangle)/2}\Big)^{1/p}\Big)^p. \tag{3.20}$$

If $\theta\in\Theta_0^*$, then

$$\|c_\theta(f)f_\theta\|_p = |\langle f,f_\theta\rangle|\,\|f_\theta\|_p \le \|f\|_p\|f_\theta\|_{p'}\|f_\theta\|_p \le c\|f\|_p$$

and by (2.1)-(2.2), $|\theta|\ge c|E|$ for $\theta\in\Theta_0$, where $c>0$ depends on the parameters of $\mathcal{T}$ (including $\#\mathcal{T}_0$). Therefore,

$$|\theta|^{-\alpha}\|c_\theta(f)f_\theta\|_p \le c|E|^{-\alpha}\|f\|_p. \tag{3.21}$$

From (3.20)-(3.21), we infer

$$(\|f\|^F_{B^\alpha_{pq}(\mathcal{T})})^q = \sum_{m=0}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}(|\theta|^{-\alpha}\|c_\theta(f)f_\theta\|_p)^p\Big)^{q/p} \le c(|E|^{-\alpha}\|f\|_p)^q$$
$$+\,c\sum_{m=1}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}|\theta|^{-\alpha p}\Big[\sum_{j=m-1}^{\infty}\Big(\sum_{\triangle\in\mathcal{T}_j}\mathbb{S}_\triangle(f)_p^p\cdot q_1^{\rho_m(\theta,\triangle)/2}\Big)^{1/p}\Big]^p\Big)^{q/p}$$
$$\le cA_E^q + c\sum_{m=1}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}\Big[\sum_{j=m-1}^{\infty}\Big(\sum_{\triangle\in\mathcal{T}_j}\Big(\frac{|\triangle|}{|\theta|}\Big)^{\alpha p}A_\triangle^p q_1^{\rho_m(\theta,\triangle)/2}\Big)^{1/p}\Big]^p\Big)^{q/p},$$

where $A_E := |E|^{-\alpha}\|f\|_p$ and $A_\triangle := |\triangle|^{-\alpha}\,\mathbb{S}_\triangle(f)_p$. As in (a) it is readily seen that, for $\theta\in\Theta_m$ and $\triangle\in\mathcal{T}_j$ with $j\ge m-1$, we have $|\triangle|/|\theta| \le c\rho^{j-m}(\rho_m(\theta,\triangle)+1)^s$. Therefore,

$$(\|f\|^F_{B^\alpha_{pq}(\mathcal{T})})^q \le cA_E^q + c\sum_{m=1}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}\Big[\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)}\sigma_j^{1/p}\Big]^p\Big)^{q/p}, \tag{3.22}$$

where $\sigma_j := \sum_{\triangle\in\mathcal{T}_j}A_\triangle^p q_*^{\rho_m(\theta,\triangle)}$ for some $q_1<q_*<1$.

Applying Hölder's inequality, we obtain

$$\Big[\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)}\sigma_j^{1/p}\Big]^p \le \Big(\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)p/2}\sigma_j\Big)\Big(\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)p'/2}\Big)^{p/p'}$$
$$\le c\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)p/2}\sigma_j.$$

Substituting this in (3.22), we find

$$(\|f\|^F_{B^\alpha_{pq}(\mathcal{T})})^q \le cA_E^q + c\sum_{m=1}^{\infty}\Big(\sum_{\theta\in\Theta_m^*}\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)p/2}\sum_{\triangle\in\mathcal{T}_j}A_\triangle^p q_*^{\rho_m(\theta,\triangle)}\Big)^{q/p}$$
$$\le cA_E^q + c\sum_{m=1}^{\infty}\Big(\sum_{j=m-1}^{\infty}\rho^{\alpha(j-m)p/2}\sum_{\triangle\in\mathcal{T}_j}A_\triangle^p\sum_{\theta\in\Theta_m^*}q_*^{\rho_m(\theta,\triangle)}\Big)^{q/p},$$

where we once switched the order of summation. Exactly as in (3.11), we have $\sum_{\theta\in\Theta^*_m} q_*^{\rho_m(\theta,\triangle)} \le c < \infty$. We insert this above and apply once again Hölder's inequality to obtain

$$(\|f\|^F_{B^\alpha_{pq}(\mathcal{T})})^q \le cA_E^q + c\sum_{m=1}^\infty \Big(\sum_{j=m-1}^\infty \rho^{\alpha(j-m)p/2} \sum_{\triangle\in\mathcal{T}_j} A_\triangle^p\Big)^{q/p}$$

$$\le cA_E^q + c\sum_{m=1}^\infty \Big(\sum_{j=m-1}^\infty \rho^{\alpha(j-m)q/4}\Big(\sum_{\triangle\in\mathcal{T}_j} A_\triangle^p\Big)^{q/p}\Big)\Big(\sum_{j=m-1}^\infty \rho^{\alpha(j-m)p\gamma/4}\Big)^{1/\gamma},$$

