DECOMPOSITION OF SPACES OF DISTRIBUTIONS INDUCED BY HERMITE EXPANSIONS

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Abstract. Decomposition systems with rapidly decaying elements (needlets) based on Hermite functions are introduced and explored. It is proved that the Triebel-Lizorkin and Besov spaces on $\mathbb{R}^d$ induced by Hermite expansions can be characterized in terms of the needlet coefficients. It is also shown that the Hermite Triebel-Lizorkin and Besov spaces are, in general, different from the respective classical spaces.

1. Introduction

The purpose of this paper is to extend the fundamental results of Frazier and Jawerth [4, 5] on the φ-transform to the case of Hermite expansions on $\mathbb{R}^d$. In the spirit of [4, 5] we will construct a pair of dual frames in terms of Hermite functions and use them to characterize the Hermite-Triebel-Lizorkin and Hermite-Besov spaces.

Let $\{h_n\}_{n=0}^\infty$ be the $L^2(\mathbb{R}^d)$ normalized univariate Hermite functions (see §2.1). The d-dimensional Hermite functions are defined by $\mathcal{H}_\alpha(x) := h_{\alpha_1}(x_1)\cdots h_{\alpha_d}(x_d)$. Then the kernel of the orthogonal projector of L^2 onto $W_n := \operatorname{span}\{\mathcal{H}_\alpha : |\alpha| = n\}$ is given by $\mathcal{H}_n(x,y) := \sum_{|\alpha|=n}\mathcal{H}_\alpha(x)\mathcal{H}_\alpha(y)$. Our construction of Hermite frames hinges on the fundamental fact that for compactly supported C^∞ functions $\widehat{a}$ the kernels $\Lambda_n(x,y) := \sum_{j=0}^\infty \widehat{a}(\frac{j}{n})\mathcal{H}_j(x,y)$ decay rapidly away from the main diagonal in $\mathbb{R}^d$. This fact was established in [3] for dimension $d = 1$ and in [1] in general. We obtain a more precise estimate in Theorem 2.2 below. We utilize kernels of such kind for the construction of a pair of dual frames $\{\varphi_\xi\}_{\xi\in\mathcal{X}}$, $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, where $\mathcal{X}$ is a multilevel index set. The frame elements have almost exponential localization (see (3.11)) which prompted us to call them "needlets". The needlet systems of this article can be viewed as an analogue of the φ-transform of Frazier and Jawerth [4, 5]. Frames of the same nature in the case $d = 1$ have been previously introduced in [3].

Our primary goal is to utilize needlets to the characterization of the Triebel-Lizorkin and Besov spaces in the context of Hermite expansions. To be more specific, assume that $\widehat{a} \in C^\infty$, $\operatorname{supp}\widehat{a} \subset [1/4, 4]$, and $|\widehat{a}| > c$ on $[1/3, 3]$, and define

$$\Phi_0 := \mathcal{H}_0 \quad \text{and} \quad \Phi_j := \sum_{\nu=0}^\infty \widehat{a}\Big(\frac{\nu}{4^{j-1}}\Big)\mathcal{H}_\nu, \quad j \ge 1. \tag{1.1}$$

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Then for all appropriate indices we define the Hermite-Triebel-Lizorkin space $F_p^{\alpha q} = F_p^{\alpha q}(H)$ as the set of all tempered distributions f such that

$$\|f\|_{F_p^{\alpha q}} := \Big\| \Big(\sum_{j=0}^{\infty} (2^{\alpha j} |\Phi_j * f(\cdot)|)^q \Big)^{1/q} \Big\|_p < \infty,$$

where $\Phi_j * f(x) := \langle f, \overline{\Phi(x,\cdot)} \rangle$ (see Definition 4.1). We define the Hermite-Besov spaces $B_p^{\alpha q} = B_p^{\alpha q}(H)$ by the norm

$$\|f\|_{B_p^{\alpha q}} := \Big(\sum_{j=0}^{\infty} \Big(2^{\alpha j} \|\Phi_j * f\|_p \Big)^q \Big)^{1/q}.$$

One normally uses binary dilations in (1.1) (see e.g. [17, §10.3] and also [1, 2]). We dilate $\widehat{a}$ by factors of 4^j instead since then the Hermite F- and B-spaces embed just as the classical F- and B-spaces.

Our main results assert that the Hermite-Triebel-Lizorkin and Hermite Besov spaces can be characterized in terms of respective sequence norms of the needlet coefficients of the distributions (Theorems 4.5, 5.7). Furthermore, we use these results to show that the Hermite-F- and B-spaces of essentially positive smoothness are different from the respective classical F- and B-spaces on $\mathbb{R}^d$.

Our development here is a part of a bigger project for needlet characterization of Triebel-Lizorkin and Besov spaces on nonclassical domains such as the unit sphere [10], the interval with Jacobi weights [7], and the unit ball [8].

The rest of the paper is organized as follows: Section 2 contains some background material. The needlets are introduced in §3. In §4 the Hermite-Triebel-Lizorlin spaces are defined and characterized via needlets. The Hermite-Besov spaces are introduced and characterized in §5. Section 6 contains the proofs of a number of lemmas and theorems from §2-§5.

Some useful notation: $\|f\|_p := \|f\|_{L^p(\mathbb{R}^d)}$; for a measurable set $E \subset \mathbb{R}^d$, $|E|$ denotes the Lebesgue measure of E and $\mathbb{1}_E$ is the characteristic function of E. Also, for $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm of x, $|x|_\infty := \max_{1 \le j \le d} |x_j|$, and $d(x, E) := \inf_{y \in E} |x - y|_\infty$ is the ℓ^∞ distance of x from $E \subset \mathbb{R}^d$. Positive constants are denoted by c, $c_1, \dots$ and they may vary at every occurrence; $A \sim B$ means $c_1 A \le B \le c_2 A$.

2. Preliminaries

2.1. **Localized kernels induced by Hermite functions.** We begin with a review of some basic properties of Hermite polynomials and functions. (For background information we refer the reader to [16].) The Hermite polynomials are defined by

$$H_n(t) = (-1)^n e^{t^2} \Big(\frac{d}{dt} \Big)^n \Big(e^{-t^2} \Big), \qquad n = 0, 1, \dots.$$

These polynomials are orthogonal with respect to e^{-t^2} on $\mathbb{R}$. We will denote the L^2-normalized Hermite functions by

$$h_n(t) := \big(2^n n! \sqrt{\pi} \big)^{-1/2} H_n(t) e^{-t^2/2}.$$

One has

$$\int_{\mathbb{R}} h_n(t) h_m(t) dt = \big(2^n n! \sqrt{\pi} \big)^{-1} \int_{\mathbb{R}} H_n(t) H_m(t) e^{-t^2} dt = \delta_{n,m}.$$

As is well known the Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$.

As already mentioned, the d-dimensional Hermite functions $\mathcal{H}_\alpha$ are defined by

$$\mathcal{H}_\alpha(x) := h_{\alpha_1}(x_1)\cdots h_{\alpha_d}(x_d), \quad \alpha = (\alpha_1, \dots, \alpha_d). \tag{2.1}$$

Evidently $e^{|x|^2/2}\mathcal{H}_\alpha(x)$ is a polynomial of degree $|\alpha| := \alpha_1 + \cdots + \alpha_d$. The Hermite functions form an orthonormal basis for $L^2(\mathbb{R}^d)$. Moreover, $\mathcal{H}_\alpha$ are eigenfunctions of the Hermite operator $D := -\Delta + |x|^2$ and

$$D\mathcal{H}_\alpha = (2|\alpha| + d)\mathcal{H}_\alpha, \tag{2.2}$$

where Δ is the Laplacian. The operator D can be written in the form

$$D = \frac{1}{2}\sum_{j=1}^{d}(A_jA_j^* + A_j^*A_j), \quad \text{where} \quad A_j = -\frac{\partial}{\partial x_j} + x_j, \quad A_j^* = \frac{\partial}{\partial x_j} + x_j. \tag{2.3}$$

Let e_j denote the jth coordinate vector in $\mathbb{R}^d$. Then the operators A_j and A_j^* satisfy

$$A_j\mathcal{H}_\alpha = (2\alpha_j + 2)^{\frac{1}{2}}\mathcal{H}_{\alpha+e_j} \quad \text{and} \quad A_j^*\mathcal{H}_\alpha = (2\alpha_j)^{\frac{1}{2}}\mathcal{H}_{\alpha-e_j}. \tag{2.4}$$

Combining these two relations shows that $\{\mathcal{H}_\alpha\}$ satisfy the recurrence relation

$$x_j\mathcal{H}_\alpha(x) = \Big(\tfrac{\alpha_j+1}{2}\Big)^{\frac{1}{2}}\mathcal{H}_{\alpha+e_j}(x) + \big(\tfrac{\alpha_j}{2}\big)^{\frac{1}{2}}\mathcal{H}_{\alpha-e_j}(x) \tag{2.5}$$

and also

$$\frac{\partial}{\partial x_j}\mathcal{H}_\alpha(x) = -\Big(\tfrac{\alpha_j+1}{2}\Big)^{\frac{1}{2}}\mathcal{H}_{\alpha+e_j}(x) + \big(\tfrac{\alpha_j}{2}\big)^{\frac{1}{2}}\mathcal{H}_{\alpha-e_j}(x). \tag{2.6}$$

Let $W_n := \operatorname{span}\{\mathcal{H}_\alpha : |\alpha| = \mathrm{n}\}$ and $V_n := \bigoplus_{j=0}^{n} W_j$. The kernels of orthogonal projectors on W_n and V_n are given by

$$\mathcal{H}_n(x,y) := \sum_{|\alpha|=n}\mathcal{H}_\alpha(x)\mathcal{H}_\alpha(y) \quad \text{and} \quad K_n(x,y) := \sum_{j=0}^{n}\mathcal{H}_j(x,y), \tag{2.7}$$

respectively.

An important role will be played by operators whose kernels are obtained by smoothing out the coefficients of the kernel K_n by sampling a compactly supported C^∞ function $\widehat{a}$. For our purposes we will be considering "smoothing" functions $\widehat{a}$ that satisfy:

Definition 2.1. *A function $\widehat{a} \in C^\infty[0,\infty)$ is said to be admissible of type*
(*a*) *if* $\operatorname{supp}\widehat{a} \subset [0, 1+v]$ $(v > 0)$ *and* $\widehat{a}(t) = 1$ *on* $[0,1]$, *and of type*
(*b*) *if* $\operatorname{supp}\widehat{a} \subset [u, 1+v]$, *where* $0 < u < 1$, $v > 0$.

For an admissible function $\widehat{a}$ we consider the kernel

$$\Lambda_n(x,y) := \sum_{j=0}^{\infty}\widehat{a}\Big(\frac{j}{n}\Big)\mathcal{H}_j(x,y). \tag{2.8}$$

It will be critical for our further development that the kernels $\Lambda_n(x,y)$ and their derivatives decay rapidly away from the main diagonal $y = x$ in $\mathbb{R}^d \times \mathbb{R}^d$:

Theorem 2.2. *Suppose $\widehat{a}$ is admissible in the sense of Definition 2.1 and let $\alpha \in \mathbb{N}_0^d$. Then for any $k \ge 1$ there exists a constant c_k depending only on k, α, d, and $\widehat{a}$ such that*

$$\Big|\frac{\partial^\alpha}{\partial x^\alpha}\Lambda_n(x,y)\Big| \le c_k \frac{n^{\frac{|\alpha|}{2}}[K_{n+[vn]+|\alpha|+k}(x,x)]^{\frac{1}{2}}[K_{n+[vn]+k}(y,y)]^{\frac{1}{2}}}{(1+n^{\frac{1}{2}}|x-y|)^k}. \tag{2.9}$$

Here the dependence of c_k on $\widehat{a}$ is of the form $c_k = c(k,|\alpha|,u,d)\max_{0\le l\le k}\|\widehat{a}^{(l)}\|_\infty$.

We relegate the somewhat lengthy proof of this theorem to §6.1.

The function

$$\lambda_n(x) := \frac{1}{K_n(x,x)} \tag{2.10}$$

is termed *Christoffel function* and it is known (see e.g. [9]) to have the following asymptotic in dimension $d = 1$:

$$\lambda_n(x) \sim n^{-1/2}\left(\max\left\{n^{-2/3}, 1-\frac{|x|}{\sqrt{2n}}\right\}\right)^{-1/2} \tag{2.11}$$

uniformly for $n \ge 1$ and $x \le \sqrt{2n}(1+c'n^{-2/3})$, where $c' > 0$ is any fixed constant. Consequently, for $d = 1$ we have

$$K_n(x,x) \sim n^{1/2}\left(\max\left\{n^{-2/3}, 1-\frac{|x|}{\sqrt{2n}}\right\}\right)^{1/2}, \quad x \le \sqrt{2n}(1+c'n^{-2/3}). \tag{2.12}$$

For $d \ge 2$ one has (see [16, p. 70])

$$|\mathcal{H}_n(x,x)| \le cn^{d/2-1}, \quad x \in \mathbb{R}^d. \tag{2.13}$$

This along with (2.7) leads to

$$K_n(x,x) \le cn^{d/2}, \quad x \in \mathbb{R}^d, \quad d \ge 1. \tag{2.14}$$

On the other hand, it is well known that (see e.g. [16, p. 26])

$$|h_n(x)| \le ce^{-\gamma x^2}, \quad |x| \ge (4n+2)^{1/2}, \quad \gamma > 0, \tag{2.15}$$

and $\|h_n\|_\infty \le cn^{-1/12}$, which readily imply

$$K_n(x,x) \le ce^{-\gamma'|x|_\infty^2}, \quad \text{if} \quad |x|_\infty := \max_{1\le j\le d}|x_j| \ge (4n+2)^{1/2}, \tag{2.16}$$

where $\gamma' > 0$ depends only on d.

Now, combining (2.9) with (2.14) and (2.16) (setting $\gamma^* := \gamma'/2$) we arrive at

Corollary 2.3. *Under the hypothesis of Theorem 2.2 we have*

$$\Big|\frac{\partial^\alpha}{\partial x^\alpha}\Lambda_n(x,y)\Big| \le c_k\frac{n^{\frac{|\alpha|+d}{2}}}{(1+n^{\frac{1}{2}}|x-y|)^k}, \quad x \in \mathbb{R}^d, \tag{2.17}$$

$$\Big|\frac{\partial^\alpha}{\partial x^\alpha}\Lambda_n(x,y)\Big| \le c_k\frac{e^{-\gamma^*|x|_\infty^2}}{(1+n^{\frac{1}{2}}|x-y|)^k}, \ \textit{if}\ |x|_\infty \ge (4(n+[vn]+|\alpha|+k)+2)^{1/2}, \tag{2.18}$$

and

$$\Big|\frac{\partial^\alpha}{\partial x^\alpha}\Lambda_n(x,y)\Big| \le c_k\frac{e^{-\gamma^*|y|_\infty^2}}{(1+n^{\frac{1}{2}}|x-y|)^k}, \ \textit{if}\ |y|_\infty \ge (4(n+[vn]+k)+2)^{1/2}. \tag{2.19}$$

Note that an estimate similar to (2.17) is proved in [3] when $d = 1$ and $\alpha = 0$ and in the general case in [1]. Estimate (2.9) is new.

We now turn to a lower bound estimate.

Theorem 2.4. *Let $\widehat{a}$ be admissible in the sense of Definition 2.1 and $|\widehat{a}(t)| > c_* > 0$ on $[1, 1+\tau]$, $\tau > 0$. Then for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}^d} |\Lambda_n(x,y)|^2 dy \ge c\, n^{d/2} \quad \textit{for} \quad |x| \le (1-\varepsilon)\sqrt{2(1+\tau)n},$$

where $c > 0$ depends only on τ, ε, c_*, and d.

This theorem provides a lower bound for the range where estimate (2.17) (with $\alpha = 0$) is sharp. To indicate the dependence of K_n on d, we write $K_{n,d} = K_n$. Theorem 2.4 is an immediate consequence of (2.12) and the following lemma.

Lemma 2.5. *If $0 < \lambda < 1$, $0 < \rho < 1$, and $d \ge 1$, then there exists a constant $c > 0$ such that for $n \ge 2/\lambda$*

$$\sum_{m=[(1-\lambda)n]}^{n} \mathcal{H}_m^2(x,x) \ge cn^{\frac{d-1}{2}} K_{[\rho n],1}(t,t) \quad \textit{if } t := |x| \le 2\sqrt{2n+1}\ . \tag{2.20}$$

The proof of this lemma is given in §6.1.

2.2. **Norm relation.** For future use we give here the well known relation between different norms of functions from V_n (see e.g. [9]): For $0 < p, q \le \infty$

$$\|g\|_p \le cn^{\frac{d}{2}|1/q-1/p|}\|g\|_q \quad \text{for} \quad g \in V_n, \tag{2.21}$$

with $c > 0$ depending only on p, q, and d.

This estimate can be proved by means of the kernels from (2.8) with $\widehat{a}$ admissible of type (a).

2.3. **Cubature formula.** In order to define our frame elements, we need a cubature formula exact for products fg with $f, g \in V_n$. Such a formula, however, is readily available using the Gaussian quadrature formula.

Proposition 2.6. [15] *Denote by $t_{\nu,n}$, $\nu = 1, 2, \dots, n$, the zeros of the Hermite polynomial $H_n(t)$. The Gaussian quadrature formula*

$$\int_{\mathbb{R}} f(t)e^{-t^2} dt \sim \sum_{\nu=1}^{n} w_{\nu,n} f(t_{\nu,n}), \quad w_{\nu,n} := \lambda_n(t_{\nu,n})e^{-t_{\nu,n}^2}, \tag{2.22}$$

is exact for all polynomials of degree $2n-1$. Here $\lambda_n(\cdot)$ is the Christoffel function defined in (2.10).

