Anisotropic Franklin bases on polygonal domains

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Abstract

Franklin systems induced by Courant elements over multilevel nested triangulations of polygonal domains in $\mathbb{R}^2$ are explored. Mild conditions are imposed on the triangulations which prevent them from deterioration and at the same time allow for a lot of flexibility and, in particular, arbitrarily sharp angles. It is shown that such anisotropic Franklin systems are Schauder bases for C and L_1, and unconditional bases for L_p $(1 < p < \infty)$ and the corresponding Hardy spaces H_1. It is also proved that the anisotropic H_1 is exactly the space of all functions in L_1 for which the corresponding Franklin system expansions converge unconditionally in L_1. Finally, it is shown that the Franklin bases characterize the corresponding anisotropic BMO spaces.

1 Introduction

The Franklin systems in the univariate case as well as in the multivariate case in regular setups are thoroughly studied and well-known. We refer the reader to [11], [1], and [8] as references for Franklin systems.

In this article, we consider Franklin systems generated by sequences of Courant elements, i.e. piecewise linear elements, induced by multilevel nested triangulations of compact polygonal domains in $\mathbb{R}^2$. For a given polygonal domain E in $\mathbb{R}^2$ we consider a sequence of nested triangulations $\mathcal{T}_0, \mathcal{T}_1, \dots$ of a general nature. Mild conditions are imposed on the triangulations which prevent them from deterioration. At the same time these conditions allow for a great deal of flexibility and, in particular, arbitrarily sharp angles.

We show that the Franklin systems obtained by applying the Gram-Schmidt orthogonalization process to the corresponding Courant elements are Schauder bases for C and L_1, and unconditional bases for L_p $(1 < p < \infty)$ and the corresponding Hardy space H_1. Further, we prove that H_1 is exactly the space of all functions in L_1 for which the corresponding Franklin system expansions converge unconditionally in L_1. Finally, we show that the anisotropic Franklin systems characterize the corresponding BMO spaces. Thus we show that the basic and well-known results on Franklin bases in the regular case have analogues in the anisotropic case. We do not consider anisotropic atomic Hardy spaces H_p with $0 < p < 1$ in this article.

The motivation for this article is two-fold. On the one hand the spaces induced by general multilevel nested triangulations are an example of spaces on homogeneous spaces. There are no bases available for such spaces in general and hence such bases are worthy to be studied.

On the other hand the Franklin bases that we explore are the only anisotropic bases over general sequences of nested triangulations. There are no constructions of spline wavelet or prewavelet bases over such triangulations available as for now.

In [12] we show that the anisotropic Franklin systems considered in this article characterize the anisotropic B-space (generalized Besov spaces) which are naturally associated with hierarchical sequences of nested triangulations. These spaces play a fundamental role in nonlinear spline approximation (see [4, 5, 9, 10]).

The paper is organized as follows. In §2 we give all auxiliary results needed for the development of the anisotropic Franklin bases. In §3 we introduce the anisotropic Franklin systems and state and proof our main results. Section §4 is an appendix where we give an example which shows that the anisotropic H_1 spaces essentially depend on the triangulations which are used.

Notation. Throughout this article for a set $G \subset \mathbb{R}^2$, $|G|$ denotes the Lebesgue measure of G, while G° means the interior of G; $\mathbb{1}_G$ denotes the characteristic function of G, and $\tilde{\mathbb{1}}_G := |G|^{-1/2}\mathbb{1}_G$. For a finite set G, $\#G$ denotes the cardinality of G. Positive constants are denoted by $c, c_1, \dots$ (if not specified, they may vary at every occurrence), $A \approx B$ means $c_1 A \le B \le c_2 B$, and $A := B$ or $B =: A$ stands for "A is by definition equal to B". We set $\langle f, g\rangle := \int_E fg$.

2 Preliminaries

In this section we collect all prerequisites regarding triangulations, maximal operators, Hardy spaces on spaces of homogeneous type, and other results, which will be needed in the development of the Franklin bases. Most of these facts are well known and we give only the essentials and suitable references for them.

2.1 Multilevel triangulations

We call $E \subset \mathbb{R}^2$ a *bounded polygonal domain* if its interior E° is connected and E is the union of a finite set $\mathcal{T}_0$ of closed triangles with disjoint interiors: $E = \bigcup_{\triangle \in \mathcal{T}_0} \triangle$. Following [9] we call

$$\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$$

a *locally regular triangulation* of E or briefly an *LR-triangulation* with levels $(\mathcal{T}_m)_{m\ge 0}$ if the following conditions are fulfilled:

(a) Every level $\mathcal{T}_m$ is a partition of E, that is, $E = \bigcup_{\triangle\in\mathcal{T}_m} \triangle$ and $\mathcal{T}_m$ consists of closed triangles with disjoint interiors.

(b) The levels $(\mathcal{T}_m)$ of $\mathcal{T}$ are nested, i.e. $\mathcal{T}_{m+1}$ is a refinement of $\mathcal{T}_m$.

(c) Each triangle $\triangle \in \mathcal{T}_m$ has at least two and at most M_0 children (subtriangles) in $\mathcal{T}_{m+1}$, where $M_0 \ge 2$ is a constant.

(d) The valence N_v of each vertex v of any triangle $\triangle \in \mathcal{T}_m$ (the number of the triangles from $\mathcal{T}_m$ which share v as a vertex) is at most N_0, where N_0 is a constant.

(e) *No hanging vertices condition*: No vertex of any triangle $\triangle \in \mathcal{T}_m$ which belongs to the interior of E lies in the interior of an edge of another triangle from $\mathcal{T}_m$.

(f) There exist constants $0 < r < \rho < 1$ $(r \le \frac{1}{2})$ such that for each $\triangle \in \mathcal{T}_m$ $(m \ge 0)$ and any child $\triangle' \in \mathcal{T}_{m+1}$ of $\triangle$,

$$r|\triangle| \le |\triangle'| \le \rho|\triangle|. \tag{2.1}$$

(g) There exists a constant $0 < \delta \le 1$ such that for $\triangle', \triangle'' \in \mathcal{T}_m$ $(m \ge 0)$ with a common vertex,

$$\delta \le |\triangle'|/|\triangle| \le \delta^{-1}. \tag{2.2}$$

The notion of a *regular triangulation* will be needed later on. We call $\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$ a regular triangulation of a bounded polygonal domain $E \subset \mathbb{R}^2$ if $\mathcal{T}$ satisfies conditions (a)-(e) of LR-triangulations and also the *minimal angle condition*, that is, $\min\text{angle}\,(\triangle) \ge \beta$ for every triangle $\triangle \in \mathcal{T}$, where $\beta > 0$ is a constant. Evidently, every regular triangulation is locally regular but not the other way around. For other types of triangulations, see [9].

We denote by $\mathcal{V}_m$ the set of all vertices of triangles from $\mathcal{T}_m$, where if v is on the boundary of E we include in $\mathcal{V}_m$ as many different copies of v as is its multiplicity. The multiplicity of a vertex $v \in \mathcal{T}_0$ on the boundary of E can be bigger than one if the interior of the union of all triangles which share v as a vertex is not connected. Also, cuts in E along edges of triangles from $\mathcal{T}_0$ are possible; such edges belong to the boundary of E.

We let $\mathcal{E}_m$ denote the set of all edges of triangles in $\mathcal{T}_m$. We also set $\mathcal{V} := \cup_{m\ge0}\mathcal{V}_m$ and $\mathcal{E} := \cup_{m\ge0}\mathcal{E}_m$.

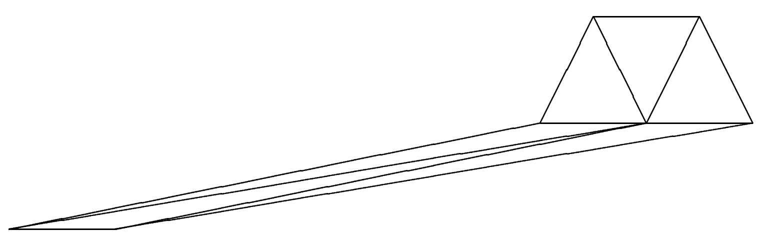

Figure 1: A skewed cell in an LR-triangulation

We next clarify a number of issues concerning LR-triangulations, which are discussed in detail in [9] (see also [5, 10]).