where $p/q + 1/\gamma = 1$. Since $0<\rho<1$, the last sum above is $\le c < \infty$. We switch one last time the order of summation and obtain

$$(\|f\|^F_{B^\alpha_{pq}(\mathcal{T})})^q \le cA_E^q + c\sum_{j=0}^\infty \Big(\sum_{\triangle\in\mathcal{T}_j} A_\triangle^p\Big)^{q/p} \sum_{m=1}^{j+1} \rho^{\alpha(j-m)q/4}$$

$$\le cA_E^q + c\sum_{j=0}^\infty \Big(\sum_{\triangle\in\mathcal{T}_j} A_\triangle^p\Big)^{q/p} \le c\|f\|^q_{B^\alpha_{pq}},$$

which completes the proof of (3.17). □

4. Nonlinear Approximation from Franklin Bases

In this section, we apply the characterization of B-spaces via Franklin bases from the previous section to nonlinear n-approximation.

Suppose that $\mathcal{F}_\mathcal{T}$ is a Franklin basis generated by an LR-triangulation $\mathcal{T}$ of a compact polygonal domain E in $\mathbb{R}^2$. We let F_n denote the nonlinear set of all functions g of the form

$$g = \sum_{\theta\in\Lambda} a_\theta f_\theta,$$

where $\Lambda \subset \Theta^*$, $\#\Lambda \le n$, and Λ is allowed to vary with g. We denote by $\sigma_n^F(f)_p$ the error of best L_p-approximation to $f\in L_p(E)$ from F_n:

$$\sigma_n^F(f)_p := \inf_{g\in F_n} \|f-g\|_p.$$

We shall use the machinery of Jackson-Bernstein estimates to characterize the approximation spaces generated by $(\sigma_n^F(f)_p)$, $1<p<\infty$. To this end we need the B-spaces $B^\alpha_\tau(\mathcal{T}) := B^\alpha_{\tau\tau}(\mathcal{T})$, where $\alpha>0$ and τ is determined by $1/\tau = \alpha + 1/p$.

Theorem 3 (Jackson estimate). *If $f\in B^\alpha_\tau(\mathcal{T})$, then*

$$\sigma_n^F(f)_p \le cn^{-\alpha}\|f\|_{B^\alpha_\tau(\mathcal{T})} \tag{4.1}$$

where c depends only on α, p, and the parameters of $\mathcal{T}$.

Here it is crucial that the space $B^\alpha_\tau(\mathcal{T})$ is embedded in L_p, namely, if $f \in B^\alpha_\tau(\mathcal{T})$, then $f \in L_p$ and $\|f\|_p \le c\|f\|_{B^\alpha_\tau(\mathcal{T})}$, see [7, 8]. In fact, $B^\alpha_\tau(\mathcal{T})$ lies on the Sobolev embedding line.

For the proof of Theorem 3 one uses the scheme of the proof of Theorem 3.4 from [7] combined with the vector valued maximal inequality (2.7) from Proposition 1. The following (embedding) estimate plays an important role: If $f = \sum_{\theta\in\Theta^*} c_\theta f_\theta$ and $0 < s < 1$, then

$$\|f\|_p \le c\bigg\|\sum_{\theta\in\Theta^*}(\mathcal{M}^s_{\mathcal{T}} c_\theta \tilde{\mathbf{1}}_\theta)(\cdot)\bigg\|_p \le c\bigg\|\sum_{\theta\in\Theta^*}|c_\theta|\tilde{\mathbf{1}}_\theta(\cdot)\bigg\|_p,$$

where we used (2.19) and the maximal inequality (2.7). We now invoke Theorem 3.3 from [7] and obtain

$$\|f\|_p \le c\bigg(\sum_{\theta\in\Theta^*}\|c_\theta\tilde{\mathbf{1}}_\theta\|_p^\tau\bigg)^{1/\tau} \le c\bigg(\sum_{\theta\in\Theta^*}\|c_\theta f_\theta\|_p^\tau\bigg)^{1/\tau} \le c\|f\|_{B^\alpha_\tau(\mathcal{T})},$$

which is the above mentioned embedding. We skip further details.

Theorem 4 (Bernstein estimate). *If $g \in F_n$, then*

$$\|g\|_{B^\alpha_\tau(\mathcal{T})} \le cn^\alpha\|g\|_p \tag{4.2}$$

where c depends only on α, p, and the parameters of the $\mathcal{T}$.

The proof of this theorem can be carried out exactly as in the wavelet case by utilizing the fact that $\mathcal{F}_{\mathcal{T}}$ is a unconditional bases for L_p $(1 < p < \infty)$ and the localization properties of the Franklin functions given in Proposition 2 (see e.g. [4], Theorem 6). The proof relys on the important fact [9] that if $f \in L_p(E)$, $1 < p < \infty$, and $f = \sum_{\theta\in\Theta^*} c_\theta f_\theta$, then

$$\|f\|_p \approx \bigg\|\bigg(\sum_{\theta\in\Theta^*}|c_\theta|^2\tilde{\mathbf{1}}_\theta(\cdot)\bigg)^{1/2}\bigg\|_p.$$

We omit the details.