The product nature of $e^{-|x|^2}$ enables us to obtain the desired cubature formula on $\mathbb{R}^d$ right away.

Proposition 2.7. *Let $\xi_{\alpha,n} := (t_{\alpha_1,n}, \dots, t_{\alpha_d,n})$ and $\lambda_{\alpha,n} := \prod_{\nu=1}^d \lambda_n(t_{\alpha_\nu,n})$. The cubature formula*

$$\int_{\mathbb{R}^d} f(x)g(x)dx \sim \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_d=1}^{n} \lambda_{\alpha,n} f(\xi_{\alpha,n})g(\xi_{\alpha,n}) \tag{2.23}$$

is exact for all $f \in V_\ell$, $g \in V_m$ with $\ell + m \le 2n-1$.

We next record some well known properties of the zeros of Hermite polynomials. Suppose $\{\xi_\nu\}$ are the zeros of $H_n(t)$ (with n even) ordered so that

$$\xi_{-\frac{n}{2}} < \cdots < \xi_{-1} < 0 < \xi_1 < \cdots < \xi_{\frac{n}{2}}, \quad \xi_{-\nu} = \xi_\nu. \tag{2.24}$$

From [9] we have $\xi_{\frac{n}{2}} \le \sqrt{2n+1} - n^{-1/6}$ and uniformly for $|\nu| \le n/2 - 1$

$$\xi_{\nu+1} - \xi_{\nu-1} \sim n^{-1/2}\left(\max\left\{n^{-2/3}, 1 - \frac{|\xi_\nu|}{\sqrt{2n}}\right\}\right)^{-1/2}. \tag{2.25}$$

Consequently, on account of (2.11)

$$\lambda_n(\xi_\nu) \sim \xi_{\nu-1} - \xi_{\nu+1}, \quad |\nu| \le n/2 - 1 \quad (\xi_0 := 0). \tag{2.26}$$

By [15, (6.31.19)]

$$\frac{\pi(\nu - \frac{1}{2})}{(2n+1)^{1/2}} < \xi_\nu < \frac{4\nu + 3}{(2n+1)^{1/2}}, \quad \nu = 1, \dots, n/2. \tag{2.27}$$

From this and (2.25) we have, for any $\varepsilon > 0$,

$$\xi_{\nu+1} - \xi_{\nu-1} \sim n^{-1/2} \quad \text{if} \quad |\nu| \le (1/2 - \varepsilon)n, \tag{2.28}$$

and

$$c_1 n^{-1/2} \le \xi_\nu - \xi_{\nu-1} \le c_2 n^{-1/6} \quad \text{if} \quad (1/2 - \varepsilon)n < |\nu| \le n/2. \tag{2.29}$$

Here the constants depend on ε.

It also follows by (2.25) that

$$\xi_{\nu+1} - \xi_{\nu-1} \sim \xi_\nu - \xi_{\nu-2}, \quad -n/2 + 2 \le \nu \le n/2 - 1. \tag{2.30}$$

For the construction of our frames in Section 3 we need the cubature formulae from Proposition 2.7 with

$$n = 2N_j, \quad \text{where} \quad N_j := [(1 + 11\delta)(4/\pi)^2 4^j] + 3 \tag{2.31}$$

and $0 < \delta < 1/37$ is an arbitrary (but fixed) constant.

Given $j \ge 0$, let as above ξ_ν, $\nu = \pm 1, \dots, \pm N_j$, be the zeros of $H_{2N_j}(t)$. Let $\mathcal{X}_j$ be the set of all nodes of cubature (2.23) with $n = 2N_j$, i.e. $\mathcal{X}_j$ is the set of all points $\xi_\alpha := (\xi_{\alpha_1}, \dots, \xi_{\alpha_d})$, where $0 < |\alpha_\nu| \le N_j$. Also, for $\xi = \xi_\alpha$ we denote briefly $\lambda_\xi := \lambda_{\alpha, N_j}$. Note that $\#\mathcal{X}_j = (2N_j)^d \sim 4^{jd}$.

An immediate consequence of Proposition 2.7 is the following

Corollary 2.8. *The cubature formula*

$$\int_{\mathbb{R}^d} f(x)g(x)dx \sim \sum_{\xi \in \mathcal{X}_j} \lambda_\xi f(\xi) g(\xi), \quad \lambda_\xi := \prod_{\nu=1}^{d} \lambda_{2N_j}(\xi_{\alpha_\nu}), \tag{2.32}$$

is exact for all $f \in V_\ell$, $g \in V_m$ with $\ell + m \le 4N_j - 1$.

For later use we now introduce *tiles* $\{R_\xi\}$ induced by the points of $\mathcal{X}_j$. Set

$$I_1 := [0, (\xi_1 + \xi_2)/2], \ I_{-1} := -I_1,$$
$$I_\nu := [(\xi_{\nu-1} + \xi_\nu)/2, (\xi_\nu + \xi_{\nu+1})/2], \quad \nu = \pm 2, \dots, \pm N_{j-1}, \quad \text{and}$$
$$I_{N_j} := [(\xi_{N_{j-1}} + \xi_{N_j})/2, \xi_{N_j} + 2^{-j/6}], \ I_{-N_j} := -I_{N_j}.$$

For each $\xi = \xi_\alpha = (\xi_{\alpha_1}, \dots, \xi_{\alpha_d})$ in $\mathcal{X}_j$ we set

$$R_\xi := I_{\alpha_1} \times I_{\alpha_2} \times \cdots \times I_{\alpha_d}, \tag{2.33}$$

and also

$$Q_j := [\xi_{-N_j} - 2^{-j/6}, \xi_{N_j} + 2^{-j/6}]^d = \cup_{\xi \in \mathcal{X}_j} R_\xi. \tag{2.34}$$

Thus we have associated to each $\xi \in \mathcal{X}_j$ $(j \geq 0)$ a tile R_ξ so that different tiles do not overlap (have disjoint interiors) and they cover the cube $Q_j \sim [-2^j, 2^j]^d$.

Observe that by the construction of the tiles $\{R_\xi\}$ and (2.26) we have

$$\lambda_\xi \sim |R_\xi|, \quad \xi \in \mathcal{X}_j. \tag{2.35}$$

By (2.28) $|R_\xi| \sim 2^{-jd}$ if $\xi = \xi_\alpha \in \mathcal{X}_j$ with $|\alpha|_\infty \leq (1/2 - \delta/2)2N_j = (1-\delta)N_j$. Assume that $|\xi_\alpha| \leq (1+4\delta)2^{j+1}$. By (2.27)

$$|\xi_\alpha|_\infty > \frac{\pi(|\alpha|_\infty - 1/2)}{(4N_j+1)^{1/2}} \quad \text{and hence} \quad \frac{\pi(|\alpha|_\infty - 1/2)}{(4N_j+1)^{1/2}} < (1+4\delta)2^{j+1}.$$

Using the definition of N_j in (2.31) it is easy to show that the above inequality implies $|\alpha|_\infty \leq (1-\delta)N_j$. Consequently, for $\xi \in \mathcal{X}_j$,

$$R_\xi \sim \xi + [-2^{-j}, 2^{-j}]^d \quad \text{and} \quad |R_\xi| \sim 2^{-jd} \quad \text{if} \quad |\xi|_\infty \leq (1+4\delta)2^{j+1}. \tag{2.36}$$

On the other hand, by (2.28)-(2.29) it follows that, in general,

$$\xi + [-c_1 2^{-j}, c_1 2^{-j}]^d \subset R_\xi \subset \xi + [-c_2 2^{-j/3}, c_2 2^{-j/3}]^d, \quad \xi \in \mathcal{X}_j, \tag{2.37}$$

and hence

$$c' 2^{-jd} \leq |R_\xi| \leq c'' 2^{-jd/3}. \tag{2.38}$$

Finally, note that since the zeros of H_n and H_{n+1} interlace, each $R_\eta \in \mathcal{X}_{j+\ell}$, $\ell \geq 1$, may intersect at most finitely many (depending only on d) tiles R_ξ, $\xi \in \mathcal{X}_j$.

2.4. **Maximal operator.** Let $\mathcal{M}_s$ be the maximal operator, defined by

$$\mathcal{M}_s f(x) := \sup_{Q:\, x \in Q} \left(\frac{1}{|Q|} \int_Q |f(y)|^s \, dy \right)^{1/s}, \quad x \in \mathbb{R}^d, \tag{2.39}$$

where the sup is over all cubes Q in $\mathbb{R}^d$ with sides parallel to the coordinate axes which contain x.

We will need the Fefferman-Stein vector-valued maximal inequality (see [14]): If $0 < p < \infty$, $0 < q \leq \infty$, and $0 < s < \min\{p, q\}$, then for any sequence of functions $f_1, f_2, \dots$ on $\mathbb{R}^d$

$$\Big\| \Big(\sum_{j=1}^{\infty} [\mathcal{M}_s f_j(\cdot)]^q \Big)^{1/q} \Big\|_p \leq c \Big\| \Big(\sum_{j=1}^{\infty} |f_j(\cdot)|^q \Big)^{1/q} \Big\|_p, \tag{2.40}$$

where $c = c(p, q, s, d)$.

2.5. **Distributions on $\mathbb{R}^d$.** As is customary, we will denote by $\mathcal{S}$ the Schwartz class of all functions $\phi \in C^\infty(\mathbb{R}^d)$ such that

$$P_{\beta,\gamma}(\phi) := \sup_x |x^\gamma D^\beta \phi(x)| < \infty \quad \text{for all } \gamma, \beta. \tag{2.41}$$

The topology on $\mathcal{S}$ is defined by the semi-norms $P_{\beta,\gamma}$. Then the space $\mathcal{S}'$ of all temperate distributions is defined as the set of all continuous linear functionals on $\mathcal{S}$. The pairing of $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ will be denoted by $\langle f, \phi \rangle := f(\overline{\phi})$ which is consistent with the inner product $\langle f, g \rangle := \int_{\mathbb{R}^d} f \overline{g} dx$ in $L^2(\mathbb{R}^d)$.

As a convenient notation we introduce the following "convolution":

Definition 2.9. *For functions $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ and $f : \mathbb{R}^d \to \mathbb{C}$, we write*

$$\Phi * f(x) := \int_{\mathbb{R}^d} \Phi(x,y) f(y)\,dy. \tag{2.42}$$

More generally, assuming that $f \in \mathcal{S}'$ and $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is such that $\Phi(x,y)$ belongs to $\mathcal{S}$ as a function of y ($\Phi(x,\cdot) \in \mathcal{S}$), we define $\Phi * f$ by

$$\Phi * f(x) := \langle f, \overline{\Phi(x,\cdot)}\rangle, \tag{2.43}$$

where on the right f acts on $\overline{\Phi(x,y)}$ as a function of y.

We next record some properties of the above "convolution" that are well known and easy to prove.

Lemma 2.10. (*a*) *If $f \in \mathcal{S}'$ and $\Phi(\cdot,\cdot) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, then $\Phi * f \in \mathcal{S}$. Furthermore $\mathcal{H}_n * f \in V_n$.*

(*b*) *If $f \in \mathcal{S}'$, $\Phi(\cdot,\cdot) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, and $\phi \in \mathcal{S}$, then $\langle \Phi * f, \phi\rangle = \langle f, \overline{\Phi} * \phi\rangle$.*

(*c*) *If $f \in \mathcal{S}'$, $\Phi(\cdot,\cdot), \Psi(\cdot,\cdot) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, and $\Phi(y,x) = \Phi(x,y)$, $\Psi(y,x) = \Psi(x,y)$, then*

$$\Psi * \overline{\Phi} * f(x) = \langle \Psi(x,\cdot), \Phi(\cdot,\cdot)\rangle * f. \tag{2.44}$$

Evidently the Hermite functions $\{\mathcal{H}_\alpha\}$ belong to the space of test functions $\mathcal{S}$. More importantly the functions in $\mathcal{S}$ can be characterized by the coefficients in their Hermite expansions. Denote

$$P_r^*(\phi) := \sum_{n=0}^\infty (n+1)^r \|\mathcal{H}_n * \phi\|_2 = \sum_{n=0}^\infty (n+1)^r \Big(\sum_{|\alpha|=n} |\langle \phi, \mathcal{H}_\alpha\rangle|^2\Big)^{1/2}, \quad r \ge 0. \tag{2.45}$$

Lemma 2.11. *We have*

$$\phi \in \mathcal{S} \iff |\langle \phi, \mathcal{H}_\alpha\rangle| \le c_k(|\alpha|+1)^{-k} \quad \textit{for all } \alpha \textit{ and all } k. \tag{2.46}$$

Moreover, the topology in $\mathcal{S}$ can be equivalently defined by the semi-norms P_r^* from above.

Proof. (a) Assume first that the right-hand side estimates in (2.46) hold. Applying repeatedly identities (2.5)-(2.6) one easily derives the estimate

$$\sup_x |x^\gamma D^\beta \mathcal{H}_\alpha(x)| \le c(|\alpha|+1)^{(|\gamma|+|\beta|)/2} \max_{|\omega| \le |\alpha|+|\beta|+|\gamma|} \|\mathcal{H}_\omega\|_\infty \tag{2.47}$$

for all indices β and γ, which implies $\phi \in \mathcal{S}$ using that $\phi = \sum_{n=0}^\infty \sum_{|\alpha|=n} \langle \phi, \mathcal{H}_\alpha\rangle \mathcal{H}_\alpha$ in L^2.

(b) Suppose $\phi \in \mathcal{S}$. Using (2.2) we have

$$\begin{aligned}\langle \phi, \mathcal{H}_\alpha\rangle &= \int_{\mathbb{R}^d} \mathcal{H}_\alpha(x)\phi(x)dx = \frac{1}{2|\alpha|+d} \int_{\mathbb{R}^d} (-\Delta + |x|^2)\mathcal{H}_\alpha(x)\phi(x)dx \\ &= \frac{1}{2|\alpha|+d} \int_{\mathbb{R}^d} \mathcal{H}_\alpha(x)(-\Delta\phi + |x|^2\phi(x))dx,\end{aligned}$$

where for the last equality we used integration by parts. Repeating the above procedure k times we obtain a representation for $\langle \phi, \mathcal{H}_\alpha\rangle$ of the form

$$\langle \phi, \mathcal{H}_\alpha\rangle = \frac{1}{(2|\alpha|+d)^k} \int_{\mathbb{R}^d} \mathcal{H}_\alpha(x) \sum_{|\beta|\le 2k,\, |\gamma|\le 2k} C_{\beta,\gamma} x^\gamma D^\beta \phi(x)dx \tag{2.48}$$

which yields the right-hand side estimates in (2.46).

The equivalence of the topologies in $\mathcal{S}$ induced by the semi-norms from (2.41) and (2.45) follows easily by (2.47) and (2.48). $\square$

3. Construction of building blocks (Needlets)

We utilize the localized kernels from Theorem 2.2 and the cubature formula from Corollary 2.8 to the construction of a pair of dual frames consisting of localized functions on $\mathbb{R}^d$.

Let $\widehat{a}$, $\widehat{b}$ satisfy the conditions:

$$\widehat{a}, \widehat{b} \in C^\infty(\mathbb{R}), \quad \operatorname{supp} \widehat{a}, \operatorname{supp} \widehat{b} \subset [1/4, 4], \tag{3.1}$$

$$|\widehat{a}(t)|, |\widehat{b}(t)| > c > 0 \quad \text{if } t \in [1/3, 3], \tag{3.2}$$

$$\overline{\widehat{a}(t)}\, \widehat{b}(t) + \overline{\widehat{a}(4t)}\, \widehat{b}(4t) = 1 \quad \text{if } t \in [1/4, 1]. \tag{3.3}$$

Consequently,

$$\sum_{\nu=0}^{\infty} \overline{\widehat{a}(4^{-\nu}t)}\, \widehat{b}(4^{-\nu}t) = 1, \quad t \in [1, \infty). \tag{3.4}$$

It is easy to see that (see e.g. [5]) if $\widehat{a}$ satisfies (3.1)-(3.2), then there exists $\widehat{b}$ satisfying (3.1)-(3.2) such that (3.3) holds true.

Assuming that $\widehat{a}$, $\widehat{b}$ satisfy (3.1)-(3.3), we define

$$\Phi_0 := \mathcal{H}_0, \quad \Phi_j := \sum_{\nu=0}^{\infty} \widehat{a}\Big(\frac{\nu}{4^{j-1}}\Big) \mathcal{H}_\nu, \quad j \ge 1, \quad \text{and} \tag{3.5}$$

$$\Psi_0 := \mathcal{H}_0, \quad \Psi_j := \sum_{\nu=0}^{\infty} \widehat{b}\Big(\frac{\nu}{4^{j-1}}\Big) \mathcal{H}_\nu, \quad j \ge 1. \tag{3.6}$$

Let $\mathcal{X}_j$ be the set of the nodes of cubature formula (2.32) from Corollary 2.8 and let λ_ξ be the coefficients of that cubature formula. We now define the jth level *needlets* by

$$\varphi_\xi(x) := \lambda_\xi^{1/2} \Phi_j(x, \xi) \quad \text{and} \quad \psi_\xi(x) := \lambda_\xi^{1/2} \Psi_j(x, \xi), \qquad \xi \in \mathcal{X}_j. \tag{3.7}$$

Write $\mathcal{X} := \cup_{j=0}^{\infty} \mathcal{X}_j$, where equal points from different levels $\mathcal{X}_j$ are considered as distinct elements of $\mathcal{X}$. We use $\mathcal{X}$ as an index set to define a pair of dual needlet systems Φ and Ψ by

$$\Phi := \{\varphi_\xi\}_{\xi \in \mathcal{X}}, \quad \Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}. \tag{3.8}$$

According to their further roles, we will call $\{\varphi_\xi\}$ *analysis needlets* and $\{\psi_\xi\}$ *synthesis needlets*.