The constants M_0, N_0, r, ρ, δ, and $\#\mathcal{T}_0$ (the cardinality of $\mathcal{T}_0$) associated with an LR-triangulation $\mathcal{T}$ are assumed fixed. We refer to them as *parameters* of $\mathcal{T}$.

It is an important observation that the collection of all LR-triangulations with given (fixed) parameters is invariant under affine transforms. More precisely, if $\mathcal{T}$ is an LR-triangulation of $E \subset \mathbb{R}^2$ and $\mathbf{A}$ is an affine transform of $\mathbb{R}^2$, then $\mathbf{A}(\mathcal{T}) := \{\mathbf{A}(\triangle) : \triangle \in \mathcal{T}\}$ is an LR-triangulation of the polygonal domain $\mathbf{A}(E)$ with the same parameters.

The most important conditions (f)-(g) on LR-triangulations involve only areas of triangles but not angles. Consequently, if $\mathcal{T}$ is an LR-triangulation and $\triangle', \triangle'' \in \mathcal{T}_m$ have a common edge, then it may happen that $\triangle'$ is an equilateral triangle (or close to an equilateral triangle) but $\triangle''$ has a uncontrollably sharp angle (see Figure 1).

In an LR-triangulation $\mathcal{T}$ there can be an equilateral (or close to such) triangle $\triangle^\diamond$ at any level T_m with descendants $\triangle_1 \supset \triangle_2 \supset \dots$ such that $\min\text{angle}(\triangle_j) \to 0$ as $j \to \infty$.

It is important to know how fast the area $|\triangle|$ of a triangle $\triangle \in \mathcal{T}_m$ may change when $\triangle$ moves away from a fixed triangle within the same level. Condition (f) suggests a geometric rate of change but in fact it is polynomial.

Lemma 2.1. *If $\triangle$, $\triangle' \in \mathcal{T}_m$ can be connected by n intermediate edges from $\mathcal{E}_m$, then*

$$c_1^{-1}(n+1)^{-s} \le |\triangle'|/|\triangle''| \le c_1(n+1)^s, \tag{2.3}$$

where $s, c_1 > 0$ depend only on the parameters of $\mathcal{T}$.

This result follows easily by the following lemma (see [9], Lemma 2.4):

Lemma 2.2. *Let $\mathcal{T}$ be an LR-triangulation of $E \subset \mathbb{R}^2$. Suppose that $\triangle', \triangle'' \in \mathcal{T}_m$ (with m sufficiently large), and $\triangle'$ and $\triangle''$ can be connected by $< 2^\nu$ intermediate edges from $\mathcal{E}_m$ with (pairwise) common vertices. Then there exist $\triangle_1, \triangle_2 \in \mathcal{T}_{m-2N_0\nu}$ with a common vertex such that $\triangle' \subset \triangle_1$ and $\triangle'' \subset \triangle_2$.*

From the above discussion (see Figure 1) it follows that for two triangles $\triangle', \triangle'' \in \mathcal{T}_m$ which share a vertex $|\max\text{edge}\,(\triangle')|/|\max\text{edge}\,(\triangle'')|$ can by uncontrollably large (or small). However, when going in depth the maximal edges of the triangles behave similarly as their areas.

Lemma 2.3. *If $\mathcal{T}$ is an LR-triangulation of E, there exist constants $0 < r_1 < \rho_1 < 1$ depending only on the parameters of $\mathcal{T}$ such that if $\triangle' \subset \triangle$, $\triangle \in \mathcal{T}_m$ $(m \ge 0)$, and $\triangle' \in \mathcal{T}_{m+3N_0\nu}$, $\nu \ge 1$, then*

$$r_1^\nu \le \frac{|\max\text{edge}\,(\triangle')|}{|\max\text{edge}\,(\triangle)|} \le \rho_1^\nu. \tag{2.4}$$

Proof. Evidently, it suffices to prove (2.4) for $\nu = 1$ only. Let $\triangle \in \mathcal{T}_m$ and let e be an edge of $\triangle$. If it is also an edge of a child of $\triangle$, then the valence of at least one of the two endpoints of e will increase by one at level $m+1$. (Recall that there are always at least two children, so that a child and a parent cannot be the same triangle.) Therefore, e will be subdivided at least once after at most $S := 2(N_0 - 3) + 1$ steps of refinement. By (2.1) it readily follows that any edge e' obtained by subdividing e satisfies $|e'| \le \rho|e|$.

We say that an edge of a descendant of $\triangle$ is a *cutting edge* for $\triangle$ if one of its endpoints is a vertex of $\triangle$ and the other lies in the interior of the opposite edge of $\triangle$. Since all cutting edges must emanate from the same vertex of $\triangle$, there are totally no more than $M := N_0 - 3$ such edges for $\triangle$. Therefore, no new cutting edges for $\triangle$ will be created at levels $l > m + N_0 - 3$. (It is easy to see that, as long as no new cutting edges are created at a level l, they cannot be created at any further level.) Using this and the above observation, we conclude that there will be no cutting edges at levels $l > m + M + S$ since they all will be subdivided. Therefore, each edge e' inside $\triangle$ at these levels is either a proper part of an edge of $\triangle$, or has both of its endpoints in the interiors of two different edges of $\triangle$, or it has at least one endpoint in the interior of $\triangle$. In all cases, condition (2.1) ensures that $|e'| \le \rho\,|\max\text{edge}\,(\triangle)|$. Consequently, if $\triangle' \in \mathcal{T}_{m+3N_0}$ then $|\max\text{edge}\,(\triangle')| \le \rho\,|\max\text{edge}\,(\triangle)|$, since $3N_0 > M+S+1$. Thus the upper bound in (2.4) is established.

The argument for the proof of the lower bound in (2.4) is simpler. Suppose $\triangle \in \mathcal{T}_m$, $\triangle' \in \mathcal{T}_{m+1}$, and $\triangle' \subset \triangle$. Let $e_{\max}$ and $e'_{\max}$ be the largest edges of $\triangle$ and $\triangle'$, respectively. Denote by h the length of the height to $e_{\max}$ in $\triangle$ and by h' the length of the height to $e'_{\max}$ in $\triangle'$. Further, let R and R' be the radii of the circles inscribed in $\triangle$ and $\triangle'$ respectively. A simple geometric argument shows that $R < h < 3R$ as well as $R' < h' < 3R'$. Since $\triangle' \subset \triangle$, then $R' \le R$ and hence $h' < 3h$. We use this and (2.1) to obtain

$$(1/2)r|e_{\max}|h = r|\triangle| \le |\triangle'| \le (1/2)|e'_{\max}|h' \le (3/2)|e'_{\max}|h$$

which implies $|e'_{\max}| \ge (r/3)|e_{\max}|$. This obviously yields the lower bound in (2.4). $\square$

Graph distance. We next introduce the mth level graph distance between vertices, which will play a vital role in our further development: For any two vertices $v', v'' \in \mathcal{T}_m$, $m \ge 0$, we define the *graph distance* $\rho_m(v', v'')$ as the minimum number of edges from $\mathcal{E}_m$ needed to connect v' and v''.

By the conditions on LR-triangulations, in particular condition (d), it follows that every edge in $\mathcal{E}$ is divided at least once after $2N_0$ steps of refinement. This immediately implies the important inequality:

$$\rho_{m+2N_0\nu}(v', v'') \ge 2^\nu \rho_m(v', v''), \quad v', v'' \in \mathcal{V}_m, \quad m, \nu \ge 0. \tag{2.5}$$

The following lemma is a consequence of Lemma 2.2.