One can now follow the standard lines to obtain direct and inverse estimates for $\sigma^F_n(f)_p$. To this end, denote by $K(f,t)_p := K(f,t;L_p,B^\alpha_\tau(\mathcal{T}))$ the K-functional defined by $K(f,t)_p := \inf_{g\in B^\alpha_\tau(\mathcal{T})}\|f-g\|_p + t\|g\|_{B^\alpha_\tau(\mathcal{T})}$, $t > 0$.

By standard arguments (see e.g. [10]), the Jackson and Bernstein estimates (4.1)-(4.2) imply the following direct and inverse estimates: For $f \in L_p(E)$ one has

$$\sigma^F_n(f)_p \le cK(f,n^{-\alpha})_p \tag{4.3}$$

and

$$K(f,n^{-\alpha})_p \le cn^{-\alpha}\bigg(\bigg[\sum_{\nu=1}^n\frac{1}{\nu}(\nu^\alpha\sigma^F_\nu(f)_p)^{\tau^*}\bigg]^{1/\tau^*} + \|f\|_p\bigg), \tag{4.4}$$

where $\tau^* := \min\{\tau, 1\}$.

We define the approximation space $A_q^\gamma = A_q^\gamma(\mathcal{F}_\mathcal{T}, L_p)$ to be the set of all functions $f \in L_p(E)$ such that

$$\|f\|_{A_q^\gamma} := \|f\|_p + \left(\sum_{n=1}^{\infty} (n^\gamma \sigma_n^F(f)_p)^q \, \frac{1}{n}\right)^{1/q} < \infty \tag{4.5}$$

with the usual modification when $q = \infty$.

The following characterization of the approximation spaces A_q^γ is immediate from estimates (4.3)-(4.4).

Theorem 5. *If $0 < \gamma < \alpha$ and $0 < q \le \infty$, then*

$$A_q^\gamma(\mathcal{F}_\mathcal{T}, L_p) = (L_p, B_\tau^\alpha(\mathcal{T}))_{\frac{\gamma}{\alpha},q}$$

with equivalent (quasi-)norms, where $(L_p, B_\tau^\alpha(\mathcal{T}))_{\frac{\gamma}{\alpha},q}$ is the real interpolation space between L_p and $B_\tau^\alpha(\mathcal{T})$ (see e.g. [1]).

In one specific case the approximation space $A_q^\alpha(\mathcal{F}_\mathcal{T}, L_p)$ can be identified as a B-space:

Theorem 6. *Assuming that $1 < p < \infty$, $\alpha > 0$, and $1/\tau := \alpha + 1/p$, we have*

$$A_\tau^\alpha(\mathcal{F}_\mathcal{T}, L_p) = B_\tau^\alpha(\mathcal{T}) \tag{4.6}$$

with equivalent norms.

The proof is a mere repetition of the proof of Theorem 3.4 in [3] and will be omitted.

Finally, we want to compare the nonlinear n-term approximation from $\mathcal{F}_\mathcal{T}$ with the n-term approximation from $\Phi_\mathcal{T}$ (Courant elements). Let Σ_n denote the set of all functions g of the form $g = \sum_{\theta\in\mathcal{M}} a_\theta \varphi_\theta$, where $\mathcal{M} \subset \Theta$, $\#\mathcal{M} \le n$. We denote by $\sigma_n^\Phi(f)_p$ the error of best L_p-approximation to $f \in L_p(E)$ from Σ_n:

$$\sigma_n^\Phi(f)_p := \inf_{g\in\Sigma_n} \|f - g\|_p.$$

Let $A_q^\gamma(\Phi_\mathcal{T}, L_p)$ be the approximation space generated by $(\sigma_n^\Phi(f)_p)$, defined similarly as in (4.5). As is shown in [7], the approximation space $A_q^\gamma(\Phi_\mathcal{T}, L_p)$ has precisely the same characterization as the one from Theorem 5. Consequently, $A_q^\gamma(\mathcal{F}_\mathcal{T}, L_p) = A_q^\gamma(\Phi_\mathcal{T}, L_p)$ with equivalent (quasi-)norms, if $1 < p < \infty$, for all $\gamma > 0$ and $0 < q \le \infty$.

We close by noting that results similar to the results from Theorems 3 - 6 hold true for nonlinear n-term approximation from $\mathcal{F}_\mathcal{T}$ in $H_1(E, \mathcal{T})$ [9], the Hardy space generated by an LR-triangulation $\mathcal{T}$ of E. For the proofs one utilizes the fact that the Franklin system $\mathcal{F}_\mathcal{T}$ is a unconditional basis for $H_1(E, \mathcal{T})$ (see [9]) and the techniques used above.

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