The almost exponential localization of the needlets will be critical for our further development. Indeed, by (2.17) we have

$$|\Phi_j(\xi, x)|, |\Psi_j(\xi, x)| \le \frac{c_k 2^{jd}}{(1 + 2^j |x - \xi|)^k}, \quad x \in \mathbb{R}^d, \quad \forall k, \tag{3.9}$$

Fix $L > 0$. Then by (2.19) it follows that for any $k > 0$

$$|\Phi_j(\xi, x)|, |\Psi_j(\xi, x)| \le \frac{c_k 2^{-jL}}{(1 + 2^j |x - \xi|)^k}, \quad x \in \mathbb{R}^d, \text{ if } |\xi|_\infty > (1 + \delta) 2^{j+1}. \tag{3.10}$$

Here c_k depends on L and δ as well.

From above and (2.35)-(2.37) we infer

$$|\varphi_\xi(x)|, |\psi_\xi(x)| \le \frac{c_k 2^{jd/2}}{(1+2^j|x-\xi|)^k}, \quad \text{if } |\xi|_\infty \le (1+\delta)2^{j+1}, \tag{3.11}$$

and

$$|\varphi_\xi(x)|, |\psi_\xi(x)| \le \frac{c_k 2^{-jL}}{(1+2^j|x-\xi|)^k}, \quad \text{if } |\xi|_\infty > (1+\delta)2^{j+1}. \tag{3.12}$$

The following proposition provides a discrete decomposition of $\mathcal{S}'$ and $L^p(\mathbb{R}^d)$ via needlets.

Proposition 3.1. *(a) If $f \in \mathcal{S}'$, then*

$$f = \sum_{j=0}^{\infty} \Psi_j * \overline{\Phi}_j * f \quad \textit{in } \mathcal{S}' \textit{ and} \tag{3.13}$$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \varphi_\xi\rangle \psi_\xi \quad \textit{in } \mathcal{S}'. \tag{3.14}$$

(b) If $f \in L^p$, $1 \le p \le \infty$, then (3.13) $-$ (3.14) *hold in L^p. Moreover, if $1 < p < \infty$, then the convergence in* (3.13) $-$ (3.14) *is unconditional.*

Proof. (a) By the definition of Φ_j and Ψ_j in (3.5)-(3.6) it follows that $\Psi_0 * \overline{\Phi}_0 = \mathcal{H}_0$ and

$$\Psi_j * \overline{\Phi}_j(x,y) = \sum_{\nu=4^{j-2}}^{4^j} \overline{\widehat{a}\Big(\frac{\nu}{4^{j-1}}\Big)}\,\widehat{b}\Big(\frac{\nu}{4^{j-1}}\Big)\mathcal{H}_\nu(x,y), \quad j \ge 1.$$

Note that $\Psi_j(x,y)$ and $\Phi_j(x,y)$ are symmetric functions (e.g. $\Psi_j(y,x) = \Psi_j(x,y)$) since $\mathcal{H}_\nu(x,y)$ are symmetric and hence $\Psi_j * \overline{\Phi}_j(x,y)$ is well defined. Now, (3.4) and Lemma 2.11 yield (3.13).

To establish (3.14), we note that $\Psi_j(x,\cdot)$ and $\overline{\Phi_j(y,\cdot)}$ belong to V_{4^j} and applying the cubature formula from Corollary 2.8, we obtain

$$\begin{aligned}
\Psi_j * \overline{\Phi}_j(x,y) &= \int_{\mathbb{R}^d} \Psi_j(x,u)\overline{\Phi_j(y,u)}\,dy \\
&= \sum_{\xi\in\mathcal{X}_j} \lambda_\xi \Psi_j(x,\xi)\overline{\Phi_j(y,\xi)} = \sum_{\xi\in\mathcal{X}_j} \psi_\xi(x)\overline{\varphi_\xi(y)}.
\end{aligned}$$

Consequently,

$$\Psi_j * \overline{\Phi}_j * f = \sum_{\xi\in\mathcal{X}_j} \langle f, \varphi_\xi\rangle \psi_\xi.$$

This along with (3.13) implies (3.14).

(b) Representation (3.13) in L^p follows easily by the rapid decay of the kernels of the nth partial sums. We omit the details. Then (3.14) in L^p follows as above. The unconditional convergence in L^p, $1 < p < \infty$, follows by Proposition 4.3 and Theorem 4.5 below. $\square$

Remark 3.2. *It is well known that there exists a function $\widehat{a} \ge 0$ satisfying* (3.1) $-$ (3.2) *such that $\widehat{a}^2(t) + \widehat{a}^2(4t) = 1$, $t \in [1/4, 1]$. Suppose that in the above construction*

$\widehat{b} = \widehat{a}$ and $\widehat{a} \geq 0$. Then $\varphi_\xi = \psi_\xi$. Now (3.14) becomes $f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle\psi_\xi$. It is easy to see that this representation holds in L^2 and

$$\|f\|_{L^2} = \Big(\sum_{\xi\in\mathcal{X}} |\langle f, \psi_\xi\rangle|^2\Big)^{1/2}, \quad f \in L^2,$$

i.e. $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ is a tight frame for $L^2(\mathbb{R}^d)$.

4. Hermite-Triebel-Lizorkin spaces (F-spaces)

In this section we introduce the analogue of Triebel-Lizorkin spaces in the context of Hermite expansions following the general approach described in [17, §10.3] and show that they can be characterized via needlets. In our treatment of Hermite-Triebel-Lizorkin spaces we will utilize the scheme of Frazier and Jawerth from [5] (see also [6]).

4.1. **Definition of Hermite-Triebel-Lizorkin spaces.** Let the kernels $\{\Phi_j\}$ be defined by

$$\Phi_0 := \mathcal{H}_0 \quad \text{and} \quad \Phi_j := \sum_{\nu=0}^{\infty} \widehat{a}\Big(\frac{\nu}{4^{j-1}}\Big)\mathcal{H}_\nu, \quad j \geq 1, \tag{4.1}$$

where $\{\mathcal{H}_\nu\}$ are from (2.7) and $\widehat{a}$ obeys the conditions:

$$\widehat{a} \in C^\infty[0,\infty), \quad \operatorname{supp} \widehat{a} \subset [1/4, 4], \tag{4.2}$$

$$|\widehat{a}(t)| > c > 0, \quad \text{if } t \in [1/3, 3]. \tag{4.3}$$

Definition 4.1. *The Hermite-Triebel-Lizorkin space $F_p^{\alpha q} := F_p^{\alpha q}(H)$, where $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, is defined as the set of all $f \in \mathcal{S}'$ such that*

$$\|f\|_{F_p^{\alpha q}} := \Big\|\Big(\sum_{j=0}^{\infty}(2^{\alpha j}|\Phi_j * f(\cdot)|)^q\Big)^{1/q}\Big\|_p < \infty, \tag{4.4}$$

where the ℓ^q-norm is replaced by the sup norm when $q = \infty$.

As will be shown in Theorem 4.5, the above definition of Triebel-Lizorkin spaces is independent of the specific selection of $\widehat{a}$ satisfying (4.2)-(4.3) in the definition of Φ_j in (4.1).

Proposition 4.2. *The Hermite-Triebel-Lizorkin space $F_p^{\alpha q}$ is a quasi-Banach space which is continuously embedded in $\mathcal{S}'$ ($F_p^{\alpha q} \hookrightarrow \mathcal{S}'$).*

Proof. We will only establish that $F_p^{\alpha q} \hookrightarrow \mathcal{S}'$. Then the completeness of $F_p^{\alpha q}$ follows by a standard argument using in addition Fatou's lemma and Proposition 3.1.

As in Definition 4.1, let $\{\Phi_j\}$ be defined by a function $\widehat{a}$ obeying (4.2)-(4.3). As already indicated there exists a function $\widehat{b}$ such that (3.1)-(3.3) hold. Let $\{\Psi_j\}$ be defined as in (3.6) using this function. After this preparation, let $\{\varphi_\xi\}$ and $\{\psi_\xi\}$ be needlet systems defined as in (3.7)-(3.7) using these $\{\Phi_j\}$ and $\{\Psi_j\}$.

Let $f \in F_p^{\alpha q}$. By Proposition 3.1 $f = \sum_{j=0}^{\infty} \Psi_j * \overline{\Phi}_j * f$ in $\mathcal{S}'$ and hence

$$\langle f, \phi\rangle = \sum_{j=0}^{\infty}\langle \Psi_j * \overline{\Phi}_j * f, \phi\rangle = \sum_{j=0}^{\infty}\langle \overline{\Phi}_j * f, \overline{\Psi}_j * \phi\rangle, \quad \phi \in \mathcal{S}.$$

Applying the Cauchy-Schwarz inequality and (2.21) we obtain, for $j \ge 2$,

$$|\langle \overline{\Phi}_j * f, \overline{\Psi}_j * \phi\rangle| \le \|\Phi_j * f\|_2 \|\Psi_j * \phi\|_2 \le c2^{jd/p}\|\Phi_j * f\|_p \sum_{\nu=4^{j-2}}^{4^j} \|\mathcal{H}_j * \phi\|_2$$
$$\le c2^{-j}\|f\|_{F_p^{\alpha q}} P_r^*(\phi),$$

whenever $r \ge |\alpha| + d/p + 1$. This leads to $|\langle f, \phi\rangle| \le c\|f\|_{F_p^{\alpha q}} P_r^*(\phi)$, which yields the claimed embedding. $\Box$

Proposition 4.3. *We have the following identification:*

$$F_p^{02} \sim L^p, \quad 1 < p < \infty, \tag{4.5}$$

with equivalent norms.

The proof of this proposition can be carried out as the proof of Proposition 4.3 in [10] in the case of spherical harmonics and will be omitted. It employs the existing L^p multipliers for Hermite expansions (see e.g. [16]).

4.2. **Needlet decomposition of Hermite-Triebel-Lizorkin spaces.** In the following we will use the multilevel set $\mathcal{X} := \cup_{j=0}^{\infty}\mathcal{X}_j$ from §3 and the tiles $\{R_\xi\}$ introduced in (2.33).

Definition 4.4. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. The Hermite-Triebel-Lizorkin sequence space $f_p^{\alpha q}$ is defined as the set of all sequences of complex numbers $s = \{s_\xi\}_{\xi\in\mathcal{X}}$ such that*

$$\|s\|_{f_p^{\alpha q}} := \Big\|\Big(\sum_{j=0}^{\infty} 2^{j\alpha q}\sum_{\xi\in\mathcal{X}_j}\Big[|s_\xi||R_\xi|^{-1/2}\mathbb{1}_{R_\xi}(\cdot)\Big]^q\Big)^{1/q}\Big\|_p < \infty \tag{4.6}$$

with the usual modification when $q = \infty$.

Assuming that $\{\varphi_\xi\}$, $\{\psi_\xi\}$ is a dual pair of analysis and synthesis needlets (see (3.7)-(3.8)), we introduce the operators: $S_\varphi : f \to \{\langle f, \varphi_\xi\rangle\}_{\xi\in\mathcal{X}}$ (*Analysis operator*) and $T_\psi : \{s_\xi\}_{\xi\in\mathcal{X}} \to \sum_{\xi\in\mathcal{X}} s_\xi\psi_\xi$ (*Synthesis operator*).

We now come to our main result on Hermite-Triebel-Lizorkin spaces.

Theorem 4.5. *If $\alpha \in \mathbb{R}$ and $0 < p < \infty$, $0 < q \le \infty$, then the operators $S_\varphi : F_p^{\alpha q} \to f_p^{\alpha q}$ and $T_\psi : f_p^{\alpha q} \to F_p^{\alpha q}$ are bounded and $T_\varphi \circ S_\varphi = \mathrm{Id}$. Consequently, assuming that $f \in \mathcal{S}'$, we have $f \in F_p^{\alpha q}$ if and only if $\{\langle f, \varphi_\xi\rangle\} \in f_p^{\alpha q}$ and*

$$\|f\|_{F_p^{\alpha q}} \sim \|\{\langle f, \varphi_\xi\rangle\}\|_{f_p^{\alpha q}}. \tag{4.7}$$

Furthermore, the definition of $F_p^{\alpha q}$ is independent of the specific selection of $\widehat{a}$ satisfying (4.2) – (4.3).

For the proof of this theorem we adapt some techniques from [5].

Definition 4.6. *For any collection of complex numbers $\{a_\xi\}_{\xi\in\mathcal{X}_j}$, we define*

$$a_j^*(x) := \sum_{\eta\in\mathcal{X}_j}\frac{|a_\eta|}{(1 + 2^j|\eta - x|)^\sigma} \tag{4.8}$$

and

$$a_\xi^* := a_j^*(\xi), \quad \xi \in \mathcal{X}_j, \tag{4.9}$$

where $\sigma > d$ is sufficiently large and will be specified later on.

We will need a couple of lemmas whose proofs are given in §6.

Lemma 4.7. *Suppose $s > 0$ and $\sigma > d\max\{2, 1/s\}$. Let $\{b_\omega\}_{\omega\in\mathcal{X}_j}$, $j \ge 0$, be a set of complex numbers. Then*

$$b_j^*(x) \le c\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_j} |b_\omega|\mathbb{1}_{R_\omega}\Big)(x), \quad x \in \mathbb{R}^d. \tag{4.10}$$

Moreover, for $\xi \in \mathcal{X}_j$,

$$b_\xi^*\mathbb{1}_{R_\xi}(x) \le c\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_j} |b_\omega|\mathbb{1}_{R_\omega}\Big)(x), \quad x \in \mathbb{R}^d. \tag{4.11}$$

Here the constants depend only on d, δ, σ, and s.

Lemma 4.8. *Let $g \in V_{4^j}$ and denote*

$$M_\xi := \sup_{x\in R_\xi} |g(x)|, \quad \xi \in \mathcal{X}_j, \quad \textit{and} \quad m_\lambda := \inf_{x\in R_\lambda} |g(x)|, \quad \lambda \in \mathcal{X}_{j+\ell}.$$

Then there exists $\ell \ge 1$, depending only d, δ, and σ, such that for any $\xi \in \mathcal{X}_j$

$$M_\xi^* \le cm_\lambda^* \quad \textit{for all} \quad \lambda \in \mathcal{X}_{j+\ell},\ R_\lambda \cap R_\xi \ne \emptyset, \tag{4.12}$$

and hence

$$M_\xi^*\mathbb{1}_{R_\xi}(x) \le c \sum_{\lambda\in\mathcal{X}_{j+\ell}, R_\lambda\cap R_\xi\ne\emptyset} m_\lambda^*\mathbb{1}_{R_\lambda}(x), \quad x \in \mathbb{R}^d, \tag{4.13}$$

where $c > 0$ depends only on d, δ, and σ.

Proof of Theorem 4.5. Suppose $q < \infty$ (the case $q = \infty$ is easier) and pick s, σ, and k so that $0 < s < \min\{p, q\}$ and $k \ge \sigma > d\max\{1, 1/s\}$.

Let $\{\Phi_j\}$ be from the definition of Hermite-Triebel-Lizorkin spaces (see (4.1)-(4.3)). As already indicated in the beginning of §3, there exists a function $\widehat{b}$ satisfying (3.1)-(3.2) such that (3.3) holds as well. We use this function to define $\{\Psi_j\}$ exactly as in (3.6). We further use $\{\Phi_j\}$ and $\{\Psi_j\}$ to define just as in (3.7) a pair of dual needlet systems $\{\varphi_\eta\}$ and $\{\psi_\eta\}$.

Let $\{\widetilde{\varphi}_\eta\}$, $\{\widetilde{\psi}_\eta\}$ be a second pair of needlet systems, defined as in (3.5)-(3.7) from another pair of kernels $\{\widetilde{\Phi}_j\}$, $\{\widetilde{\Psi}_j\}$.

Our first step is to establish the boundedness of the operator $T_{\widetilde{\psi}} : f_p^{\alpha q} \to F_p^{\alpha q}$, defined by $T_{\widetilde{\psi}}s := \sum_{\xi\in\mathcal{X}} s_\xi\widetilde{\psi}_\xi$. Proposition 4.2 and the fact that finitely supported sequences are dense in $f_p^{\alpha q}$ imply that it suffices to prove the boundedness of $T_{\widetilde{\psi}}$ only for finitely supported sequence. So, assume $s = \{s_\xi\}_{\xi\in\mathcal{X}}$ is a finitely supported sequence and let $f := T_{\widetilde{\psi}}s$. Evidently $\Phi_j * \widetilde{\psi}_\xi = 0$ if $\xi \in \mathcal{X}_\nu$ and $|j - \nu| \ge 2$, and hence

$$\Phi_j * f = \sum_{\nu=j-1}^{j+1} \sum_{\xi\in\mathcal{X}_\nu} s_\xi\Phi_j * \widetilde{\psi}_\xi \qquad (\mathcal{X}_{-1} := \emptyset).$$

Let $\xi \in \mathcal{X}_\nu$, $j - 1 \le \nu \le j + 1$, and $|\xi|_\infty \le (1 + \delta)2^{\nu+1}$. Then using (3.9)-(3.11) we get

$$|\Phi_j * \widetilde{\psi}_\xi(x)| \le c2^{3jd/2} \int_{\mathbb{R}^d} \frac{1}{(1 + 2^j|x - y|)^k(1 + 2^j|\xi - y|)^k}\, dy \le \frac{c2^{jd/2}}{(1 + 2^j|\xi - x|)^k}.$$

Hence, on account of (2.36)

$$|\Phi_j * \widetilde{\psi}_\xi(x)| \le \frac{c|R_\xi|^{-1/2}}{(1+2^j|\xi - x|)^k}, \quad x \in \mathbb{R}^d. \tag{4.14}$$

If $\xi \in \mathcal{X}_\nu$, $j-1 \le \nu \le j+1$, and $|\xi|_\infty > (1+\delta)2^{\nu+1}$, then by (3.9)-(3.12)

$$|\Phi_j * \widetilde{\psi}_\xi(x)| \le c2^{-jL} \int_{\mathbb{R}^d} \frac{2^{jd}}{(1+2^j|x-y|)^k(1+2^j|\xi - y|)^k}\, dy \le \frac{c2^{-jL}}{(1+2^j|\xi - x|)^k}$$

for any $L > 0$. Consequently, in view of (2.37), estimate (4.14) holds again.