Lemma 2.4. *There exist constants $c > 0$ and $t > 0$ depending only on the parameters of $\mathcal{T}$ such that for any $v^\diamond \in \mathcal{V}_m$*

$$\#\{v \in \mathcal{V}_m : \rho_m(v, v^\diamond) \le n\} \le cn^t, \quad n \ge 1. \tag{2.6}$$

Furthermore, for any $v', v'' \in \mathcal{V}_m$ with $\rho_m(v', v'') = n$,

$$\#\{v \in \mathcal{V}_m : \rho_m(v, v') + \rho_m(v, v'') = n + k\} \le c(n+k)^t, \quad k \ge 0. \tag{2.7}$$

Proof. To prove (2.6) choose $\nu \ge 1$ so that $2^{\nu-1} \le n < 2^\nu$. Assume first that $m \ge 2N_0\nu$. Denote by $\mathcal{T}^\diamond$ the set of all triangles $\triangle \in \mathcal{T}_m$ which have at least one vertex in the set $\{v \in \mathcal{V}_m : \rho_m(v, v^\diamond) \le n\}$. Let $\triangle^\diamond \in \mathcal{T}^\diamond$ have $v^\diamond$ as a vertex. Applying Lemma 2.2 with $\triangle^\diamond$ and any other triangle in $\mathcal{T}^\diamond$, it readily follows that there exists a set $\mathcal{T}^0$, say, consisting of a triangle $\triangle_0 \in \mathcal{T}_{m-2N_0\nu}$ and its neighbors (triangles in $\mathcal{T}_{m-2N_0\nu}$ which share a vertex with $\triangle_0$) such that $\cup_{\triangle \in \mathcal{T}^\diamond} \triangle \subset \cup_{\triangle \in \mathcal{T}^0} \triangle$. Evidently $\#\mathcal{T}^0 < 3N_0$.

Let $\triangle_{\min}$ be a triangle in $\mathcal{T}^\diamond$ with maximum area. By (2.1)-(2.2), we infer $\delta r^{2N_0\nu}|\triangle_0| \le |\triangle_{\min}|$. We use this to obtain

$$\#\mathcal{T}^\diamond|\triangle_{\min}| \le \sum_{\triangle \in \mathcal{T}^0} |\triangle| \le 3N_0\delta^{-1}|\triangle_0| \le 3N_0\delta^{-2}r^{-2N_0\nu}|\triangle_{\min}|$$

and hence

$$\#\{v \in \mathcal{V}_m : \rho_m(v, v^\diamond) \le n\} \le 3\#\mathcal{T}^\diamond \le 9N_0\delta^{-2}r^{-2N_0\nu} \le cn^t$$

for some $c, t > 0$ depending on N_0, r, and δ.

If $m < 2N_0\nu$ one proceeds in the same way using $\mathcal{T}_0$ in place of $\mathcal{T}^0$.

Estimate (2.7) follows easily by (2.6). $\square$

Cells. For any vertex $v \in \mathcal{V}_m$ ($m \geq 0$), we denote by θ_v the union of all triangles from $\mathcal{T}_m$ which have v as a common vertex. (Here we take into account the above observation about the multiplicities of vertices from $\mathcal{V}_m$). We denote by Θ_m the set of all such cells θ_v with $v \in \mathcal{V}_m$ and set $\Theta = \bigcup_{m\geq 0} \Theta_m$. For a given cell $\theta \in \Theta$, we shall denote by v_θ the "central" vertex of θ. We let $l(\theta)$ denote the *level* of θ. Thus $l(\theta) = m$ if $\theta \in \Theta_m$.

For given $\theta', \theta'' \in \Theta_m$, we define the *graph distance* $\rho_m(\theta', \theta'')$ between θ' and θ'' by $\rho_m(\theta', \theta'') := \rho_m(v_{\theta'}, v_{\theta''})$, where $v_{\theta'}, v_{\theta''} \in \mathcal{V}_m$ are the "central vertices" of θ', θ''. Evidently, $\rho_m(\cdot, \cdot)$ is a true distance on Θ_m.

Furthermore, if $\theta', \theta'' \in \Theta_{m-1} \cup \Theta_m$ then $v_{\theta'}, v_{\theta''} \in \mathcal{V}_m$, and we define the *$m$th level graph distance* $\rho_m(\theta', \theta'')$ between θ' and θ'' by $\rho_m(\theta', \theta'') := \rho_m(v_{\theta'}, v_{\theta''})$. Evidently, if $\Lambda \subset \Theta_{m-1} \cup \Theta_m$ consists of cells with distinct "central" points, then $\rho_m(\cdot, \cdot)$ is a true distance on Λ. This will be needed in §3.2.

Definition of θ_x^m. We want to associate with each $x \in E$ a cell $\theta_x^m \in \Theta_m$, $m \geq 0$, which contains x. Since the cells from Θ_m overlap this needs some care. We first associate with each triangle $\triangle \in \mathcal{T}_m$ a cell $\theta_\triangle^m \in \Theta_m$ such that $\triangle \subset \theta_\triangle^m$. Such a cell can be selected in three different ways. We choose one of them for each $\triangle \in \mathcal{T}_m$. Then for each $x \in E$ such that $x \in \triangle^\circ$ with $\triangle \in \mathcal{T}_m$, we define $\theta_x^m := \theta_\triangle^m$. If x lies on the edge of a triangle from $\mathcal{T}_m$, we define θ_x^m as any cell from θ_m such that x belongs to its interior, but if $x = v_\theta$ for some $\theta \in \Theta_m$, we set $\theta_x^m := \theta$.

From the above definition of θ_x^m it readily follows that for any $\theta \in \Theta_m$ the function $h(x) := \rho_m(\theta, \theta_x^m)$ is piecewise constant over $\mathcal{T}_m$ and hence it is measurable, which will be needed later on.

We now introduce the mth level graph "distance" between any two points $x, y \in E$ by

$$\rho_m(x, y) := \rho_m(\theta_x^m, \theta_y^m). \tag{2.8}$$

The following useful inequality is immediate from (2.5): For $m \geq 0$, $\nu \geq 1$, and $x, y \in E$,

$$\rho_{m+2N_0\nu}(x, y) \geq 2^{\nu-2}\rho_m(x, y), \quad \text{if} \quad \rho_m(x, y) \geq 3. \tag{2.9}$$

Stars. In order to deal with graph distances and neighborhood relations it is convenient to employ the notion of the mth *level star of a set*: For any set $G \subset E$ and $m \geq 0$, we define the first mth level star of G by

$$\operatorname{Star}_m(G) := \operatorname{Star}_m^1(G) := \cup\{\theta \in \Theta_m : \theta^\circ \cap G \neq \emptyset\} \tag{2.10}$$

and inductively

$$\operatorname{Star}_m^k(G) := \operatorname{Star}_m^1(\operatorname{Star}_m^{k-1}(G)), \qquad k > 1. \tag{2.11}$$

When G consists of a single point x, in slight abuse of notation, we shall write $\operatorname{Star}_m^k(x)$ instead of $\operatorname{Star}_m^k(\{x\})$. For instance, $\operatorname{Star}_m^1(v) = \theta_v$ if $v \in \mathcal{V}_m$.

Courant elements. The no-hanging-vertices condition (e) on LR-triangulations guarantees the existence of *Courant elements*, that is, for every cell $\theta \in \Theta_m$ there exists a unique continuous piecewise linear function φ_θ on E which is supported on θ and satisfies $\varphi_\theta(v_\theta) = 1$. We denote $\Phi_m := \Phi_{m,\mathcal{T}} := (\varphi_\theta)_{\theta\in\Theta_m}$.

We let $\mathcal{S}_m$ denote the space of all continuous piecewise linear functions over $\mathcal{T}_m$. Clearly, $S \in \mathcal{S}_m$ if and only if $S = \sum_{v\in\mathcal{V}_m} S(v)\varphi_{\theta_v}$. Evidently, $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ \dots$ and by Lemma 2.3 it follows that $\overline{\cup_{m\geq 0}\mathcal{S}_m} = L_p(E)$, $0 < p \leq \infty$.

We shall frequently use the obvious fact that all norms of a polynomial on a triangle are equivalent, namely, if P is a polynomial of degree $\leq k$ and $\triangle$ is a triangle in $\mathbb{R}^2$, then

$$\|P\|_{L_p(\triangle)} \approx |\triangle|^{1/p-1/q}\|P\|_{L_q(\triangle)}, \quad 0 < p, q \leq \infty, \tag{2.12}$$

with constants of equivalence depending only on k, p, and q.