Denote $S_\xi := s_\xi |R_\xi|^{-1/2}$. Then by (4.14) we have

$$\begin{aligned} |\Phi_j * f(x)| &\le \sum_{\nu=j-1}^{j+1} \sum_{\xi \in \mathcal{X}_\nu} |s_\xi||\Phi_j * \widetilde{\psi}_\xi(x)| \le c \sum_{\nu=j-1}^{j+1} \sum_{\xi \in \mathcal{X}_\nu} \frac{|s_\xi||R_\xi|^{-1/2}}{(1+2^\nu|\xi - x|)^k} \\ &\le c \sum_{\nu=j-1}^{j+1} S_\nu^*(x) \qquad (S_{-1} := 0), \end{aligned} \tag{4.15}$$

where $S_\nu^*(x)$ is defined as in (4.8). We insert this in (4.4) and apply Lemma 4.7 and the maximal inequality (2.40) to obtain

$$\begin{aligned} \|f\|_{F_p^{\alpha q}} &\le \Big\| \Big(\sum_{j=0}^\infty (2^{j\alpha}|S_j^*(\cdot)|)^q \Big)^{1/q} \Big\|_p \\ &\le \Big\| \Big(\sum_{j=0}^\infty \Big[\mathcal{M}_s \Big(2^{j\alpha} \sum_{\xi \in \mathcal{X}_j} |s_\xi||R_\xi|^{-1/2} \mathbb{1}_{R_\xi} \Big) \Big]^q \Big)^{1/q} \Big\|_p \le c\|\{s_\eta\}\|_{f_p^{\alpha q}}. \end{aligned}$$

Hence the operator $T_{\widetilde{\psi}} : f_p^{\alpha q} \to F_p^{\alpha q}$ is bounded.

Assuming that the space $F_p^{\alpha q}$ is defined via $\{\overline{\Phi}_j\}$ instead of $\{\Phi_j\}$ we next prove the boundedness of the operator $S_\varphi : F_p^{\alpha q} \to f_p^{\alpha q}$. Let $f \in F_p^{\alpha q}$ and set

$$M_\xi := \sup_{x \in R_\xi} |\overline{\Phi}_j * f(x)|, \quad \xi \in \mathcal{X}_j, \quad \text{and} \quad m_\lambda := \inf_{x \in R_\lambda} |\overline{\Phi}_j * f(x)|, \quad \lambda \in \mathcal{X}_{j+\ell},$$

where ℓ is the constant from Lemma 4.8. We have

$$|\langle f, \varphi_\xi \rangle| \le c|R_\xi|^{1/2} |\overline{\Phi}_j * f(\xi)| \le c|R_\xi|^{1/2} M_\xi \le c|R_\xi|^{1/2} M_\xi^*.$$

By Lemma 2.10, $\overline{\Phi}_j * f \in V_{4^j}$, and applying Lemma 4.8 (see (4.13)), we have

$$M_\xi^* \mathbb{1}_{R_\xi}(x) \le c \sum_{\lambda \in \mathcal{X}_{j+\ell}, R_\lambda \cap R_\xi \ne \emptyset} m_\lambda^* \mathbb{1}_{R_\lambda}(x), \quad x \in \mathbb{R}^d.$$

We use the above, Lemma 4.7, and the maximal inequality (2.40) to obtain

$$\begin{aligned}
\|\{\langle f,\varphi_\xi\rangle\}\|_{f_p^{\alpha q}} &\le c\Big\|\Big(\sum_{j=0}^{\infty}2^{\alpha jq}\Big(\sum_{\xi\in\mathcal{X}_j}M_\xi^*\mathbb{1}_{R_\xi}\Big)^q\Big)^{1/q}\Big\|_p\\
&\le c\Big\|\Big(\sum_{j=0}^{\infty}2^{\alpha jq}\Big(\sum_{\lambda\in\mathcal{X}_{j+\ell}}m_\lambda^*\mathbb{1}_{R_\lambda}\Big)^q\Big)^{1/q}\Big\|_p\\
&\le c\Big\|\Big(\sum_{j=0}^{\infty}\mathcal{M}_s\Big(2^{\alpha j}\sum_{\lambda\in\mathcal{X}_{j+\ell}}m_\lambda\mathbb{1}_{R_\lambda}\Big)^q\Big)^{1/q}\Big\|_p\\
&\le c\Big\|\Big(\sum_{j=0}^{\infty}\Big(2^{\alpha j}\sum_{\lambda\in\mathcal{X}_{j+\ell}}m_\lambda\mathbb{1}_{R_\xi}\Big)^q\Big)^{1/q}\Big\|_p\\
&\le c\Big\|\Big(\sum_{j=0}^{\infty}2^{\alpha jq}|\overline{\Phi}_j*f|^q\Big)^{1/q}\Big\|_p=c\|f\|_{F_p^{\alpha q}}.
\end{aligned}$$

Here for the second inequality we used that each tile R_λ, $\lambda\in\mathcal{X}_{j+l}$, intersects no more that finitely many (depending only on d) tiles R_η, $\eta\in\mathcal{X}_j$. The above confirms the boundedness of the operator $S_\varphi: F_p^{\alpha q}\to f_p^{\alpha q}$.

The identity $T_\psi\circ S_\varphi=Id$ follows by Theorem 3.1.

We finally show the independence of the definition of Triebel-Lizorkin spaces from the specific selection of $\widehat{a}$ satisfying (4.2)-(4.3). Let $\{\Phi_j\}$, $\{\widetilde{\Phi}_j\}$ be two sequences of kernels as in the definition of Triebel-Lizorkin spaces defined by two different functions $\widehat{a}$ satisfying (4.2)-(4.3). As in the beginning of this proof, there exist two associated needlet systems $\{\Phi_j\}$, $\{\Psi_j\}$, $\{\varphi_\xi\}$, $\{\psi_\xi\}$ and $\{\widetilde{\Phi}_j\}$, $\{\widetilde{\Psi}_j\}$, $\{\widetilde{\varphi}_\xi\}$, $\{\widetilde{\psi}_\xi\}$. Denote by $\|f\|_{F_p^{\alpha q}(\Phi)}$ and $\|f\|_{F_p^{\alpha q}(\widetilde{\Phi})}$ the F-norms defined via $\{\Phi_j\}$ and $\{\widetilde{\Phi}_j\}$. Then from above it follows that

$$\|f\|_{F_p^{\alpha q}(\Phi)}\le c\|\{\langle f,\widetilde{\varphi}_\xi\rangle\}\|_{f_p^{\alpha q}}\le c\|f\|_{F_p^{\alpha q}(\overline{\widetilde{\Phi}})}.$$

The claimed independence of the definition of $F_p^{\alpha q}$ of the specific selection of $\widehat{a}$ in the definition of the functions $\{\Phi_j\}$ follows by interchanging the roles of $\{\Phi_j\}$ and $\{\widetilde{\Phi}_j\}$ and their complex conjugates. □

The Hermite-F-spaces embed in one another similarly as the classical F-spaces.

Proposition 4.9. *(a) If $0<p<\infty$, $0<q,q_1\le\infty$, $\alpha\in\mathbb{R}$ and $\varepsilon>0$, then*

$$F_p^{\alpha+\varepsilon,q}\hookrightarrow F_p^{\alpha q_1}. \tag{4.16}$$

(b) Let $0<p<p_1<\infty$, $0<q,q_1\le\infty$, and $-\infty<\alpha_1<\alpha<\infty$. Then we have the continuous embedding

$$F_p^{\alpha q}\hookrightarrow F_{p_1}^{\alpha_1 q_1}\quad \text{if}\quad \alpha-d/p=\alpha_1-d/p_1. \tag{4.17}$$

The proof of this embedding result uses estimate (2.21) and Theorem 4.5 and can be carried out exactly as in the classical case on $\mathbb{R}^n$ (see e.g. [17], p. 47 and p. 129). We omit it.

4.3. **Comparison of Hermite-F-spaces with classical F-spaces.**

We next use needlet decompositions to show that the Hemite-Triebel-Lizorkin spaces of essentially positive smoothness are different from the corresponding classical Triebel-Lizorkin spaces on $\mathbb{R}^d$.

Theorem 4.10. *Let $0 < p < \infty$, $0 < q \le \infty$, and $\alpha > d(1/p-1)_+$. Then there exists a function $f \in F_p^{\alpha q}$ such that $\|f\|_{F_p^{\alpha q}(H)} = \infty$ and hence $f \notin F_p^{\alpha q}(H)$. Here $F_p^{\alpha q}$ and $F_p^{\alpha q}(H)$ are the respective classical and Hermite Triebel-Lizorkin spaces.*

Proof. For any $y \in \mathbb{R}^d$ and a function f we define

$$\|f\|_{F_y^*} := \Big\|\Big(\sum_{j=0}^{\infty} 2^{j\alpha q} \sum_{\xi\in\mathcal{X}_j,\, |\xi-y|>|y|/2} \Big(|R_\xi|^{-1/2}|\langle f,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(\cdot)\Big)^q\Big)^{1/q}\Big\|_p. \tag{4.18}$$

Choose a function $h \in C^\infty(\mathbb{R}^d)$ such that $\|h\|_\infty = 1$ and $\operatorname{supp} h \subset B(0,1)$, where $B(0,1) := \{x \in \mathbb{R}^d : |x| < 1\}$.

Theorem 4.10 will follow easily by the following lemma whose proof is given in §6.2.

Lemma 4.11. *With the notation from above, we have*

$$\|h(\cdot - y)\|_{F_p^{\alpha q}(H)} \to \infty \quad \textit{as} \quad |y| \to \infty, \quad \textit{and} \tag{4.19}$$

$$\|h(\cdot - y)\|_{F_y^*} \to 0 \quad \textit{as} \quad |y| \to \infty. \tag{4.20}$$

By this lemma it follows that there exists a sequence $\{y_j\}_{j\ge1} \subset \mathbb{R}^d$ such that $0 < |y_1| < |y_2| < \dots$ and $|y_{j+1}| > 3|y_j|$, $\|h(\cdot - y_j)\|_{F_q^{\alpha p}(H)} > 2^{2j}$, and $\|h(\cdot - y_j)\|_{F_{y_j}^*} < 1$, $j = 1, 2, \dots$.

We now define $f(x) := \sum_{j=1}^\infty f_j(x)$, where $f_j(x) := 2^{-j}h(x-y_j)$, $x \in \mathbb{R}^d$. Set $\tau := \min\{p,q,1\}$. Evidently, h belongs to all classical Triebel-Lizorkin spaces, which are shift invariant, and hence

$$\|f\|_{F_q^{\alpha p}}^\tau \le \sum_{j=1}^\infty 2^{-j\tau}\|h(\cdot - y_j)\|_{F_q^{\alpha p}}^\tau = \|h\|_{F_q^{\alpha p}}^\tau \sum_{j=1}^\infty 2^{-j\tau} \le c\|h\|_{F_q^{\alpha p}}^\tau < \infty.$$

Here we use that $\|\sum_j g_j\|_{F_q^{\alpha p}}^\tau \le \sum_j \|g_j\|_{F_q^{\alpha p}}^\tau$. Thus $f \in F_q^{\alpha p}$.

On the other hand, for any $\ell \ge 1$,

$$\begin{aligned}
\|f\|_{F_q^{\alpha p}(H)}^\tau &\ge c\Big\|\Big(\sum_{\nu=0}^\infty 2^{\nu\alpha q} \sum_{\xi\in\mathcal{X}_\nu,\, |\xi-y_\ell|\le|y_\ell|/2} \Big(|R_\xi|^{-1/2}|\langle f,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(\cdot)\Big)^q\Big)^{1/q}\Big\|_p^\tau \\
&\ge c\Big(\|f_\ell\|_{F_q^{\alpha p}(H)}^\tau - \sum_{j=1}^\infty \|f_j\|_{F_{y_j}^*}^\tau\Big) \\
&= c2^{-\ell\tau}\|h(\cdot - y_\ell)\|_{F_q^{\alpha p}(H)}^\tau - c\sum_{j=1}^\infty 2^{-j\tau}\|h(\cdot - y_j)\|_{F_{y_j}^*}^\tau \\
&> c2^{\ell\tau} - c\sum_{j=1}^\infty 2^{-j\tau} \ge c2^{\ell\tau} - c'.
\end{aligned}$$

Here for the second inequality we used that if $|\xi - y_\ell| \le |y_\ell|/2$, then $|\xi - y_j| > |y_j|/2$ for all $j \ne \ell$. Consequently, $\|f\|_{F_q^{\alpha p}(H)}^\tau = \infty$. $\square$

5. Hermite-Besov spaces (B-spaces)

Besov type spaces are natural to introduce in the context of Hermite expansions (see e.g. [17, §10.3]). We will call them Hermite-Besov spaces. To characterize these space via needlets we use the approach of Frazier and Jawerth [4] (see also

[6]) to the classical Besov spaces. We refer to [11, 17] as general references for Besov spaces.

5.1. **Definition of Hermite-Besov spaces.**

Definition 5.1. *Let the kernels* $\{\Phi_j\}$ *be defined by* (4.1) *with* $\widehat{a}$ *satisfying* (4.2) – (4.3). *The Hermite-Besov space* $B_p^{\alpha q} := B_p^{\alpha q}(H)$, *where* $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, *is defined as the set of all* $f \in \mathcal{S}'$ *such that*

$$\|f\|_{B_p^{\alpha q}} := \Big(\sum_{j=0}^{\infty}\Big(2^{\alpha j}\|\Phi_j * f\|_p\Big)^q\Big)^{1/q} < \infty, \tag{5.1}$$

where the ℓ^q*-norm is replaced by the sup-norm if* $q = \infty$.

Similarly as for Hermite-Triebel-Lizorkin spaces (§4) Theorem 5.3 below implies that the above definition of Hermite-Besov spaces is independent of the specific selection of $\widehat{a}$; also $B_p^{\alpha q}$ is a quasi-Banach space which is continuously embedded in $\mathcal{S}'$.

5.2. **Needlet decomposition of Hermite-Besov spaces.** As for the Hermite-Triebel-Lizorkin spaces we employ the tiles $\{R_\xi\}$ introduced in (2.33) in the following. Also as before $\mathcal{X} := \cup_{j=0}^{\infty}\mathcal{X}_j$.

Definition 5.2. *The Hermite-Besov sequence space* $b_p^{\alpha q}$, *where* $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, *is defined as the set of all sequences of complex numbers* $s = \{s_\xi\}_{\xi\in\mathcal{X}}$ *such that*

$$\|s\|_{b_p^{\alpha q}} := \Big(\sum_{j=0}^{\infty}\Big[2^{j\alpha}\Big(\sum_{\xi\in\mathcal{X}_j}|R_\xi|^{1-p/2}|s_\xi|^p\Big)^{1/p}\Big]^q\Big)^{1/q} < \infty \tag{5.2}$$

with obvious modifications when $p = \infty$ *or* $q = \infty$.

In the following, we assume that $\{\Phi_j\}$, $\{\Psi_j\}$, $\{\varphi_\xi\}$, $\{\psi_\xi\}$ is a needlet system defined by (3.5)-(3.8). Recall the *analysis operator:* $S_\varphi : f \to \{\langle f, \varphi_\xi\rangle\}_{\xi\in\mathcal{X}}$, and the *synthesis operator:* $T_\psi : \{s_\xi\}_{\xi\in\mathcal{X}} \to \sum_{\xi\in\mathcal{X}} s_\xi\psi_\xi$.

Theorem 5.3. *If* $\alpha \in \mathbb{R}$ *and* $0 < p, q \le \infty$, *then the operators* $S_\varphi : B_p^{\alpha q} \to b_p^{\alpha q}$ *and* $T_\psi : b_p^{\alpha q} \to B_p^{\alpha q}$ *are bounded and* $T_\varphi \circ S_\varphi = \mathrm{Id}$. *Consequently, assuming that* $f \in \mathcal{S}'$, *we have* $f \in B_p^{\alpha q}$ *if and only if* $\{\langle f, \varphi_\xi\rangle\} \in b_p^{\alpha q}$ *and*

$$\|f\|_{B_p^{\alpha q}} \sim \|\{\langle f, \varphi_\xi\rangle\}\|_{b_p^{\alpha q}}. \tag{5.3}$$

Furthermore, the definition of $B_p^{\alpha q}$ *is independent of the choice of* $\widehat{a}$ *satisfying* (4.2) – (4.3).

For the proof of this theorem we need one additional lemma.

Lemma 5.4. *For any* $g \in V_{4^j}$, $j \ge 0$, *and* $0 < p \le \infty$

$$\Big(\sum_{\xi\in\mathcal{X}_j}|R_\xi|\max_{x\in R_\xi}|g(x)|^p\Big)^{1/p} \le c\|g\|_p. \tag{5.4}$$

The proof of this lemma is given in §6.

Proof of Theorem 5.3. Let $0 < s < p$ and $\sigma > d\max\{1, 1/s\}$. Just as in the proof of Theorem 4.5 we assume that $\{\Phi_j\}$, $\{\Psi_j\}$, $\{\varphi_\eta\}$, $\{\psi_\eta\}$ and $\{\widetilde{\Phi}_j\}$, $\{\widetilde{\Psi}_j\}$, $\{\widetilde{\varphi}_\eta\}$, $\{\widetilde{\psi}_\eta\}$ are two needlet systems, defined as in (3.5)-(3.7), which originate from two completely different functions $\widehat{a}$ satisfying (4.2)-(4.3).