The L_p-stability of $\Phi_m = (\varphi_\theta)_{\theta\in\Theta_m}$ is immediate from (2.12). In fact we shall need the following obvious modification of this fact: Let $\Lambda \subset \Theta_{m-1} \cup \Theta_m$ consists of cells with distinct "central points". If $(a_\theta)_{\theta\in\Lambda}$, is an arbitrary sequence of real numbers and $S := \sum_{\theta\in\Lambda} a_\theta\varphi_\theta$, then

$$||S||_p \approx \Big(\sum_{\theta\in\Lambda} \|a_\theta\varphi_\theta\|_p^p\Big)^{1/p} \approx \Big(\sum_{\theta\in\Lambda} |\theta||a_\theta|^p\Big)^{1/p}, \quad 0 < p \leq \infty. \tag{2.13}$$

2.2 Quasi-distance and maximal operators

Here we introduce a quasi-distance and maximal operators induced by LR-triangulations.

We begin by recalling the definition of a quasi-distance on a set X: The mapping $d : X \times X \to [0, \infty)$ is called a *quasi-distance* on X if for $x, y, z \in X$,

(a) $d(x, y) = 0 \Longleftrightarrow x = y$,
(b) $d(y, x) = d(x, y)$,
(c) $d(x, z) \leq K(d(x, y) + d(y, z))$ with $K \geq 1$.

Assuming that $\mathcal{T}$ is an LR-triangulation of a polygonal domain $E \subset \mathbb{R}^2$, we define the quasi-distance $d_\mathcal{T} : E \times E \to [0, \infty)$ by

$$d_\mathcal{T}(x, y) := \min\{|\theta| : \theta \in \Theta \ \text{ and } \ x, y \in \theta\}, \tag{2.14}$$

if x, y belong to at least one cell from Θ, and by $d_\mathcal{T}(x, y) := |E|$ otherwise.

Lemma 2.5. *The mapping $d_\mathcal{T} : E \times E \to [0, \infty)$ defined above is a quasi-distance on E.*

Proof. Condition (a)-(b) on quasi-distances are apparently satisfied by $d_\mathcal{T}(\cdot, \cdot)$. To prove that condition (c) holds let x, y, z be three distinct points in E. Assume that $d(x, z) = |\theta'|$, where $\theta' \in \Theta_m$ is a cell containing x, z and let $d(y, z) = |\theta''|$ for some cell $\theta'' \in \Theta_n$ containing y, z. Suppose $m \leq n$. Since $x, z \in \theta'$, x and z lie in two triangles from $\mathcal{T}_m$ with a common vertex or in the same triangle. Since $m \leq n$, the same is true for y and z. In other words there exist triangles $\triangle_1, \triangle_2 \in \mathcal{T}_m$ which can be connected by $< 2^2$ edges from $\mathcal{T}_m$, so that $x \in \triangle_1$, $y \in \triangle_2$.

Assume that $m \geq 4N_0$. Then by Lemma 2.2 there exists $\theta \in \Theta_{m-4N_0}$ such that $\triangle_1, \triangle_2 \subset \theta$ and hence $d(x, y) \leq |\theta|$. By (2.1)-(2.2) there exists a constant c depending on the parameters of $\mathcal{T}$ such that $|\theta| \le c|\theta'|$. Therefore, $d(x,y) \le c(d(x,z)+d(z,y))$. If $m < 4N_0$, we use E instead of θ with the same result. □

With the following lemma we relate the quasi-distance $d_{\mathcal{T}}(\cdot,\cdot)$ to the mth level graph distance introduced in §2.1.

Lemma 2.6. *There exist constants $\beta > 0$ and $c > 0$ such that for $\theta \in \Theta_m$ $(m \ge 0)$ and $x \in E$,*

$$d_{\mathcal{T}}(v_\theta, x) \le c|\theta|\rho_m(\theta, \theta_x^m)^\beta \quad \text{if} \quad \rho_m(\theta, \theta_x^m) \ge 2. \tag{2.15}$$

Proof. Clearly, if we prove (2.15) when $x = v'$ with $v' \in \mathcal{V}_m$, then it will hold in general with a different constant c. Let $d_{\mathcal{T}}(v_\theta, v') = |\omega|$ with $\omega \in \Theta_\ell$ $(\ell \ge 0)$, i.e. the cell ω is of minimum area and $v_\theta, v' \in \omega$. Let $\omega^* \in \Omega$ be of maximum level such that $v_\theta, v' \in \omega^*$. Evidently, $|\omega| \le |\omega^*|$. Since $\omega \cap \omega^* \ne \emptyset$, it follows by (2.1)-(2.2) that

$$l(\omega) \le l(\omega^*) \le l(\omega) + \nu_0$$

where $\nu_0 > 0$ is a constant depending only on the parameters of $\mathcal{T}$.

Our next claim is that $\rho_m(v_\theta, v') \ge 2^n$, where $n := [(m - l(\omega^*))/2N_0] + 1$. Assume to the contrary that $\rho_m(v_\theta, v') < 2^n$. Then by Lemma 2.2 it follows that there is $\eta \in \Theta_{m-2N_0 n}$ such that $v_\theta, v' \in \eta$ and hence $m - 2N_0 n \ge l(\omega^*)$ or $n \le (m - l(\omega^*)/2N_0$, which is a contradiction. Therefore,

$$\rho_m(v_\theta, v') \ge 2^{(m-l(\omega^*))/2N_0} \ge 2^{(m-l(\omega)-\nu_0)/2N_0} \ge c2^{(m-l(\omega))/2N_0}. \tag{2.16}$$

Evidently, by (2.1)-(2.2), $|\omega|/|\theta| \le c(1/r)^{m-l(\omega)}$. Combining this with (2.16), there exists $\beta > 1$ such that

$$d_{\mathcal{T}}(v_\theta, v') = |\omega| \le c|\theta|\left(2^{(m-l(\omega))/2N_0}\right)^\beta \le c|\theta|\rho_m(v_\theta, v')^\beta.$$

Above it may happen that $\omega = \theta_0 := E$. The proof is the same. □

The quasi-distance $d_{\mathcal{T}}(\cdot,\cdot)$ induces a maximal operator. Denote by $B(y,a)$ the "ball" centered at y of radius $a > 0$ with respect to this quasi-distance, i.e. $B(y,a) := \{x : d_{\mathcal{T}}(x,y) < a\}$. Then for any $s > 0$ the maximal operator $\mathcal{M}^s_{d_{\mathcal{T}}}$ is defined by

$$(\mathcal{M}^s_{d_{\mathcal{T}}} f)(x) := \sup_{B:\, x \in B} \left(\frac{1}{|B|}\int_B |f(y)|^s\, dy\right)^{1/s}, \quad x \in E, \tag{2.17}$$

where the supremum is over all balls B containing x.

For our purposes it is more convenient to use the equivalent maximal operator $\mathcal{M}^s_{\mathcal{T}}$ defined by

$$(\mathcal{M}^s_{\mathcal{T}} f)(x) := \sup_{\theta:\, x \in \theta} \left(\frac{1}{|\theta|}\int_\theta |f(y)|^s dy\right)^{1/s} \tag{2.18}$$

where the supremum is over all cells $\theta \in \Theta$ containing x or $\theta = E$.

Lemma 2.7. *For any measurable function f*

$$\mathcal{M}^s_{\mathcal{T}} f(x) \approx \mathcal{M}^s_{d_{\mathcal{T}}} f(x), \quad x \in \mathbb{R}^2, \tag{2.19}$$

where the constants of equivalence depend only on s and the parameters of $\mathcal{T}$.

This equivalence is immediate from the following lemma which will be needed later on as well.

Lemma 2.8. (*a*) *Given a ball* $B = B(x, r)$, $x \in E$, $r > 0$, *there exist* $\theta' \in \Theta$ *and* $\theta'' \in \Theta$ *or* $\theta'' = E$ *such that*

$$\theta' \subset B \subset \theta'' \quad \text{and} \quad r \leq |\theta''| \leq c_1|B| \leq c_2|\theta'| < c_2 r. \tag{2.20}$$

(*b*) *For any* $\theta \in \Theta$ *there exists a ball* $B \subset E$ *(with respect to the quasi-distance* $d_{\mathcal{T}}$*) such that*

$$\theta \subset B \quad \text{and} \quad |B| \leq c|\theta| \tag{2.21}$$

Here the constants depend only on the parameters of $\mathcal{T}$.