We first prove the boundedness of the operator $T_{\widetilde{\psi}}: b_p^{\alpha q} \to B_p^{\alpha q}$, defined by $T_{\widetilde{\psi}} s := \sum_{\xi\in\mathcal{X}} s_\xi \widetilde{\psi}_\xi$, assuming that $B_p^{\alpha q}$ is defined by $\{\Phi_j\}$. As in the Triebel-Lizorkin case due to the embedding $B_p^{\alpha q} \hookrightarrow \mathcal{S}'$ it suffices to consider only the case of a finitely supported sequence $s = \{s_\xi\}_{\xi\in\mathcal{X}}$. Let $f := T_{\widetilde{\psi}} s$. By (4.15) and Lemma 4.7 we get

$$\begin{aligned}\|\Phi_j * f\|_p &\le c \sum_{\nu=j-1}^{j+1} \Big\|\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_\nu} |R_\omega|^{-1/2}|s_\omega|\mathbb{1}_{R_\omega}\Big)\Big\|_p \\ &\le c \sum_{\nu=j-1}^{j+1} \Big(\sum_{\omega\in\mathcal{X}_\nu} |R_\omega|^{1-p/2}|s_\omega|^p\Big)^{1/p} \quad (\mathcal{X}_{-1} := \emptyset),\end{aligned}$$

which leads to $\|f\|_{B_p^{\alpha q}} \le c\|\{s_\eta\}\|_{b_p^{\alpha q}}$ and hence to the boundedness of $T_{\widetilde{\psi}}$.

To prove the boundedness of the operator $S_\varphi : B_p^{\alpha q} \to b_p^{\alpha q}$ we assume that $B_p^{\alpha q}$ is defined in terms of $\{\overline{\Phi}_j\}$. Observing that

$$|\langle f, \varphi_\xi\rangle| = \lambda_\xi^{1/2}|\overline{\Phi}_j * f(\xi)| \sim |R_\xi|^{1/2}|\overline{\Phi}_j * f(\xi)|, \quad \xi \in \mathcal{X}_j,$$

and $\overline{\Phi}_j * f \in V_{4^j}$, we get using Lemma 5.4

$$\Big(\sum_{\xi\in\mathcal{X}_j} |R_\xi|^{1-p/2}|\langle f,\varphi_\xi\rangle|^p\Big)^{1/p} \le c\Big(\sum_{\xi\in\mathcal{X}_j} |R_\xi||\overline{\Phi}_j * f(\xi)|^p\Big)^{1/p} \le c\|\overline{\Phi}_j * f\|_p.$$

This yields $\|\{\langle f,\varphi_\xi\rangle\}\|_{b_p^{\alpha q}} \le c\|f\|_{B_p^{\alpha q}}$ and hence the operator S_φ is bounded.

The identity $T_\psi \circ S_\varphi = Id$ is a consequence of Proposition 3.1.

The independence of the definition of $B_p^{\alpha q}$ from the particular selection of $\widehat{a}$ follows from above exactly as in the case of Triebel-lizorkin spaces (see the proof of Theorem 4.5). $\square$

The Hermite-Besov spaces embed similarly as the classical Besov spaces.

Proposition 5.5. *(a) If $0 < p, q, q_1 \le \infty$, $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, then*

$$B_p^{\alpha+\varepsilon,q} \hookrightarrow B_p^{\alpha q_1}. \tag{5.5}$$

(b) Let $0 < p < p_1 < \infty$, $0 < q \le \infty$, and $-\infty < \alpha_1 < \alpha < \infty$. Then we have the continuous embedding

$$B_p^{\alpha q} \hookrightarrow B_{p_1}^{\alpha_1 q} \quad \text{if} \quad \alpha - d/p = \alpha_1 - d/p_1. \tag{5.6}$$

(c) If $0 < p < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$, then

$$B_p^{\alpha,\min\{p,q\}} \hookrightarrow F_p^{\alpha q} \hookrightarrow B_p^{\alpha,\max\{p,q\}}. \tag{5.7}$$

Part (b) of this proposition follows readily by estimate (2.21). The proofs of parts (a) and (c) are as in the classical case.

We now show that under some restriction on the indices the Hemite-Besov spaces are essentially different from the classical Besov spaces on $\mathbb{R}^d$.

Theorem 5.6. *Let $0 < p, q < \infty$, and $\alpha > d(1/p-1)_+$. Then there exists a function $f \in B_p^{\alpha q}$ such that $\|f\|_{B_p^{\alpha q}(H)} = \infty$ and then $f \notin B_p^{\alpha q}(H)$. Here $B_p^{\alpha q}$ and $B_p^{\alpha q}(H)$ are the respective classical and Hermite Besov spaces.*

Proof. We proceed quite similarly as in the proof of Theorem 4.10. Given $y \in \mathbb{R}^d$ and a function f we define

$$\|f\|_{B^*_y} := \Big(\sum_{j=0}^{\infty} 2^{j\alpha q} \Big(\sum_{\xi \in \mathcal{X}_j,\, |\xi - y| > |y|/2} |R_\xi|^{1-p/2} |\langle f, \varphi_\xi \rangle|^p \Big)^{q/p} \Big)^{1/q}. \tag{5.8}$$

Pick $h \in C^\infty(\mathbb{R}^d)$ such that $\|h\|_\infty = 1$ and $\operatorname{supp} h \subset B(0,1)$.

The theorem follows easily by the following:

$$\|h(\cdot - y)\|_{B^{\alpha q}_p(H)} \to \infty \quad \text{as} \quad |y| \to \infty, \quad \text{and} \tag{5.9}$$

$$\|h(\cdot - y)\|_{B^*_y} \to 0 \quad \text{as} \quad |y| \to \infty. \tag{5.10}$$

To prove (5.9) we will show that there exist $\varepsilon > 0$ and $r > 1$ such that

$$B^{\alpha q}_p(H) \hookrightarrow F^{\varepsilon 2}_r(H). \tag{5.11}$$

Then the result follows by the argument from the proof of Lemma 4.11.

Let $p > 1$. Pick $\varepsilon > 0$ so that $\alpha > 2\varepsilon$. Then by Propositions 4.9, 5.5 we have the following embeddings $B^{\alpha q}_p \hookrightarrow B^{2\varepsilon, p}_p \hookrightarrow F^{2\varepsilon, p}_p \hookrightarrow F^{\varepsilon 2}_p$ which confirms (5.11).

Let $p \le 1$. Then $\alpha > d(1 - 1/p)$ and hence (as in the proof of Lemma 4.11) there exist $\varepsilon, \delta > 0$ such that $\alpha - d/p = 3\varepsilon - d/(1+\delta)$. Then Propositions 4.9, 5.5 give us the following embeddings $B^{\alpha q}_p \hookrightarrow B^{3\varepsilon, q}_{1+\delta} \hookrightarrow B^{2\varepsilon, 1+\delta}_{1+\delta} \hookrightarrow F^{2\varepsilon, 1+\delta}_{1+\delta} \hookrightarrow F^{\varepsilon 2}_{1+\delta}$, which leads again to (5.11).

The proof of (5.10) is similar to the proof of (4.20) and will be omitted. $\square$

We finally want to link the Hermite-Besov spaces with the L^p-approximation from linear combinations of Hermite functions. Denote by $E_n(f)_p$ the best approximation of $f \in L^p$ from V_n, i.e.

$$E_n(f)_p := \inf_{g \in V_n} \|f - g\|_p. \tag{5.12}$$

Let $A^{\alpha q}_p$ be the approximation space of all functions $f \in L^p$ for which

$$\|f\|_{A^{\alpha q}_p} := \|f\|_p + \Big(\sum_{j=0}^{\infty} (2^{\alpha j} E_{2^j}(f)_p)^q \Big)^{1/q} < \infty \tag{5.13}$$

with the usual modification when $q = \infty$.

Proposition 5.7. *If $\alpha > 0$, $1 \le p \le \infty$, $0 < q \le \infty$, then $B^{\alpha q}_p = A^{\alpha/2, q}_p$ with equivalent norms.*

Proof. Let $f \in B^{\alpha q}_p$. We first observe that under the conditions on α, p, and q, $B^{\alpha q}_p$ is continuously imbedded in L^p, i.e. f can be identified as a function in L^p and $\|f\|_p \le c\|f\|_{B^{\alpha q}_p}$. The proof of this is easy and standard and will be omitted.

By a well known and easy construction there exists a function $\widehat{a} \ge 0$ satisfying (4.2)-(4.3) such that $\widehat{a}(t) + \widehat{a}(4t) = 1$ for $t \in [1/4, 1]$ and hence $\sum_{\nu=0}^{\infty} \widehat{a}(4^{-\nu} t) = 1$ for $t \in [1, \infty)$. Assume that $\{\Phi_j\}$ are defined by (4.1) using this function $\widehat{a}$. By Theorem 5.3 the definition of the Besov spaces $B^{\alpha q}_p$ is independent of the selection of $\widehat{a}$ and hence they can be defined via these functions $\{\Phi_j\}$. Similarly as in Proposition 3.1 $f = \sum_{j=0}^{\infty} \Phi_j * f$ for $f \in L^p$.

Now, using that $\Phi_j * f \in V_{4^j}$, we have $E_{4^m}(f)_p \le \sum_{j=m+1}^{\infty} \|\Phi_j * f\|_p$, $m \ge 0$. A standard argument employing this leads to the estimate $\|f\|^A_{A^{\alpha/2,q}_p} \le c\|f\|_{B^{\alpha q}_p}$.

To prove the estimate in the other direction, let $g \in V_{4^{j-2}}$ ($j \ge 2$). Evidently, $\Phi_j * f = \Phi_j * (f - g)$ and the rapid decay of Φ_j yields $\|\Phi_j * f\|_p \le c\|f - g\|_p$.

Consequently, $\|\Phi_j * f\|_p \le cE_{4^{j-2}}(f)_p$, $j \ge 2$, and $\|\Phi_j * f\|_p \le c\|f\|_p$. These lead to $\|f\|_{B_p^{\alpha q}} \le c\|f\|^A_{A_p^{\alpha/2,q}}$. $\square$

6. Proofs

6.1. Proofs for Section 2.1.

Proof of Theorem 2.2. This proof hinges on an important lemma from [16]. Let ψ be a univariate function. The forward differences of ψ are defined by

$$\Delta\psi(t) = \psi(t+1) - \psi(t) \quad \text{and} \quad \Delta^k\psi = \Delta(\Delta^{k-1}\psi), \quad k \ge 2.$$

For a given function ψ we define

$$M_\psi(x,y) := \sum_{\nu=0}^{\infty} \psi(\nu)\mathcal{H}_\nu(x,y), \quad \text{and then} \quad M_{\Delta^k\psi} = \sum_{\nu=0}^{\infty} \Delta^k\psi(\nu)\mathcal{H}_\nu.$$

Lemma 6.1. [16, p. 72] *Let $A_j^{(x)}$ and $A_j^{(y)}$ denote the operator A_j applied to the x and y variables, respectively. Then for any $k \ge 1$,*

$$2^k(x_j - y_j)^k M_\psi(x,y) = \sum_{k/2 \le l \le k} c_{l,k}\left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} M_{\Delta^l\psi}(x,y), \tag{6.1}$$

where $c_{l,k}$ are constants given by

$$c_{l,k} = (-1)^{k-l}4^{k-l}(2k-2l-1)!!\binom{k}{2l-k}.$$

Proof. This lemma is proved in [16] except that the constants $c_{l,k}$ are not determined explicitly there. For $k = 1$ one has [16, (3.2.20)]

$$2(x_j - y_j)M_\psi(x,y) = \left(A_j^{(y)} - A_j^{(x)}\right) M_{\Delta\psi}(x,y). \tag{6.2}$$

The general result is obtained by induction using the identity [16, (3.2.23)]

$$(x_j - y_j)\left(A_j^{(y)} - A_j^{(x)}\right)^r - \left(A_j^{(y)} - A_j^{(x)}\right)^r (x_j - y_j) = -2r\left(A_j^{(y)} - A_j^{(x)}\right)^{r-1}. \tag{6.3}$$

Assume that (6.1) holds for some $k \ge 0$. Then using (6.3) we get

$$\begin{aligned}
2^{k+1}(x_j - y_j)^{k+1}M_\psi &= 2(x_j - y_j)\sum_{k/2 \le l \le k} c_{l,k}\left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} M_{\Delta^l\psi} \\
&= \sum_{k/2 \le l \le k} c_{l,k}\left[\left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} 2(x_j - y_j)M_{\Delta^l\psi}\right. \\
&\qquad \left. -4(2l-k)\left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k-1} M_{\Delta^l\psi}\right] \\
&= c_{k,k}\left(A_j^{(y)} - A_j^{(x)}\right)^{k+1} M_{\Delta^{k+1}\psi} \\
&\qquad + \sum_{(k+1)/2 \le l \le k} [c_{l-1,k} - 4(2l-k)c_{l,k}]\left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k-1} M_{\Delta^l\psi},
\end{aligned}$$

where $c_{l,k} := 0$ if $l < k/2$. Consequently, the coefficients satisfy the recurrence relations

$$c_{k+1,k+1} = c_{k,k}, \qquad c_{l,k+1} = c_{l-1,k} - 4(2l-k)c_{l,k}, \quad (k+1)/2 \le l \le k.$$

It follows from this and (6.2) that $c_{k,k} = 1$ for all k. Furthermore, the recurrence relation above shows that

$$c_{k-j,k} = -4 \sum_{\nu=2j-1}^{k-1} (\nu - 2j + 2) c_{\nu-j+1,\nu},$$

from which one uses induction and the fact that $\sum_{\nu=j}^{k-1} \binom{\nu}{j} = \binom{k}{j+1}$ to derive the stated identity for $c_{l,k}$. $\square$

The case $\boldsymbol{\alpha = (0, \ldots, 0)}$. Assume $k \geq 2$. By Lemma 6.1, we have

$$2^k (x_j - y_j)^k \Lambda_n(x,y) = \sum_{k/2 \leq l \leq k} c_{l,k} \sum_{\nu=0}^{\infty} \Delta^l \widehat{a}\Big(\frac{\nu}{n}\Big) \left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} \mathcal{H}_\nu(x,y),$$

where $\Delta^l \widehat{a}(\frac{\nu}{n})$ is the lth forward difference applied with respect to ν.

Note first that applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big| \sum_{|\alpha|=n} \mathcal{H}_{\alpha+\beta}(x) \mathcal{H}_{\alpha+\gamma}(y) \Big|^2 &\leq \sum_{|\alpha|=n} |\mathcal{H}_{\alpha+\beta}(x)|^2 \sum_{|\alpha|=n} |\mathcal{H}_{\alpha+\gamma}(y)|^2 \\ &\leq \mathcal{H}_{n+|\beta|}(x,x) \mathcal{H}_{n+|\gamma|}(y,y). \end{aligned} \tag{6.4}$$

We next consider the action of A_j on $\mathcal{H}_j(x,y)$. Using repeatedly (2.4) we get

$$\left(A_j^{(x)}\right)^m \mathcal{H}_\beta = \prod_{r=0}^{m} [2(\beta_j + r) + 2]^{1/2} \mathcal{H}_{\beta + m e_j}. \tag{6.5}$$

This along with (6.4) leads to

$$\begin{aligned} |(A_j^{(y)})^i (A_j^{(x)})^m \mathcal{H}_\nu(x,y)|^2 &= \Big| \sum_{|\alpha|=\nu} \prod_{r=0}^{i-1} [2(\alpha_j + r) + 2]^{1/2} \prod_{r=0}^{m-1} [2(\alpha_j + r) + 2]^{1/2} \\ &\qquad \times \mathcal{H}_{\alpha + i e_j}(y) \mathcal{H}_{\alpha + m e_j}(x) \Big|^2 \\ &\leq [2(\nu + i + m)]^{i+m} \left[\sum_{|\alpha|=\nu} \left| \mathcal{H}_{\alpha + i e_j}(y) \mathcal{H}_{\alpha + m e_j}(x) \right| \right]^2 \\ &\leq [2(\nu + i + m)]^{i+m} \mathcal{H}_{\nu+i}(y,y) \mathcal{H}_{\nu+m}(x,x). \end{aligned}$$

Hence the binomial theorem and the Cauchy-Schwarz inequality give

$$\begin{aligned} \Big| \left(A_j^{(y)} - A_j^{(x)}\right)^{2l-k} \mathcal{H}_\nu(x,y) \Big| &\leq \sum_{i=0}^{2l-k} \binom{2l-k}{i} \Big| (A_j^{(y)})^i (A_j^{(x)})^{2l-k-i} \mathcal{H}_\nu(x,y) \Big| \\ &\leq (2\nu + 4l - 2k)^{(2l-k)/2} \sum_{i=0}^{2l-k} \binom{2l-k}{i} [\mathcal{H}_{\nu+i}(y,y)]^{\frac12} [\mathcal{H}_{\nu+2l-k-i}(x,x)]^{\frac12} \\ &\leq c \nu^{(2l-k)/2} \left[\sum_{i=0}^{2l-k} \mathcal{H}_{\nu+i}(y,y) \right]^{\frac12} \left[\sum_{i=0}^{2l-k} \mathcal{H}_{\nu+i}(x,x) \right]^{\frac12}. \end{aligned}$$

A well known property of the difference operator gives

$$\Big| \Delta^l \widehat{a}\Big(\frac{\nu}{n}\Big) \Big| = n^{-l} |\widehat{a}^{(l)}(\xi)| \leq n^{-l} \|\widehat{a}^{(l)}\|_\infty. \tag{6.6}$$

By Definition 2.1 it follow that $\Delta^l \widehat{a}(\frac{\nu}{n}) = 0$ if $0 \le \nu \le un - l$ or $\nu \ge n + vn$, where $0 < u \le 1$ and $v > 0$. (Here $u = 1$ if $\widehat{a}$ is of type (a).) Using this, the above estimates, and the Cauchy-Schwarz inequality we infer

$$\begin{aligned}\left|(x_j - y_j)^k \Lambda_n(x,y)\right| &\le c \sum_{k/2 \le l \le k} |c_{l,k}| n^{-l} \|\widehat{a}^{(l)}\|_\infty n^{(2l-k)/2} \\ &\qquad \times \sum_{\nu=[un]-l}^{n+[vn]} \left[\sum_{i=0}^{2l-k} \mathcal{H}_{\nu+i}(y,y)\right]^{\frac12} \left[\sum_{i=0}^{2l-k} \mathcal{H}_{\nu+i}(x,x)\right]^{\frac12} \\ &\le c_k n^{-k/2} \left[K_{n+[vn]+k}(y,y)\right]^{\frac12} \left[K_{n+[vn]+k}(x,x)\right]^{\frac12},\end{aligned}$$

where $c_k > 0$ is of the form $c_k = c(k,u,d)\max_{0\le l\le k} \|\widehat{a}^{(l)}\|_\infty$. Consequently,

$$|\Lambda_n(x,y)| \le c_k \frac{\left[K_{n+[vn]+k}(y,y)\right]^{\frac12} \left[K_{n+[vn]+k}(x,x)\right]^{\frac12}}{(\sqrt{n}|x-y|)^k}, \qquad x \ne y. \tag{6.7}$$

We also need estimate $|\Lambda_n(x,y)|$ whenever x and y are close to one another. Applying the Cauchy-Schwarz inequality to the sum in (2.7) which defines $\mathcal{H}_\nu$ we get

$$\begin{aligned}|\Lambda_n(x,y)| &\le \sum_{\nu=0}^{n+[vn]} \left|\widehat{a}\left(\frac{\nu}{n}\right)\right| \mathcal{H}_\nu(x,x)^{1/2}\mathcal{H}_\nu(y,y)^{1/2} \\ &\le c\|\widehat{a}\|_\infty \left[K_{n+[vn]}(y,y)\right]^{\frac12} \left[K_{n+[vn]}(x,x)\right]^{\frac12},\end{aligned}$$

which coupled with (6.7) yields (2.9) in the case under consideration.