Proof. (a) Fix a ball $B = B(x, r)$, $x \in E$, $r > 0$. Let θ' be a cell of minimal level, say m, such that $x \in \theta' \subset B$. Clearly,

$$B \subset \bigcup_{\theta \in \Theta_m:\, x \in \theta} \theta \subset \operatorname{Star}_m^2(x)$$

and by Lemma 2.2 there exists $\theta'' \in \Theta_{m-4N_0}$ or $\theta'' = E$ if $m < 4N_0$ such that

$$\theta' \subset B \subset \operatorname{Star}_m^2(x) \subset \theta''.$$

By properties (f)-(g) of LR-triangulations (§2.1), it follows that $|\theta''| \leq c|\theta'|$. Evidently $|\theta'| < r$ and $|\theta''| \geq r$, and (2.20) follows.

(b) Suppose that $\theta \in \Theta_n$ ($n \geq 0$) is with "central" vertex v. Let $\delta := \max\{|\theta| : \theta \subset \operatorname{Star}_n^2(v)\}$. Then for sufficiently small $\varepsilon > 0$,

$$\theta \subset B(v, \delta + \varepsilon) = \bigcup_{|\tilde{\theta}| < \delta + \varepsilon:\, v \in \tilde{\theta}} \tilde{\theta} \subset \operatorname{Star}_n^2(v),$$

which yields $|B(v, \delta + \varepsilon)| \leq c|\theta|$. □

We now come to the main point in this subsection. It is well-known that the Fefferman-Stein vector valued maximal inequality holds for maximal functions generalized by quasi-distances as in our case (see [15]). This combined with Lemma 2.7 gives the needed maximal inequality:

Proposition 2.9. *Let* $\mathcal{T}$ *be an LR-triangulation of* $E \subset \mathbb{R}^2$. *If* $0 < p < \infty$, $0 < q \leq \infty$, *and* $0 < s < \min\{p, q\}$, *then for any sequence of functions* $(f_j)_{j=1}^{\infty}$ *on* E,

$$\Big\| \Big(\sum_{j=1}^{\infty} |\mathcal{M}_{\mathcal{T}}^s f_j|^q \Big)^{1/q} \Big\|_p \leq c \Big\| \Big(\sum_{j=1}^{\infty} |f_j|^q \Big)^{1/q} \Big\|_p, \tag{2.22}$$

where c *depends only on* p, q, s, *and the parameters of* $\mathcal{T}$.

2.3 Spaces of homogeneous type on polygonal domains

Spaces of homogeneous type were first introduced in [2] as a means to extend the Calderon-Zygmund theory of singular integral operators to more general settings.

Let X be a topological space endowed with a Borel measure μ and a quasi-distance $d(\cdot,\cdot)$ (see §2.2). Assume that the balls $B(x,r) := \{y \in X : d(x,y) < r\}$, $x \in X$, $r > 0$, form a basis for the topology T in X, and $\mu(B(x,r)) > 0$ if $r > 0$. The space (X, d, μ) is said to be of *homogeneous type* if there exists a constant A such that for all $x \in X$ and $r > 0$,

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x,r)). \tag{2.23}$$

The space of homogeneous type $(E, d_{\mathcal{T}}, m)$. Suppose that E is a bounded polygonal domain and let $\mathcal{T}$ be a LR-triangulation on E. Also, let $d_{\mathcal{T}}(\cdot,\cdot)$ be the quasi-distance on E, defined in (2.14). Finally, denote by m the Lebesgue measure on E. It is easy to see that $(E, d_{\mathcal{T}}, m)$ is a space of homogeneous type, so that we can utilize the machinery developed in [2]. Indeed, by Lemma 2.5, $d_{\mathcal{T}}(\cdot,\cdot)$ is a quasi-distance on E and evidently $m(B(x,r)) = |B(x,r)| > 0$ for $x \in E$ and $r > 0$. Further, it follows by Lemma 2.8 that condition (2.23) is fulfilled as well.

The Hardy space $H_1(E, \mathcal{T})$. We next define the Hardy space $H_1 := H_1(E, \mathcal{T})$ associated with the space $(E, d_{\mathcal{T}}, m)$ by means of atomic representations (see [3]).

According to Coifmann and Weiss [3], a function $a(x)$ is said to be a q-atom ($1 < q \leq \infty$) if there exist $x_0 \in E$ and $r > 0$ such that

(i) $\operatorname{supp} a \subset B(x_0, r)$, (ii) $\|a\|_q \leq |B(x_0,r)|^{1/q-1}$, (iii) $\int a(x)\,dx = 0$.

In addition, $|E|^{-1}\mathbb{1}_E$ is by definition a q-atom as well.

We adopt the following slightly different but equivalent definition for a q-atom which better suits our purposes.

Definition 2.10. A function $a(x)$ is said to be a q-atom ($1 < q \leq \infty$) for $H_1(E, \mathcal{T})$ if there is $\theta \in \Theta$ or $\theta = E$ such that

(a) $\operatorname{supp} a \subset \theta$,

(b) $\|a\|_q \leq |\theta|^{1/q-1}$,

(c) $\int_E a(x)\,dx = 0$.

We also postulate $|E|^{-1}\mathbb{1}_E$ to be a q-atom.

The equivalence of the two definitions for a q-atom is immediate by Lemma 2.8,(a).

Definition 2.11. The space $H_1^q := H_1^q(E, \mathcal{T})$ ($1 < q \leq \infty$) is defined as the set of all functions $f \in L_1(E)$ admitting an atomic decomposition

$$f = \sum_{j=0}^{\infty} \lambda_j a_j,$$

where the a_j's are q-atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. Moreover, the norm of $f \in H_1^q$ is given by

$$\|f\|_{H_1^q} := \inf\Big\{ \sum_{j=0}^{\infty} |\lambda_j| : f = \sum_{j=0}^{\infty} \lambda_j a_j,\ a_j\ q\text{-atoms}\Big\}.$$

A fundamental fact in the theory of Hardy spaces is that $H_1^q = H_1^\infty$ whenever $1 < q \leq \infty$ with equivalent norms (see [3], Theorem A). Thus all spaces H_1^q are the same and we shall drop the index q. In the following we shall only work with the norm in H_1 defined by using 2-atoms.

An important fact is that the spaces $H_1(E, \mathcal{T})$ essentially depend on the triangulations $\mathcal{T}$. We call $H_1(E, \mathcal{T}^*)$ a *regular* H_1*-space* if $\mathcal{T}^*$ is a regular multilevel triangulation of E (see §2.1). It is readily seen that if $H_1(E, \mathcal{T}^*)$ is regular, then it is the same (with equivalent norms) as the space $H_1(E)$ defined using atoms generated by the Euclidean distance on E. Thus all regular spaces $H_1(E, \mathcal{T})$ are the same. Consider the case when $E := [-1, 1]^2$ and denote by $H_1(E)$ the regular H_1-space on $[-1, 1]^2$. As will be shown in the appendix there exists an LR-triangulation $\mathcal{T}$ such that $H_1(E, \mathcal{T}) \neq H_1(E)$. The reason for this is that there exist LR-triangulations on $[-1, 1]^2$ containing triangles with uncontrollably sharp angles (see §2.1). The fact that the spaces $H_1(E, \mathcal{T})$, where $\mathcal{T}$ is allowed to vary, are not all the same is not a surprise since as is well known the norm in $H_1(\mathbb{R}^d)$ $(d > 1)$ is not invariant (like the L_1-norm) under linear transforms with determinant one. We do not explore in more detail the relationship between the various spaces $H_1(E, \mathcal{T})$ in this article.

It is not hard to prove that $H_1(E, \mathcal{T})$ is a Banach space and $\|f\|_{L_1(E)} \leq c\|f\|_{H_1(E,\mathcal{T})}$ for $f \in H_1(E, \mathcal{T})$.