The case $|\boldsymbol{\alpha}| > 0$. We will make use of the relation $\partial_j = x_j - A_j$, where as usual $\partial_j := \frac{\partial}{\partial x_j}$. In the following we again denote by $A_j^{(x)}$ the operator A_j acting on the x variables, and A_j^0 is understood as the identity operator; by definition $(A^{(x)})^\alpha := (A_1^{(x)})^{\alpha_1} \dots (A_d^{(x)})^{\alpha_d}$. We also identify the operator of multiplication by x_j with x_j. In order to use Lemma 6.1, we will need two commuting relations.

Lemma 6.2. *Let k, r, s be nonnegative integers. Then*

$$x_j^r \left(A_j^{(x)} - A_j^{(y)}\right)^k = \sum_{i=0}^{r} \binom{r}{i} \frac{k!}{(k-i)!} \left(A_j^{(x)} - A_j^{(y)}\right)^{k-i} x_j^{r-i} \tag{6.8}$$

and

$$(x_j - y_j)^k \left(A_j^{(x)}\right)^s = \sum_{i=0}^{s} \binom{s}{i} \frac{k!}{(k-i)!} \left(A_j^{(x)}\right)^{s-i} (x_j - y_j)^{k-i}, \tag{6.9}$$

where $k!/(k-i)! := 0$ if $k < i$.

Proof. To prove (6.8) we start from the identity:

$$x_j \left(A_j^{(x)} - A_j^{(y)}\right)^k = k\left(A_j^{(x)} - A_j^{(y)}\right)^{k-1} + \left(A_j^{(x)} - A_j^{(y)}\right)^k x_j. \tag{6.10}$$

For $k = 1$ this follows from the obvious identities

$$x_j A_j^{(x)} = Id + A_j^{(x)} x_j \quad \text{and} \quad x_j A_j^{(y)} = A_j^{(y)} x_j. \tag{6.11}$$

In general it follows readily by induction.

We now proceed by induction on r. Suppose (6.8) holds for some $r \geq 1$ and all $k \geq 1$. Then

$$x_j^{r+1}\left(A_j^{(x)}-A_j^{(y)}\right)^k = x_j \sum_{i=0}^{r}\binom{r}{i}\frac{k!}{(k-i)!}\left(A_j^{(x)}-A_j^{(y)}\right)^{k-i} x_j^{r-i}$$
$$= \sum_{i=0}^{r}\binom{r}{i}\frac{k!}{(k-i)!}\left[\left(A_j^{(x)}-A_j^{(y)}\right)^{k-i} x_j^{r+1-i} + (k-i)\left(A_j^{(x)}-A_j^{(y)}\right)^{k-i-1} x_j^{r-i}\right]$$
$$= \sum_{i=0}^{r+1}\left[\binom{r}{i}+\binom{r}{i-1}\right]\frac{k!}{(k-i)!}\left(A_j^{(x)}-A_j^{(y)}\right)^{k-i} x_j^{r+1-i},$$

which completes the induction step as $\binom{r}{i}+\binom{r}{i-1}=\binom{r+1}{i}$. Thus (6.8) is established.

To proof (6.9), we start from

$$(x_j-y_j)^k A_j^{(x)} = k(x_j-y_j)^{k-1} + A_j^{(x)}(x_j-y_j)^k.$$

For $k=1$ this identity follows from (6.11) and, in general, by induction. Finally, one proves (6.9) by induction on s similarly as above. We omit the details. □

The next lemma is instrumental in the proof of Theorem 2.2 in the case $|\alpha|>0$.

Lemma 6.3. *If $\alpha, \beta \in \mathbb{N}_0^d$ and $k \geq 1$, then*

$$\left|\left(A^{(x)}\right)^\alpha x^\beta \Lambda_n(x,y)\right| \leq c_k \frac{n^{\frac{|\alpha|+|\beta|}{2}}\left[K_{n+[vn]+|\alpha|+|\beta|+k}(x,x)\right]^{\frac12}\left[K_{n+[vn]+k}(y,y)\right]^{\frac12}}{(1+\sqrt{n}|x-y|)^k}.$$

Proof. We first show that for $1 \leq i \leq d$

$$\left|(x_i-y_i)^k\left(A^{(x)}\right)^\alpha x^\beta \Lambda_n(x,y)\right| \tag{6.12}$$
$$\leq c_k n^{(-k+|\alpha|+|\beta|)/2}\left[K_{n+[vn]+|\alpha|+|\beta|+k}(x,x)\right]^{\frac12}\left[K_{n+[vn]+k}(y,y)\right]^{\frac12}.$$

Clearly $\left(A^{(x)}\right)^\alpha x^\beta = \left(A^{(x)}\right)^{\alpha-\alpha_i e_i} x^{\beta-\beta_i e_i}\cdot\left(A_i^{(x)}\right)^{\alpha_i} x_i^{\beta_i}$ and the two operators separated by a dot commute. Using (6.9) and Lemma 6.1, we have

$$2^k(x_i-y_i)^k\left(A^{(x)}\right)^\alpha x^\beta\Lambda_n(x,y) = 2^k\left(A^{(x)}\right)^{\alpha-\alpha_i e_i} x^{\beta-\beta e_i} \tag{6.13}$$
$$\times\sum_{j=0}^{\alpha_i}\binom{\alpha_i}{j}\frac{k!}{(k-j)!}\left(A_i^{(x)}\right)^{\alpha_i-j} x_i^{\beta_i}(x_i-y_i)^{k-j}\Lambda_n(x,y)$$
$$=\sum_{j=0}^{\alpha_i}\binom{\alpha_i}{j}\frac{2^j k!}{(k-j)!}\sum_{(k-j)/2\leq l\leq k-j} c_{l,k-j}\sum_{\nu=0}^{\infty}\Delta^j\widehat{a}\Big(\frac{\nu}{n}\Big)$$
$$\times\left(A^{(x)}\right)^{\alpha-je_i} x^\beta\left(A_i^{(y)}-A_i^{(x)}\right)^{2l-k+j}\mathcal{H}_\nu(x,y).$$

Furthermore, by (6.8),

$$\left(A_i^{(x)}\right)^{\alpha_i-j} x_i^{\beta_i}\left(A_i^{(y)}-A_i^{(x)}\right)^{2l-k+j} = \sum_{\mu=0}^{\beta_i}\binom{\beta_i}{\mu}\frac{(-1)^\mu(2l-k+j)!}{(2l-k+j-\mu)!}$$
$$\times\left(A_i^{(x)}\right)^{\alpha_i-j}\left(A_i^{(y)}-A_i^{(x)}\right)^{2l-k+j-\mu} x_i^{\beta_i-\mu}.$$

As $\left(A^{(x)}\right)^{\alpha-\alpha_i e_i} x^{\beta-\beta_i e_i}$ commutes with $A_i^{(x)}$ and x_i, we then conclude that

$$\left(A^{(x)}\right)^{\alpha-je_i} x^\beta \left(A_i^{(y)} - A_i^{(x)}\right)^{2l-k+j} \mathcal{H}_\nu(x,y) = \sum_{\mu=0}^{\beta_i} \binom{\beta_i}{\mu} \frac{(-1)^\mu (2l-k+j)!}{(2l-k+j-\mu)!} \times \left(A^{(x)}\right)^{\alpha-je_i} \left(A_i^{(y)} - A_i^{(x)}\right)^{2l-k+j-\mu} x^{\beta-\mu e_i} \mathcal{H}_\nu(x,y).$$

Using relation (2.5) repeatedly, it follows readily that

$$x_i^r \mathcal{H}_\lambda(x) = \sum_{m=0}^{r} b_{m,r}(\lambda_i) \mathcal{H}_{\lambda+(r-2m)e_i}(x),$$

where $b_{m,r}(\lambda_i)$ are positive numbers satisfying $b_{m,r}(\lambda_i) \sim \lambda_i^{r/2}$. Applying this identity to all variables we obtain

$$x^{\beta-\mu e_i} \mathcal{H}_\nu(x,y) = \sum_{\omega_1=0}^{\gamma_1} \cdots \sum_{\omega_d=0}^{\gamma_d} b_{\omega,\gamma}(\lambda) \sum_{|\lambda|=\nu} \mathcal{H}_{\lambda+\gamma-2\omega}(x) \mathcal{H}_\lambda(y), \tag{6.14}$$

where $\gamma_j = \beta_j$ for $j \neq i$ and $\gamma_i = \beta_i - \mu$, and $b_{\omega,\gamma}(\lambda) = b_{\omega_1,\gamma_1}(\lambda_1) \dots b_{\omega_d,\gamma_d}(\lambda_d)$. Clearly, $|b_{\omega,\gamma}(\lambda)| \le c\nu^{(|\beta|-\mu)/2}$.

We now use the binomial formula and (6.5) to obtain

$$\begin{aligned} &\left| \left(A^{(x)}\right)^{\alpha-je_i} \left(A_i^{(y)} - A_i^{(x)}\right)^{2l-k+j-\mu} \mathcal{H}_{\lambda+\gamma-2\omega}(x) \mathcal{H}_\lambda(y) \right| \\ &\le c \sum_{q=0}^{2l-k+j-\mu} \left| \left(A_i^{(y)}\right)^{2l-k+j-\mu-q} \left(A^{(x)}\right)^{\alpha-je_i+qe_i} \mathcal{H}_{\lambda+\gamma-2\omega}(x) \mathcal{H}_\lambda(y) \right| \\ &\le c \sum_{q=0}^{2l-k+j-\mu} \nu^{(|\alpha|+2l-k-\mu)/2} \left| \mathcal{H}_{\lambda+\gamma-2\omega-je_i+qe_i}(x) \mathcal{H}_{\lambda+(2l-k+j-\mu-q)e_i}(y) \right| \end{aligned}$$

and hence

$$\left| \left(A^{(x)}\right)^{\alpha-je_i} x^\beta \left(A_i^{(y)} - A_i^{(x)}\right)^{2l-k+j-\mu} \mathcal{H}_\nu(x,y) \right| \le c \sum_{\mu=0}^{\beta_i} \nu^{(2l-k+|\alpha|+|\beta|-2\mu)/2} \times \sum_{\omega_1=0}^{\gamma_1} \cdots \sum_{\omega_d=0}^{\gamma_d} \sum_{q=0}^{2l-k+j-\mu} \sum_{|\lambda|=\nu} \left| \mathcal{H}_{\lambda+\gamma-2\omega+\alpha-je_i+qe_i}(x) \mathcal{H}_{\lambda+(2l-k+j-\mu-q)e_i}(y) \right|.$$

As before $\Delta^l \widehat{a}(\frac{\nu}{n}) = 0$ if $0 \le \nu \le un - l$ or $\nu \ge n + vn$, where $0 < u \le 1$ and $v > 0$. Also, by (6.6) $|\Delta^l \widehat{a}(\frac{\nu}{n})| \le n^{-l} \|\widehat{a}^{(j)}\|_\infty$. We use all of the above to conclude that

$$\begin{aligned}
\Big|(x_i - y_i)^k \left(A_j^{(x)}\right)^\alpha x_j^\beta \Lambda_n(x,y)\Big| &\le cn^{(-k+|\alpha|+|\beta|)/2} \sum_{j=0}^{\alpha_i} \sum_{(k-j)/2 \le l \le k-j} |c_{l,k}| \\
&\times \sum_{\nu=[un]-l}^{n+[vn]} \sum_{\mu=0}^{\beta_i} \sum_{\omega_1=0}^{\gamma_1} \cdots \sum_{\omega_d=0}^{\gamma_d} \left[\sum_{q=0}^{2l-k+j-\mu} \mathcal{H}_{\nu+|\alpha|+|\gamma-2\omega|+q-j}(x,x) \right]^{\frac{1}{2}} \\
&\times \left[\sum_{q=0}^{2l-k+j-\mu} \mathcal{H}_{\nu+2l-k+j-\mu-q}(y,y) \right]^{\frac{1}{2}} \\
&\le cn^{(-k+|\alpha|+|\beta|)/2} \left[K_{n+[vn]+|\alpha|+|\beta|+k}(x,x)\right]^{\frac{1}{2}} \left[K_{n+[vn]+k}(y,y)\right]^{\frac{1}{2}},
\end{aligned}$$

where we again used the Cuachy-Schwarz inequality. This proves (6.12).

On the other hand, using (6.14) with $\mu = 0$ and (6.5) we can write

$$\begin{aligned}
\left(A^{(x)}\right)^\alpha x^\beta \Lambda_n(x,y) &= \left(A^{(x)}\right)^\alpha \sum_{\nu=0}^{n+[vn]} \widehat{a}\left(\frac{\nu}{n}\right) \\
&\times \sum_{|\lambda|=\nu} \sum_{\omega_1=0}^{\beta_1} \cdots \sum_{\omega_d=0}^{\beta_d} b_{\omega,\beta}(\lambda) c_\alpha(\lambda) |\mathcal{H}_{\lambda+\alpha+\beta-2\omega}(x) \mathcal{H}_\lambda(y)|,
\end{aligned}$$

where $c_\alpha(\lambda) \sim |\lambda|^{|\alpha|/2}$. Hence, using the fact that $b_{\omega,\beta}(\lambda) \sim |\lambda|^{|\beta|/2}$ we conclude that

$$\begin{aligned}
\Big| \left(A^{(x)}\right)^\alpha x^\beta \Lambda_n(x,y) \Big| &\le c \sum_{\nu=0}^{n+[vn]} \widehat{a}\left(\frac{\nu}{n}\right) \nu^{(|\alpha|+|\beta|)/2} \\
&\times \sum_{|\lambda|=\nu} \sum_{\omega_1=0}^{\beta_1} \cdots \sum_{\omega_d=0}^{\beta_d} |\mathcal{H}_{\lambda+\alpha+\beta-2\omega}(x) \mathcal{H}_\lambda(y)| \\
&\le cn^{(|\alpha|+|\beta|)/2} \left[K_{n+[vn]+|\alpha|+|\beta|}(x,x)\right]^{\frac{1}{2}} \left[K_{n+[vn]}(y,y)\right]^{\frac{1}{2}}.
\end{aligned}$$

This along with (6.12) completes the proof of Lemma 6.3. $\square$

The last step in the proof of Theorem 2.2 is to show that the operator ∂^α can be represented in the form

$$\partial^\alpha = \sum_{\beta+\gamma \le \alpha} c_{\beta\gamma} A^\beta x^\gamma, \tag{6.15}$$

where $\beta + \gamma \le \alpha$ means $\beta_j + \gamma_j \le \alpha_j$ for $1 \le j \le d$, and $c_{\beta\gamma}$ are constants (depending only on α, β, γ).

By (2.3) $\partial_j = x_j - A_j$ and hence $\partial_j^r = (x_j - A_j)^r$. The operators x_j (multiplication by x_j) and A_j do not commute, but it is easy to see that $x_j^s A_j = s x_j^{s-1} + A_j x_j^s$. Applying this repeatedly one finds the representation

$$\partial_j^r = \sum_{0 \le \nu+\mu \le r} c_{\nu\mu} A_j^\nu x_j^\mu.$$

Since the operator $A_j^\nu x_j^\mu$ commutes with $A_i^s x_i^\ell$ if $j \ne i$, this readily implies representation (6.15).

Evidently, Lemma 6.3 and (6.15) yield (2.9) whenever $|\alpha| > 0$. The proof of Theorem 2.2 is complete. $\square$

Proof of Lemma 2.5. Observe first that it suffices to prove (2.20) only for n sufficiently large since it holds trivially if $2/\lambda \le n \le c$.

We next prove (2.20) for $d = 1$. The Christoph-Darboux formula for the Hermite polynomials ([15, (5.59)]) shows that

$$K_m(x,x) = (2^{m+1}m!)^{-1}\left[H'_{m+1}(x)H_m(x) - H'_m(x)H_{m+1}(x)\right].$$

Using the fact that $H'_{m+1}(x) = 2(m+1)H_m(x)$ ([15, (5.5.10)]) and $H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x)$ ([15, (5.5.8)]), we can rewrite $K_m(x,x)$ as

$$K_m(x,x) = (2^{m+1}m!)^{-1}\left[2(m+1)H_m^2(x) - 4mxH_{m-1}(x)H_m(x) + 2m^2H_{m-1}^2(x)\right].$$

Written in terms of the orthonormal Hermite functions, the above identity becomes

$$(m+1)h_m^2(x) + mh_{m-1}^2(x) = K_m(x,x) + \sqrt{2m}\,x\,h_m(x)h_{m-1}(x).$$

In particular, it follows that for $|x| \le 2\sqrt{2m+1}$

$$(m+1)h_m^2(x) + mh_{m-1}^2(x) \ge K_m(x,x) - 2\sqrt{2m}\sqrt{2m+1}|h_m(x)|\cdot|h_{m-1}(x)|.$$

and hence

$$\begin{aligned}(3m+2)h_m^2(x) + 3mh_{m-1}^2(x) &\ge K_m(x,x) + \left(\sqrt{2m+1}|h_m(x)| - \sqrt{2m}|h_{m-1}(x)|\right)^2 \\ &\ge K_m(x,x).\end{aligned}$$

Consequently, for $|x| \le 2\sqrt{2n+1}$

$$\begin{aligned}\sum_{m=[(1-\lambda)n]}^{n} h_m^2(x) &\ge \frac{1}{2}\sum_{m=[(1-\lambda)n]}^{n}\left(h_m^2(x) + h_{m+1}^2(x)\right) \\ &\ge \frac{c}{n}\sum_{m=[(1-\lambda)n]}^{n} K_m(x,x) \ge c_1\,K_{[\rho n]}(x,x),\end{aligned}$$

which proves (2.20) when $d = 1$.

For $d > 1$ we need the following identity which follows from the generating function of Hermite polynomials (see e.g. [16]):

$$\sum_{k=0}^{\infty}\mathcal{H}_k(x,x)r^k = \pi^{-d/2}(1-r^2)^{-d/2}e^{-\frac{1-r}{1+r}\|x\|^2} := F_d(r,t),$$

where $t = |x|$. Let us denote $\mathcal{H}_{k,d}(x,x) = \mathcal{H}_k(x,x)$ for $x \in \mathbb{R}^d$ in order to indicate the dependence on d. Then it follows from above that

$$\begin{aligned}\sum_{k=0}^{\infty} r^k[\mathcal{H}_{k,d}(x,x) - \mathcal{H}_{k-2,d}(x,x)] &= (1-r^2)\sum_{k=0}^{\infty} r^k\mathcal{H}_{k,d}(x,x) \\ &= (1-r^2)F_d(r,t) = \pi^{-1}F_{d-2}(r,t).\end{aligned} \tag{6.16}$$

Notice that $\mathcal{H}_{n,d}(x,x)$ is a radial function and hence a function of t. Thus comparing the coefficients of r^k in both side shows that

$$\mathcal{H}_{k,d}(x,x) - \mathcal{H}_{k-2,d}(x,x) = \pi^{-1}\mathcal{H}_{k,d-2}(x,x),$$

which implies

$$\mathcal{H}_{k,d}(x,x)+\mathcal{H}_{k-1,d}(x,x)=\pi^{-1}\sum_{j=0}^{k}\mathcal{H}_{j,d-2}(x,x)=\pi^{-1}K_{k,d-2}(x,x).$$

Now, summing over k we get

$$\sum_{k=[(1-\lambda)n]}^{n}\mathcal{H}_{k,d}(x,x)\geq\frac{1}{2}\sum_{k=[(1-\lambda)n]+1}^{n}[\mathcal{H}_{k,d}(x,x)+\mathcal{H}_{k-1,d}(x,x)]$$

$$\geq c\sum_{k=[(1-\lambda)n]+1}^{n}K_{k,d-2}(x,x)\geq c\,nK_{[(1-\varepsilon)n],d-2}(x,x)\geq c\,n\sum_{k=[(1-\lambda)n]}^{[(1-\varepsilon)n]}\mathcal{H}_{k,d-2}(x,x),$$

where $\varepsilon:=(1-\rho)/d$. Evidently, by induction this estimate yields (2.20) for d odd.

To establish the result for d even, we only have to prove estimate (2.20) for $d=2$. By the definion of $F_d(r,t)$, we have

$$F_0(r,t)=e^{-\frac{1-r}{1+r}t^2}=\pi^{1/2}(1-r^2)^{1/2}F_1(r,t)$$
$$=\pi^{1/2}\sum_{j=0}^{\infty}\binom{1/2}{j}(-1)^jr^{2j}\sum_{n=0}^{\infty}h_k^2(t)r^k=\pi^{1/2}\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}a_{k-j}h_j^2(x)\right]r^k,$$

where $a_{2j}=(-1)^j\binom{1/2}{j}$ and $a_{2j-1}=0$. Hence, using (6.16)

$$\mathcal{H}_{k,2}(x,x)-\mathcal{H}_{k-2,2}(x,x)=\pi^{-1/2}\sum_{j=0}^{k}a_{k-j}h_j^2(t).$$

Consequently,

$$\mathcal{H}_{k,2}(x,x)+\mathcal{H}_{k-1,2}(x,x)=\pi^{-1/2}\sum_{l=0}^{k}\sum_{j=0}^{l}a_{l-j}h_j^2(t)=\sum_{j=0}^{k}h_j^2(t)\sum_{l=0}^{k-j}a_l.$$

A simple combinatorial formula shows that

$$\sum_{l=0}^{k-j}a_l=\sum_{l=0}^{[(k-j)/2]}(-1)^l\binom{1/2}{l}=\frac{\Gamma(\frac{1}{2}+[\frac{k-j}{2}])}{\Gamma(\frac{1}{2})\Gamma(1+[\frac{k-j}{2}])},$$

which is positive for all $0\leq j\leq k$. Furthermore, by $\Gamma(k+a)/\Gamma(k+1)\sim k^{a-1}$ it follows that $\sum_{l=0}^{k-j}a_l\geq ck^{-1/2}$ for $0\leq j\leq\alpha k$ for any $\alpha<1$. Therefore,

$$\mathcal{H}_{k,2}(x,x)+\mathcal{H}_{k-1,2}(x,x)\geq ck^{-1/2}\sum_{j=0}^{\alpha k}h_j^2(t)$$

and summing over k we get

$$\sum_{k=[(1-\lambda)n]}^{n}\mathcal{H}_{k,2}(x,x)\geq cn^{1/2}K_{[\rho n],1}(t,t),$$

which establishes (2.20) for $d=2$. $\square$

6.2. **Proofs for Sections 4-5.**

Proof of Lemma 4.7. Let

$$b_j^\diamond(x) := \sum_{\eta\in\mathcal{X}_j} \frac{|b_\eta|}{(1+2^j d(x,R_\eta))^\sigma}, \tag{6.17}$$

where $d(x,E)$ stands for the ℓ^∞ distance of x from $E\subset\mathbb{R}^d$. Evidently,

$$b_j^*(x)\le cb_j^\diamond(x) \quad\text{and}\quad b_\xi^*\mathbb{1}_{R_\xi}(x)\le cb_j^\diamond(x), \quad x\in\mathbb{R}^d, \quad \xi\in\mathcal{X}_j. \tag{6.18}$$

We will show that

$$b_j^\diamond(x)\le c\mathcal{M}_s\Big(\sum_{\omega\in\mathcal{X}_j}|b_\omega|\mathbb{1}_{R_\omega}\Big)(x), \quad x\in\mathbb{R}^d. \tag{6.19}$$

In view of (6.18) this implies (4.10)-(4.11), and hence Lemma 4.7.

By the construction of the tiles $\{R_\xi\}$ in (2.33)-(2.34) it follows that there exists a constant $c_\diamond>0$ depending only on d such that

$$Q_j := \cup_{\xi\in\mathcal{X}_j}R_\xi\subset[-c_\diamond 2^j, c_\diamond 2^j]^d.$$

Fix $x\in\mathbb{R}^d$. To prove (6.19) we consider two cases for x.

Case 1. $|x|_\infty>2c_\diamond 2^j$. Then $d(x,R_\eta)>|x|_\infty/2$ for $\eta\in\mathcal{X}_j$ and hence

$$\begin{aligned} b_j^\diamond(x) = \sum_{\eta\in\mathcal{X}_j}\frac{|b_\eta|}{(1+2^jd(x,R_\eta))^\sigma} &\le \frac{c}{(2^j|x|_\infty)^\sigma}\sum_{\eta\in\mathcal{X}_j}|b_\eta| \\ &\le \frac{c4^{jd\lambda}}{(2^j|x|_\infty)^\sigma}\Big(\sum_{\eta\in\mathcal{X}_j}|b_\eta|^s\Big)^{1/s}, \end{aligned} \tag{6.20}$$

where $\lambda:=1-\min\{1,1/s\}$ and for the last estimate we use Hölder's inequality if $s>1$ and the s-triangle inequality if $s<1$.

Denote $Q_x:=[-|x|_\infty,|x|_\infty]^d$. Notice that $Q_j\subset Q_x$. From above we infer

$$\begin{aligned} b_j^\diamond(x) &\le \frac{c4^{jd\lambda}|x|_\infty^{d/s}}{(2^j|x|_\infty)^\sigma}\Big(\frac{1}{|Q_x|}\int_{Q_x}\Big(\sum_{\eta\in\mathcal{X}_j}|b_\eta|\mathbb{1}_{R_\xi}(y)\Big)^s dy\Big)^{1/s} \\ &\le c2^{j(2d\lambda-\sigma)}|x|^{d/s-\sigma}\mathcal{M}_s\Big(\sum_{\eta\in\mathcal{X}_j}|b_\eta|\mathbb{1}_{R_\xi}\Big)(x)\le c\mathcal{M}_s\Big(\sum_{\eta\in\mathcal{X}_j}|b_\eta|\mathbb{1}_{R_\xi}\Big)(x) \end{aligned}$$

as claimed. Here we used the fact that $\sigma\ge d\max\{2,1/s\}$.

Case 2. $|x|_\infty\le 2c_\diamond 2^j$. To make the argument more transparent we first subdivide the tiles $\{R_\eta\}_{\eta\in\mathcal{X}_j}$ into boxes of almost equal sides of length $\sim 2^{-j}$. By the construction of the tiles (see (2.33)) there exists a constant $\tilde c>0$ such that the minimum side of each tile R_η is $\ge\tilde c2^{-j}$. Now, evidently each tile R_η can be subdivided into a disjoint union of boxes R_θ with centers θ such that

$$\theta+[-\tilde c2^{-j-1},\tilde c2^{-j-1}]\subset R_\theta\subset\theta+[-\tilde c2^{-j},\tilde c2^{-j}].$$

Denote by $\widehat{\mathcal{X}}_j$ the set of centers of all boxes obtained by subdividing the tiles from $\mathcal{X}_j$. Also, set $b_\theta:=b_\eta$ if $R_\theta\subset R_\eta$. Evidently,

$$b_j^\diamond(x) := \sum_{\eta\in\mathcal{X}_j}\frac{|b_\eta|}{(1+2^jd(x,R_\eta))^\sigma}\le\sum_{\theta\in\widehat{\mathcal{X}}_j}\frac{|b_\theta|}{(1+2^jd(x,R_\theta))^\sigma} \tag{6.21}$$

and

$$\sum_{\eta\in\mathcal{X}_j} |b_\eta| \mathbb{1}_{R_\eta} = \sum_{\eta\in\widehat{\mathcal{X}}_j} |b_\theta| \mathbb{1}_{R_\theta}. \tag{6.22}$$

Denote $Y_0 := \{\theta \in \widehat{\mathcal{X}}_j : 2^j|\theta - x|_\infty \le \tilde{c}\}$,

$$Y_m := \{\theta \in \widehat{\mathcal{X}}_j : \tilde{c}2^{m-1} \le 2^j|\theta - x|_\infty \le \tilde{c}2^m\}, \quad \text{and}$$
$$Q_m := \{y \in \mathbb{R}^d : |y - x|_\infty \le \tilde{c}(2^m + 1)2^{-j}\}, \quad m \ge 1.$$

Clearly, $\#Y_m \le c2^{md}$, $\cup_{\theta\in Y_m} R_\theta \subset Q_m$, and $\widehat{\mathcal{X}} = \cup_{m\ge0} Y_m$. Similarly as in (6.20)

$$\begin{aligned}\sum_{\theta\in Y_m} \frac{|b_\theta|}{(1 + 2^j d(x, R_\theta))^\sigma} &\le c2^{-m\sigma} \sum_{\theta\in Y_m} |b_\theta| \le c2^{-m\sigma} 2^{md\lambda} \Big(\sum_{\theta\in Y_m} |b_\theta|^s\Big)^{1/s} \\ &\le c2^{-m(\sigma-d\lambda-d/s)} \Big(\frac{1}{|Q_m|} \int_{Q_m} \Big(\sum_{\theta\in Y_m} |b_\theta| \mathbb{1}_{R_\theta}(y)\Big)^s dy\Big)^{1/s} \\ &\le c2^{-m(\sigma-d\max\{1,1/s\})} \mathcal{M}_s \Big(\sum_{\eta\in\mathcal{X}_j} |b_\eta| \mathbb{1}_{R_\eta}\Big)(x),\end{aligned}$$

where we used (6.22). Summing up over $m \ge 0$, taking into account that $\sigma > d\max\{2, 1/s\})$, and also using (6.21) we arrive at (6.19). $\square$

Proof of Lemma 4.8. For this proof we will need an additional lemma.

Lemma 6.4. *Suppose $g \in V_{4^j}$ and $\xi \in \mathcal{X}_j$. Then for any $k > 0$ and $L > 0$ we have for $x', x'' \in 2R_\xi$*

$$|g(x') - g(x'')| \le c2^j|x' - x''| \sum_{\eta\in\mathcal{X}_j} \frac{|g(\eta)|}{(1 + 2^j|\xi - \eta|)^k} \tag{6.23}$$

and

$$|g(x')-g(x'')| \le \hat{c}2^{-jL}|x'-x''| \sum_{\eta\in\mathcal{X}_j} \frac{|g(\eta)|}{(1 + 2^j|\xi - \eta|)^k}, \ \text{if } |\xi|_\infty > (1+2\delta)2^{j+1}. \tag{6.24}$$

Here c, $\hat{c}$ depend on k, d, and δ, and $\hat{c}$ depends on L as well; $2R_\xi \subset \mathbb{R}^d$ is the set obtained by dilating R_ξ by a factor of 2 and with the same center.

Proof. Let Λ_{4^j} be the kernel from (2.8) with $n = 4^j$, where $\widehat{a}$ is admissible of type (a) with $v := \delta$. Then $\Lambda_{4^j} * g = g$, since $g \in V_{4^j}$, and $\Lambda_{4^j}(x, \cdot) \in V_{[(1+\delta)4^j]}$. Note that $[(1 + \delta)4^j] + 4^j \le 2N_j - 1$. Therefore, we can use the cubature formula from Corollary 2.8 to obtain

$$g(x) = \int_{\mathbb{R}^d} \Lambda_{4^j}(x, y)g(y)dy = \sum_{\eta\in\mathcal{X}_j} \lambda_\eta \Lambda_{4^j}(x, \eta)g(\eta),$$

where the weights λ_ξ obey (2.35) (see also (2.36)-(2.37)). Hence, for $x', x'' \in 2R_\xi$

$$\begin{aligned}|g(x') - g(x'')| &\le \sum_{\eta\in\mathcal{X}_j} \lambda_\eta |\Lambda_{4^j}(x', \eta) - \Lambda_{4^j}(x'', \eta)||g(\eta)| \\ &\le c|x' - x''| \sum_{\eta\in\mathcal{X}_j} \lambda_\eta \sup_{x\in2R_\xi} |\nabla\Lambda_{4^j}(x, \eta)||g(\eta)|. \qquad (6.25)\end{aligned}$$

Note that $\Big(4([(1+\delta)4^j]+k+1)+2\Big)^{1/2} \le (1+\delta)2^{j+1}$ for sufficiently large j (depending on k and δ). Therefore, we have from (2.17)-(2.18)

$$|\nabla\Lambda_{4^j}(x,\eta)| \le \frac{c2^{j(d+1)}}{(1+2^j|x-\eta|)^k}, \quad x\in\mathbb{R}^d,\ \eta\in\mathcal{X}_j, \tag{6.26}$$

and for any $L>0$ (we need $L\ge k$)

$$|\nabla\Lambda_{4^j}(x,\eta)| \le \frac{c2^{-2jL}}{(1+2^j|x-\eta|)^k}, \text{ if } |x|_\infty > (1+\delta)2^{j+1} \text{ or } |\eta|_\infty > (1+\delta)2^{j+1}. \tag{6.27}$$

Suppose $|\xi|_\infty > (1+2\delta)2^{j+1}$, then $2R_\xi \subset \{x\in\mathbb{R}^d : |x|_\infty > (1+\delta)2^{j+1}\}$ for sufficiently large j. Combining (6.25) with (6.27) and (2.37) we get

$$|g(x')-g(x'')| \le c2^{-jd/3}|x'-x''| \sum_{\eta\in\mathcal{X}_j} \sup_{x\in 2R_\xi} \frac{2^{-j(k+L)}}{(1+2^j|x-\eta|)^k}|g(\eta)|, \tag{6.28}$$

where we used that $\operatorname{diam}(2R_\xi)\le c2^{-j/3}$. However, for any $x\in 2R_\xi$ we have

$$1+2^j|\xi-\eta| \le 1+2^j(|\xi-x|+|x-\eta|) \le 1+2^j(c2^{-j/3}+|x-\eta|) \le c2^j(1+2^j|x-\eta|).$$

We use this in (6.28) to obtain (6.24) for sufficiently large j.