Another fundamental result is that the dual of $H_1(E, \mathcal{T})$ is the space $BMO := BMO(E, \mathcal{T})$ which can be defined in our case as the set of all functions f on E such that

$$\|f\|_{BMO} := \Big|\int_E f(x)dx\Big| + \sup_\theta \Big(\frac{1}{|\theta|}\int_\theta |f(x) - f_\theta|^2 dx\Big)^{1/2} < \infty, \tag{2.24}$$

where $f_\theta := \frac{1}{|\theta|}\int_\theta f(x)dx$ and the supremum is taken over all $\theta \in \Theta$ or $\theta = E$. More precisely, for $g \in BMO(E, \mathcal{T})$ and $f \in H_1(E, \mathcal{T})$ with an atomic decomposition $f = \sum_{j=1}^\infty \lambda_j a_j$,

$$\lim_{m\to\infty} \sum_{j=1}^m \lambda_j \int_E g(x)a_j(x)dx \tag{2.25}$$

defines a continuous linear functional on H_1 whose norm is equivalent to $\|g\|_{BMO}$ and vice versa each continuous linear functional on H_1 is of this form.

Note that an equivalent norm in $BMO(E, \mathcal{T})$ can be defined by replacing in (2.24) $\Big(\frac{1}{|\theta|}\int_\theta |f(x) - f_\theta|^2 dx\Big)^{1/2}$ by $\frac{1}{|\theta|}\int_\theta |f(x) - f_\theta|dx$. For more details, see [3].

Finally, we observe that since $H_1(E, \mathcal{T}) \neq H_1(E)$ for some LR-triangulations $\mathcal{T}$, then by a duality argument it follows that for the same triangulations $BMO(E, \mathcal{T}) \neq BMO(E)$, where $BMO(E)$ stands for the regular BMO space on E. Thus in general $BMO(E, \mathcal{T})$ depends on the triangulation $\mathcal{T}$.

One of the advantages of introducing H_1 via atomic decompositions is that questions related to the boundedness of Calderon-Zygmund operators (CZO) on H_1 can be answered by focusing on individual atoms. Evidently, any operator T would be bounded if T maps atoms into atoms. Coifman and Weiss observed that for certain type of operators T, for every atom $a(x)$, Ta is a function with similar structure, which they term a *molecule.* We shall use the following definition of a molecule.

Definition 2.12. For a given $\varepsilon > 0$, we say that $m(x)$ is an ε-molecule for $H_1(E, \mathcal{T})$ centered at $x_0 \in E$ if

$$\Big(\int_E |m(x)|^2\, dx\Big)\Big(\int_E |m(x)|^2 d_{\mathcal{T}}(x, x_0)^{1+\varepsilon} dx\Big)^{1/\varepsilon} \le 1 \tag{2.26}$$

and $\int_E m(x)dx = 0$.

It is trivial to see that every 2-atom is an ε-molecule for any $\varepsilon > 0$. More importantly, $\|m\|_{H_1} \le c$ for each ε-molecule $m(x)$ (see [3], Theorem C). From this it follows that a linear operator mapping atoms into molecules has a bounded extension to H_1.

The following result [3] will play an important role in our further development:

Proposition 2.13. *Let $T : L_2(E) \to L_2(E)$ be a bounded linear operator given by*

$$(Tf)(x) = \int_E K(x, y) f(y)\, dy.$$

Suppose that for each 2-atom $a \neq |E|^{-1}\mathbb{1}_E$

$$\int_E (Ta)(x)\, dx = 0 \tag{2.27}$$

and there is $\varepsilon > 0$ such that for $d(x, y_0) > cd(y, y_0)$ the kernel $K(\cdot, \cdot)$ satisfies

$$|K(x, y) - K(x, y_0)| \le c\Big(\frac{d(y, y_0)}{d(x, y_0)}\Big)^{\varepsilon} \frac{1}{d(x, y_0)}. \tag{2.28}$$

Then Ta is a constant multiple of an ε-molecule for any atom $a \neq |E|^{-1}\mathbb{1}_E$.

3 Anisotropic Franklin Bases

In this section we explore the Franklin system $\mathcal{F}_{\mathcal{T}}$ generated by the Courant elements associated with an arbitrary locally regular triangulation $\mathcal{T}$ of a compact polygonal domain E in $\mathbb{R}^2$. We shall show that each such Franklin system is a Schauder basis for $C(E)$ and $L_1(E)$, and it is an unconditional basis for $H_1(E, \mathcal{T})$ and $L_p(E)$ $(1 < p < \infty)$. We also prove some related results.

3.1 Definition of the Franklin system. Main results

Throughout this section, we assume that $\mathcal{T} = \bigcup_{m=0}^{\infty} \mathcal{T}_m$ is an LR-triangulation of E. We recall that $\mathcal{V}_m$ denotes the set of all vertices of triangles from $\mathcal{T}_m$. We set $\mathcal{V}_0^* = \mathcal{V}_0$ and $\mathcal{V}_m^* = \mathcal{V}_m \setminus \mathcal{V}_{m-1}$ for $m \ge 1$ and write $\mathcal{V}^* = \bigcup_{m=0}^{\infty} \mathcal{V}_m^*$.

Let $\theta_0 := E$. Choose $\theta_{\max} \in \Theta_0$ to be of maximum area and denote $\Theta_0^* := \{\theta_0\} \cup \Theta_0 \setminus \{\theta_{\max}\}$, i.e. we replace $\theta_{\max}$ by $\theta_0 = E$. Moreover, we associate θ_0 with $v_{\theta_{\max}}$ and set $\varphi_{\theta_0} := \mathbb{1}_{\theta_0}$. For $m \ge 1$ denote by Θ_m^* the set of all cells $\theta \in \Theta_m$ with "central" vertices $v_\theta \in \mathcal{V}_m^*$ and set $\Theta^* := \bigcup_{m=0}^{\infty} \Theta_m^*$.

Note that for each m, the set $\{\varphi_\theta : \theta \in \bigcup_{i=0}^m \Theta_i^*\}$ is linearly independent. Also, $\mathcal{S}_m = \operatorname{span}\{\varphi_\theta : \theta \in \Theta_m\} = \operatorname{span}\{\varphi_\theta : \theta \in \bigcup_{i=0}^m \Theta_i^*\}$. For $\theta \in \Theta$ we denote by $\tilde{\varphi}_\theta$ the L_2-normalized version of the Courant element φ_θ, i.e. $\tilde{\varphi}_\theta := \|\varphi_\theta\|_2^{-1}\varphi_\theta \approx |\theta|^{-1/2}\varphi_\theta$.

We consider an arbitrary (but fixed) *linear order* $\preceq$ on Θ^* satisfying the following conditions:

$$\text{(i) If } \theta \in \Theta_m^* \text{ and } \theta' \in \Theta_n^* \text{ with } m < n, \text{ then } \theta \preceq \theta' \text{ and (ii) } \theta_0 \preceq \theta, \forall \theta \in \Theta^*. \tag{3.1}$$

We now define the *Franklin system* $\mathcal{F}_\mathcal{T}$ by applying the Gram-Schmidt orthogonalization process to $\{\tilde{\varphi}_\theta\}_{\theta\in\Theta^*}$ in $L_2(E)$ with respect to the order $\preceq$. We obtain an orthonormal system $\mathcal{F}_\mathcal{T} := \{f_\theta\}_{\theta\in\Theta^*}$ in $L_2(E)$ consisting of continuous piecewise linear functions. Each Franklin function f_θ is uniquely determined (up to a multiple ± 1) by the conditions:

(a) $f_\theta \in \operatorname{span}\{\varphi_{\theta'} : \theta' \preceq \theta\}$.
(b) $\langle f_\theta, \varphi_{\theta'}\rangle = 0$ for all $\theta' \prec \theta$,
(c) $\|f_\theta\|_2 = 1$.

Note that $f_{\theta_0} = \pm\tilde{\mathbb{1}}_{\theta_0} := \pm|E|^{-1/2}\mathbb{1}_E$.

We next state our main results on Franklin systems $\mathcal{F}_\mathcal{T}$, where $\mathcal{T}$ is an arbitrary LR-triangulation of a bounded polygonal domain $E \subset \mathbb{R}^2$.

Theorem 3.1. *The Franklin system $\mathcal{F}_\mathcal{T} := \{f_\theta\}_{\theta\in\Theta^*}$ is a Schauder basis for $L_p(E)$, $1 \le p \le \infty$, with $L_\infty(E) := C(E)$.*

Theorem 3.2. *The Franklin system $\mathcal{F}_\mathcal{T} := \{f_\theta\}_{\theta\in\Theta^*}$ is an unconditional basis for $H_1(E,\mathcal{T})$ and $L_p(E), 1 < p < \infty$.*

Theorem 3.3. *The following conditions are equivalent:*

(*a*) $f \in H_1(E,\mathcal{T})$;

(*b*) *The series* $\sum_{\theta\in\Theta^*}\langle f, f_\theta\rangle f_\theta$ *converges unconditionally in* L_1;

(*c*) $S_f(x) := \Big(\sum_{\theta\in\Theta^*}|\langle f, f_\theta\rangle|^2|f_\theta(x)|^2\Big)^{1/2} \in L_1$;

(*d*) $F_f(x) := \Big(\sum_{\theta\in\Theta^*}|\langle f, f_\theta\rangle|^2|\tilde{\mathbb{1}}_\theta(x)|^2\Big)^{1/2} \in L_1$.

Furthermore, if $f \in H_1(E,\mathcal{T})$, then

$$\|f\|_{H_1} \approx \|S_f\|_{L_1} \approx \|F_f\|_{L_1}. \tag{3.2}$$

Theorem 3.4. *A function $f \in BMO(E,\mathcal{T})$ if and only if*

$$\sup_\theta \Big(\frac{1}{|\theta|}\sum_{\eta\in\Theta^*:\,\eta\subset\theta}|\langle f, f_\eta\rangle|^2\Big)^{1/2} < \infty, \tag{3.3}$$

where the supremum is taken over all $\theta \in \Theta$ or $\theta = E$. Furthermore, $\|f\|_{BMO(E,\mathcal{T})}$ is equivalent to the quantity in (3.3).

3.2 Representation of the Franklin functions and proof of Theorem 3.1

The exponential decay of the Franklin functions is a central issue in the study of Franklin systems. We begin with a generalization of the well-known result of Demko [6] on the inverses of band matrices, given in [13].

Proposition 3.5. *Suppose K is a finite set of indices and let ρ be a distance on K. Let $A = [a_{k,l}]_{k,l\in K}$ be an invertible band matrix of order $r \geq 1$, i.e. $a_{k,l} = 0$ if $\rho(k,l) > r$. Let $A^{-1} = [b_{k,l}]_{k,l\in K}$ be the inverse matrix of A. Suppose that for some $1 \leq p \leq \infty$,*

$$\|A\|_{\ell_p(K)\to\ell_p(K)} \leq M_1 \quad \text{and} \quad \|A^{-1}\|_{\ell_p(K)\to\ell_p(K)} \leq M_2.$$

Then there exist constants $c > 0$ and $0 < q < 1$ depending only on M_1, M_2, r, and p such that

$$|b_{k,l}| \leq cq^{\rho(k,l)} \quad \text{for} \quad k,l \in K.$$

For any $\eta \in \Theta_m^*$ $(m \geq 0)$, denote

$$\Lambda_\eta := \Theta_{m-1} \cup \{\theta \in \Theta_m^* : \theta \preceq \eta\}. \tag{3.4}$$

Note that the cells $\theta \in \Lambda_\eta$ have distinct "central" points v_θ and hence the set $\{\varphi_\theta : \theta \in \Lambda_\eta\}$ is linearly independent. Let G_η be the Gram matrix given by

$$G_\eta = [a_{\theta\theta'}]_{\theta,\theta'\in\Lambda_\eta} \quad \text{with} \quad a_{\theta\theta'} := \langle\tilde{\varphi}_\theta, \tilde{\varphi}_{\theta'}\rangle. \tag{3.5}$$

and denote $G_\eta^{-1} =: [b_{\theta\theta'}]_{\theta,\theta'\in\Lambda_\eta}$.

Lemma 3.6. *There exist constants $0 < q < 1$ and $c > 0$ such that for any $\eta \in \Theta_m^*$ $(m \geq 0)$ we have the following estimate for the entries of G_η^{-1}:*

$$|b_{\theta\theta'}| \leq cq^{\rho_m(\theta,\theta')}, \quad \theta,\theta' \in \Lambda_\eta, \tag{3.6}$$

where $\rho_m(\theta,\theta')$ is the mth level graph distance between θ and θ', introduced in §2.1.

Proof. By condition (c) on LR-triangulations, every triangle $\triangle \in \mathcal{T}$ has at most M_0 children. Then $a_{\theta\theta'} := \langle\tilde{\varphi}_\theta, \tilde{\varphi}_{\theta'}\rangle = 0$ if $\rho_m(\theta,\theta') > 2M_0$ and hence the Gram matrix G_η is r-banded with $r := 2M_0 + 1$. Since G_η is symmetric, then

$$\|G_\eta\|_{\ell_2(\Lambda_\eta)\to\ell_2(\Lambda_\eta)} = \max\{\lambda : \lambda \text{ eigenvalue of } G_\eta\} \quad \text{and}$$

$$\|G_\eta^{-1}\|_{\ell_2(\Lambda_\eta)\to\ell_2(\Lambda_\eta)} = \max\{1/\lambda : \lambda \text{ eigenvalue of } G_\eta\}.$$

On the other hand, for any vector $x := (x_\theta)_{\theta\in\Lambda_\eta}$ we have $\|\sum_{\theta\in\Lambda_\eta} x_\theta\tilde{\varphi}_\theta\|_{L_2(E)}^2 = \langle Gx, x\rangle$ and thus by (2.13) there exist constants $c_1, c_2 > 0$ such that

$$c_1\|x\|_{\ell_2(\Lambda_\eta)}^2 \leq \langle G_\eta x, x\rangle \leq c_2\|x\|_{\ell_2(\Lambda_\eta)}^2,$$

which implies $c_1 \le \lambda \le c_2$ for every eigenvalue λ. Hence

$$\|G_\eta\|_{\ell_2(\Lambda_\eta)\to\ell_2(\Lambda_\eta)} \le c_2 \quad \text{and} \quad \|G_\eta^{-1}\|_{\ell_2(\Lambda_\eta)\to\ell_2(\Lambda_\eta)} \le c_1^{-1}.$$

Finally, note that since the "central" points of the cells from Λ_η are distinct points in $\mathcal{V}_m$, the mth level graph distance $\rho_m(\cdot,\cdot)$ is a true distance on Λ_η. Thus G_η satisfies the conditions of Lemma 3.5 with $\rho(\cdot,\cdot) = \rho_m(\cdot,\cdot)$ and hence (3.6) holds. □

We are now prepared to deduce an important representation of the Franklin functions.

Lemma 3.7. *For any $\theta \in \Theta_m^*$ $(m \ge 0)$ the Franklin function f_θ has a representation of the form*

$$f_\theta = \sum_{\eta\in\Theta_m} c_{\theta\eta}\tilde{\varphi}_\eta, \tag{3.7}$$

with coefficients $c_{\theta\eta}$ satisfying

$$|c_{\theta\eta}| \le c q^{\rho_m(\theta,\eta)}, \quad \eta \in \Theta_m, \tag{3.8}$$

where the constants $0 < q < 1$ and $c > 0$ depend only on the parameters of $\mathcal{T}$.