One proves (6.23) in a similar fashion. In the case $j\le c$ estimates (6.23)-(6.24) follow easily by (6.25). $\square$

We are now prepared to prove Lemma 4.8. Let $g\in V_{4^j}$. Pick $\ell\ge 1$ sufficiently large (to be determined later on) and denote for $\xi\in\mathcal{X}_j$

$$\mathcal{X}_{j+\ell}(\xi) := \{\eta\in\mathcal{X}_{j+\ell} : R_\eta\cap R_\xi \ne \emptyset\} \quad \text{and} \tag{6.29}$$

$$d_\xi := \sup\{|g(x')-g(x'')| : x',x''\in R_\eta \text{ for some } \eta\in\mathcal{X}_{j+\ell}(\xi)\}. \tag{6.30}$$

We first estimate d_ξ, $\xi\in\mathcal{X}_j$.

Case A: $|\xi|_\infty \le (1+3\delta)2^{j+1}$. By (2.36) it follows that for sufficiently large ℓ (depending only on d and δ) $\cup_{\eta\in\mathcal{X}_{j+\ell}(\xi)}R_\eta \subset 2R_\xi$. Hence, using Lemma 6.4 (see (6.23)) with $k\ge\sigma$, we get

$$d_\xi \le c2^{-\ell}\sum_{\eta\in\mathcal{X}_j}\frac{|g(\eta)|}{(1+2^j|\xi-\eta|)^\sigma}, \tag{6.31}$$

for sufficiently large j (depending only on d and δ), where $c>0$ is a constant independent of ℓ.

Case B: $|\xi|_\infty > (1+3\delta)2^{j+1}$. By (2.36) $|x|_\infty > (1+2\delta)2^{j+1}$ for $x\in\cup_{\eta\in\mathcal{X}_{j+\ell}(\xi)}R_\eta$ if j is sufficiently large. We apply estimate (6.24) of Lemma 6.4 with $k\ge\sigma$ and $L=1$ to obtain

$$d_\xi \le c2^{-j}\sum_{\eta\in\mathcal{X}_j}\frac{|g(\eta)|}{(1+2^j|\xi-\eta|)^\sigma}. \tag{6.32}$$

To estimate M_ξ^*, $\xi\in\mathcal{X}_j$, we consider two cases for ξ.

Case 1: $|\xi|_\infty \le (1+4\delta)2^{j+1}$. By (2.36), we have for sufficiently large j:

$$R_\xi \sim \xi + [-2^{-j},2^{-j}]^d, \quad \text{and} \quad R_\eta \sim \eta + [-2^{-j-\ell},2^{-j-\ell}]^d, \quad \eta\in\mathcal{X}_{j+\ell}(\xi). \tag{6.33}$$

By the definition of d_ξ in (6.30) it follows that $M_\xi \le m_\lambda + d_\xi$ for some $\lambda \in \mathcal{X}_{j+\ell}(\xi)$ and hence, using (6.33),

$$M_\xi \le c \sum_{\omega\in\mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^{j+\ell}|\xi-\omega|)^\sigma} + d_\xi =: G_\xi + d_\xi, \quad c = c(d,\delta,\sigma,\ell).$$

Consequently,

$$M_\xi^* \le G_\xi^* + d_\xi^*. \tag{6.34}$$

Write $\mathcal{X}_j' := \{\eta \in \mathcal{X}_j : |\eta|_\infty \le (1+3\delta)2^{j+1}\}$ and $\mathcal{X}_j'' := \mathcal{X}_j \setminus \mathcal{X}_j'$. Now, we use (6.31)-(6.32) to obtain

$$\begin{aligned} d_\xi^* = \sum_{\eta\in\mathcal{X}_j} \frac{d_\eta}{(1+2^j|\xi-\eta|)^\sigma} &\le c2^{-\ell} \sum_{\eta\in\mathcal{X}_j}\sum_{\omega\in\mathcal{X}_j'} \frac{|g(\omega)|}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} \\ &\quad + c2^{-j} \sum_{\eta\in\mathcal{X}_j}\sum_{\omega\in\mathcal{X}_j''} \frac{|g(\omega)|}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma}, \end{aligned}$$

replacing $\mathcal{X}_j'$ and $\mathcal{X}_j''$ by $\mathcal{X}_j$ above and shifting the order of summation we get

$$\begin{aligned} d_\xi^* &\le c(2^{-\ell}+2^{-j}) \sum_{\omega\in\mathcal{X}_j} |g(\omega)| \sum_{\eta\in\mathcal{X}_j} \frac{1}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} \\ &\le c(2^{-\ell}+2^{-j}) \sum_{\omega\in\mathcal{X}_j} \frac{|g(\omega)|}{(1+2^j|\xi-\omega|)^\sigma} \le c(2^{-\ell}+2^{-j})M_\xi^*. \end{aligned} \tag{6.35}$$

Here the constant c is independent of ℓ and j, and we used that

$$\begin{aligned} \sum_{\eta\in\mathcal{X}_j} \frac{1}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} &\le \int_{\mathbb{R}^d} \frac{c2^{jd}}{(1+2^j|\xi-y|)^\sigma(1+2^j|y-\omega|)^\sigma} dy \\ &\le \frac{c}{(1+2^j|\xi-\omega|)^\sigma} \quad (\sigma > d). \end{aligned} \tag{6.36}$$

These estimates are standard and easy to prove utilizing the fact that the tiles $\{R_\eta\}_{\eta\in\mathcal{X}_j}$ do not overlap and obey (2.36).

To estimate G_ξ^* we use again (2.36) and (6.36). We get

$$\begin{aligned} G_\xi^* &= \sum_{\eta\in\mathcal{X}_j} \frac{G_\eta}{(1+2^j|\xi-\eta|)^\sigma} \le c \sum_{\eta\in\mathcal{X}_j}\sum_{\omega\in\mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} \\ &\le c \sum_{\omega\in\mathcal{X}_{j+\ell}} m_\omega \sum_{\eta\in\mathcal{X}_j} \frac{1}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} \le c \sum_{\omega\in\mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^j|\xi-\omega|)^\sigma} \\ &\le c2^{\ell\sigma} \sum_{\omega\in\mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^{j+\ell}|\lambda-\omega|)^\sigma} = cm_\lambda^* \quad \text{for each } \lambda \in \mathcal{X}_{j+\ell}(\xi). \end{aligned}$$

Combining this with (6.34)-(6.35) we obtain

$$M_\xi^* \le c_1 m_\lambda^* + c_2(2^{-\ell}+2^{-j})M_\xi^* \quad \text{for } \lambda \in \mathcal{X}_{j+\ell}(\xi),$$

where $c_2 > 0$ is independent of ℓ and j. Choosing ℓ and j sufficiently large (depending only on d, δ, and σ) this yields $M_\xi^* \le cm_\lambda^*$ for all $\lambda \in \mathcal{X}_{j+\ell}(\xi)$. For $j \le c$ this relation follows as above but using only (6.23) and taking ℓ large enough. We skip the details. Thus (4.12) is established in Case 1.

Case 2: $|\xi|_\infty > (1+4\delta)2^{j+1}$. In this case for sufficiently large j (depending only on d, δ, and σ) $|x|_\infty \ge (1+3\delta)2^{j+1}$ for $x \in \cup_{\eta\in\mathcal{X}_{j+\ell}(\xi)} R_\eta$. Hence, using (6.24) with $L = 1$, we have

$$M_\xi \le m_\omega + c2^{-j} \sum_{\eta\in\mathcal{X}_j} \frac{|g(\eta)|}{(1+2^j|\xi-\eta|)^\sigma} \le m_\omega + c2^{-j} M_\xi^* \quad \text{for all } \omega \in \mathcal{X}_{j+\ell}(\xi),$$

where $c > 0$ is independent of j. Fix $\lambda \in \mathcal{X}_{j+\ell}(\xi)$ and for each $\eta \in \mathcal{X}_j$, $\eta \ne \xi$, choose $\omega_\eta \in \mathcal{X}_{j+\ell}(\eta)$ so that $|\lambda - \omega_\eta| = \min_{\omega\in\mathcal{X}_{j+\ell}(\eta)} |\lambda-\omega|$. Then from above

$$M_\xi^* \le \sum_{\eta\in\mathcal{X}_j} \frac{m_{\omega_\eta}}{(1+2^j|\xi-\eta|)^\sigma} + c2^{-j} \sum_{\eta\in\mathcal{X}_j} \frac{M_\eta^*}{(1+2^j|\xi-\eta|)^\sigma} =: A_1 + A_2. \tag{6.37}$$

By (2.30) it easily follows that ω_η from above obeys $|\lambda-\omega_\eta| \le c|\xi-\eta|$ and hence

$$A_1 \le c \sum_{\eta\in\mathcal{X}_j} \frac{m_{\omega_\eta}}{(1+2^j|\lambda-\omega_\eta|)^\sigma} \le c2^{\ell\sigma} \sum_{\omega\in\mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^{j+\ell}|\lambda-\omega|)^\sigma} \le c_1 m_\lambda^*. \tag{6.38}$$

On the other hand, using Definition 4.6 and (6.36), we have

$$\begin{aligned} A_2 &\le c2^{-j} \sum_{\eta\in\mathcal{X}_j}\sum_{\omega\in\mathcal{X}_j} \frac{M_\omega}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} \\ &\le c2^{-j} \sum_{\omega\in\mathcal{X}_j} M_\omega \sum_{\eta\in\mathcal{X}_j} \frac{1}{(1+2^j|\xi-\eta|)^\sigma(1+2^j|\eta-\omega|)^\sigma} \\ &\le c_2 2^{-j} \sum_{\omega\in\mathcal{X}_j} \frac{M_\omega}{(1+2^j|\eta-\omega|)^\sigma} = c_2 2^{-j} M_\omega^*, \end{aligned}$$

where $c_2 > 0$ is independent of j. Combining this with (6.37)-(6.38) we arrive at

$$M_\xi^* \le c_1 m_\lambda^* + c_2 2^{-j} M_\xi^* \quad \text{for } \lambda \in \mathcal{X}_{j+\ell}(\xi).$$

Choosing j sufficiently large we get $M_\xi^* \le c_1 m_\lambda^*$ for each $\lambda \in \mathcal{X}_{j+\ell}(\xi)$. For $j \le c$ this estimate follows as in Case 1 but using only (6.23). This completes the proof of Lemma 4.8.

Proof of Lemma 4.11. To prove (4.19) we first show that there exit $\varepsilon > 0$ and $r > 1$ such that

$$F_p^{\alpha q}(H) \hookrightarrow F_r^{\varepsilon 2}(H). \tag{6.39}$$

Indeed, if $p > 1$, using that $\alpha > 0$, Proposition 4.9 (a) yields $F_p^{\alpha q} \hookrightarrow F_p^{\varepsilon 2}$ for any $0 < \varepsilon < \alpha$. On the other hand, if $p \le 1$, then $\alpha - d/p > -d$ and hence there exist $\delta > 0$ and $\varepsilon > 0$ such that, first, $\alpha - d/p > -d/(1+\delta)$ and then $\alpha - d/p = \varepsilon - d/(1+\delta)$. Now, by Proposition 4.9 (b) we have $F_p^{\alpha q} \hookrightarrow F_{1+\delta}^{\varepsilon 2}$. Thus (6.39) is established.

Denote $h_y(x) := h(x-y)$. It follows by Proposition 4.3 and Theorem 4.5 that

$$\|h_y\|_r \sim \Big\|\Big(\sum_{\xi\in\mathcal{X}} \Big(|R_\xi|^{-1/2}|\langle h_y, \varphi_\xi\rangle| \mathbb{1}_{R_\xi}(\cdot)\Big)^2\Big)^{1/2}\Big\|_r =: \mathcal{N}(h_y).$$

Fix $J \ge 1$ and denote $\mathcal{Y}_J := \cup_{0\le j\le J} \mathcal{X}_j$. By the decay of needlets (see (3.11)) it follows that

$$\max_{\xi\in\mathcal{Y}_J} |\langle h_y, \varphi_\xi\rangle| \to 0 \quad \text{as } |y| \to \infty.$$

Hence there exists $A > 0$ such that if $|y| > A$,

$$\Big\|\Big(\sum_{\xi\in\mathcal{Y}}\Big(|R_\xi|^{-1/2}|\langle h_y,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(\cdot)\Big)^2\Big)^{1/2}\Big\|_r \le \frac{1}{2}\mathcal{N}(h_y). \tag{6.40}$$

Evidently, h_y being C^∞ and compactly supported belongs to all Hermite-F-spaces and by (6.39) $\|h_y\|_{F_p^{\alpha q}(H)} \ge c\|h_y\|_{F_r^{\varepsilon 2}(H)}$. We now use Theorem 4.5 and (6.40) to obtain, for $|y| > A$,

$$\begin{aligned}\|h_y\|_{F_p^{\alpha q}(H)} \ge c\|h_y\|_{F_r^{\varepsilon 2}(H)} &\ge c\Big\|\Big(\sum_{j=0}^{\infty}2^{\varepsilon j}\sum_{\xi\in\mathcal{X}_j}\Big(|R_\xi|^{-1/2}|\langle h_y,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(\cdot)\Big)^2\Big)^{1/2}\Big\|_r\\
&\ge c2^{J\varepsilon}\Big\|\Big(\sum_{j=J+1}^{\infty}\sum_{\xi\in\mathcal{X}\setminus\mathcal{Y}_J}\Big(|R_\xi|^{-1/2}|\langle h_y,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(\cdot)\Big)^2\Big)^{1/2}\Big\|_r\\
&\ge (1/2)c2^{J\varepsilon}\Big\|\Big(\sum_{\xi\in\mathcal{X}}\Big(|R_\xi|^{-1/2}|\langle h_y,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(\cdot)\Big)^2\Big)^{1/2}\Big\|_r\\
&\ge c'2^{J\varepsilon}\|h_y\|_r = c'2^{J\varepsilon}\|h\|_r \quad (\|h\|_r > 0),\end{aligned}$$

where $c' > 0$ is independent of J. Letting $J\to\infty$ the above implies (4.19).

We next prove (4.20). Choose $k > \max\{\alpha+d, d/p\}$. Using (3.11)-(3.12) we get, for $\xi\in\mathcal{X}_j$ and $|\xi-y| > |y|/2$, and sufficiently large $|y|$,

$$|\langle h_y,\varphi_\xi\rangle| \le \frac{c2^{jd/2}}{(1+2^j|y-\xi|)^k} \le \frac{c'2^{jd/2}}{(1+2^j|y-x|)^k} \quad \text{for each } x\in R_\xi.$$

Hence, using also (2.38) we have that for $|x-y|\ge|y|/4$ and $|y|$ sufficiently large

$$\begin{aligned}G(x) &:= \sum_{j=0}^{\infty}2^{j\alpha q}\sum_{\xi\in\mathcal{X}_j,|\xi-y|>|y|/2}\Big(|R_\xi|^{-1/2}|\langle h_y,\varphi_\xi\rangle|\mathbb{1}_{R_\xi}(x)\Big)^q\\
&\le c\sum_{j=0}^{\infty}\frac{2^{j(\alpha+d)q}}{(1+2^j|y-x|)^{kq}} \le \frac{c}{|y-x|^{kq}}\sum_{j=0}^{\infty}2^{-j(k-\alpha-d)q} \le \frac{c}{|y-x|^{kq}},\end{aligned}$$

while

$$G(x) = 0 \quad \text{if } |x-y| < |y|/4.$$

Hence,

$$\|h_y\|_{F_y^*} \le c\Big(\int_{|x-y|>|y|/4}\frac{dx}{|y-x|^{kp}}\Big)^{1/p} \le \frac{c}{|y|^{k-d/p}},$$

which yields (4.20). $\square$

Proof of Lemma 5.4. Let $g\in V_{4^j}$ $(j\ge 0)$ and $0<p<\infty$. We will utilize Definition 4.6 and Lemmas 4.7-4.8. To this end choose $0<s<\min\{p,1\}$ and $\sigma > d\max\{2,1/s\}$. Set $M_\xi := \sup_{x\in R_\xi}|g(x)|$, $\xi\in\mathcal{X}_j$, and $m_\lambda := \inf_{x\in R_\lambda}|g(x)|$, $\lambda\in\mathcal{X}_{j+\ell}$, where $\ell\ge 1$ is the constant from Lemma 4.8. Using Lemmas 4.7-4.8 and the maximal inequality (2.40) we get

$$\begin{aligned}\Big(\sum_{\xi\in\mathcal{X}_j}|R_\xi|\sup_{x\in R_\xi}|g(x)|^p\Big)^{1/p} &= \Big\|\sum_{\xi\in\mathcal{X}_j}M_\xi\mathbb{1}_{R_\xi}\Big\|_p \le c\Big\|\sum_{\eta\in\mathcal{X}_{j+\ell}}m_\eta^*\mathbb{1}_{R_\eta}\Big\|_p\\
&\le c\Big\|\mathcal{M}_s\Big(\sum_{\eta\in\mathcal{X}_{j+\ell}}m_\eta\mathbb{1}_{R_\eta}\Big)\Big\|_p \le c\Big\|\sum_{\eta\in\mathcal{X}_{j+\ell}}m_\eta\mathbb{1}_{R_\eta}\Big\|_p \le c\|g\|_p. \quad \square\end{aligned}$$

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