Proof. It is readily seen (and well-known) that the function g_θ defined by

$$g_\theta := \sum_{\xi\in\Lambda_\theta} b_{\theta\xi}\tilde{\varphi}_\xi = \sum_{\xi\in\Theta_{m-1}} b_{\theta\xi}\tilde{\varphi}_\xi + \sum_{\xi\in\Theta_m^*:\,\xi\preceq\theta} b_{\theta\xi}\tilde{\varphi}_\xi, \tag{3.9}$$

where $b_{\theta\xi}$ are entries of G_θ^{-1}, has the property $\langle g_\theta, \tilde{\varphi}_\eta\rangle = \delta_{\theta\eta}$. Here, we set $b_{\theta\xi} := 0$ if $\xi \in \Theta_{-1}$. Therefore, $f_\theta = \pm\|g_\theta\|_2^{-1} g_\theta$. Evidently, for $\xi \in \Theta_{m-1}$,

$$\tilde{\varphi}_\xi = \sum_{\eta\in\Theta_m:\,\eta\subset\xi} \tilde{\varphi}_\xi(v_\eta)\|\varphi_\eta\|_2\tilde{\varphi}_\eta.$$

Substituting this in (3.9) and switching the order of summation, we arrive at

$$f_\theta = \sum_{\eta\in\Theta_m} c_{\theta\eta}\tilde{\varphi}_\eta,$$

where

$$c_{\theta\eta} = \|g_\theta\|_2^{-1}\|\varphi_\eta\|_2 \sum_{\xi\in\Theta_{m-1}:\,\eta\subset\xi} b_{\theta\xi}\tilde{\varphi}_\xi(v_\eta) \quad \text{if} \quad \eta \in \Theta_m \setminus \Lambda_\theta, \quad \text{and}$$

$$c_{\theta\eta} = \|g_\theta\|_2^{-1}\Big(b_{\theta\eta} + \|\varphi_\eta\|_2 \sum_{\xi\in\Theta_{m-1}:\,\eta\subset\xi} b_{\theta\xi}\tilde{\varphi}_\xi(v_\eta)\Big) \quad \text{if} \quad \eta \in \Theta_m \cap \Lambda_\theta.$$

Note that

$$1 = \langle g_\theta, \tilde{\varphi}_\theta\rangle \le \|g_\theta\|_2\|\tilde{\varphi}_\theta\|_2 = \|g_\theta\|_2$$

and $\rho_m(\eta,\xi) \le M_0$, whenever $\eta \in \Theta_{m-1}$, $\xi \in \Theta_m$, and $\xi \subset \eta$. Also, $\|\tilde\varphi_\xi\|_\infty = \|\varphi_\xi\|_2^{-1} \approx |\xi|^{-1/2}$. We use these along with properties (2.1)-(2.2) of LR-triangulations, and the estimate for $|b_{\theta\eta}|$ from Lemma 3.6 to obtain, for $\eta \in \Theta_m \cap \Lambda_\theta$,

$$\begin{aligned} |c_{\theta\eta}| &\le |b_{\theta\eta}| + c|\eta|^{1/2} \sum_{\xi\in\Theta_{m-1}:\,\eta\subset\xi} |b_{\theta\xi}|\tilde\varphi_\xi(v_\eta) \\ &\le cq^{\rho_m(\theta,\eta)} + c(|\eta|/|\xi|)^{1/2} \sum_{\xi\in\Theta_{m-1}:\,\eta\subset\xi} q^{\rho_m(\theta,\xi)} \le cq^{\rho_m(\theta,\eta)}. \end{aligned}$$

The estimate of $|c_{\theta\eta}|$ when $\eta \in \Theta_m \setminus \Lambda_\theta$ is the same. □

Proof of Theorem 3.1. As was already mentioned in §2.1, $\overline{\cup_{m=0}^\infty \mathcal{S}_m} = L_p(E)$, $1 \le p \le \infty$, and hence $\mathcal{F}_\mathcal{T}$ is dense in $L_p(E)$. It remains to prove that the orthogonal projector operator $P_\eta f := \sum_{\theta\preceq\eta}\langle f, f_\theta\rangle f_\theta$ is bounded on $L_p(E)$ ($1 \le p \le \infty$), i.e. considered as an operator from L_p into L_p. It is easy to see that

$$P_\eta f(x) = \int_E \sum_{\theta,\theta'\in\Lambda_\eta} b_{\theta,\theta'}\tilde\varphi_\theta(x)\tilde\varphi_{\theta'}(y)f(y)dy, \quad \Lambda_\eta := \Theta_{m-1} \cup \{\theta \in \Theta_m^* : \theta \preceq \eta\}.$$

Denote $K_\eta(x,y) := \sum_{\theta,\theta'\in\Lambda_\eta} b_{\theta,\theta'}\tilde\varphi_\theta(x)\tilde\varphi_{\theta'}(y)$. Then for each $x \in E$,

$$\begin{aligned} \|K_\eta(x,\cdot)\|_{L_1} &\le \sum_{\theta,\theta'\in\Lambda_\eta} |b_{\theta,\theta'}||\tilde\varphi_\theta(x)|\|\tilde\varphi_{\theta'}\|_{L_1} \\ &\le c \sum_{\theta,\theta'\in\Lambda_\eta} q^{\rho_m(\theta,\theta')}(|\theta|/|\theta'|)^{1/2}\mathbb{1}_\theta(x), \end{aligned}$$

where we used (3.6) and that $\|\tilde\varphi_\theta\|_2 = 1$. Our goal is to show that $\|K_\eta(x,\cdot)\|_{L_1} \le C < \infty$. Fix $x \in E$. Since there are at most 6 cells $\theta \in \Lambda_\eta$ such that $x \in \theta^\circ$, it suffices to show that for each $\theta \in \Lambda_\eta$, $\sum_{\theta'\in\Lambda_\eta} q^{\rho_m(\theta,\theta')}(|\theta|/|\theta'|)^{1/2} \le c < \infty$. By (2.3) it follows that if $\rho_m(\theta,\theta') = n$ ($n \ge 1$), then $|\theta|/|\theta'| \le cn^s$. On the other hand, Lemma 2.4 yields that $\#\{\theta' \in \Lambda_\eta : \rho_m(\theta,\theta') = n\} \le cn^t$. Using the above, we obtain

$$\sum_{\theta'\in\Lambda_\eta} q^{\rho_m(\theta,\theta')}(|\theta|/|\theta'|)^{1/2} \le c\sum_{n=1}^\infty q^n n^{t+s/2} \le c < \infty$$

since $0 < q < 1$. Consequently, $\|K_\eta(x,\cdot)\|_{L_1} \le C$. This estimate implies $\|P_\eta\|_{L_1\to L_1} < \infty$ and $\|P_\eta\|_{L_\infty\to L_\infty} < \infty$. By interpolation it follows that $\|P_\eta\|_{L_p\to L_p} < \infty$, $1 < p < \infty$. □

3.3 Localization and smoothness of the Franklin functions

Here we show that the Franklin functions belong to Lip ε (for some $\varepsilon > 0$) with respect to the quasi-distance $d_\mathcal{T}(\cdot,\cdot)$ introduced in (2.14) and have exponential rate of decay with respect to the corresponding graph distance.

We shall systematically use the notation introduced in §2.1. We recall, in particular, that for $x \in E$, θ_x^m is a cell from Θ_m containing x, and $\rho_m(\cdot,\cdot)$ is the mth level graph distance.

Theorem 3.8. *There exist constants $\varepsilon > 0$, $0 < q_1 < 1$, and $c > 0$ depending only on the parameters of $\mathcal{T}$ such that for any $\theta \in \Theta_m^*$ $(m \geq 0)$,*

$$|f_\theta(x) - f_\theta(y)| \leq c\frac{d_{\mathcal{T}}(x,y)^\varepsilon}{|\theta|^{1/2+\varepsilon}}\Big(q_1^{\rho_m(\theta,\theta_x^m)} + q_1^{\rho_m(\theta,\theta_y^m)}\Big), \quad x, y \in E. \tag{3.10}$$

Moreover,

$$|f_\theta(x)| \leq c|\theta|^{-1/2}q_1^{\rho_m(\theta,\theta_x^m)}, \quad x \in E, \tag{3.11}$$

and for any $s > 0$ there exist a constant c_s such that

$$|f_\theta(x)| \leq c_s|\theta|^{-1/2}(\mathcal{M}_{\mathcal{T}}^s \mathbb{1}_\theta)(x), \quad x \in E, \tag{3.12}$$

where $\mathcal{M}_{\mathcal{T}}^s$ is the maximal operator defined in (2.18).

In addition,

$$c_p^{-1}|\theta|^{1/p-1/2} \leq \|f_\theta\|_{L_p(\theta)} \leq \|f_\theta\|_p \leq c_p|\theta|^{1/p-1/2}, \quad 0 < p \leq \infty, \tag{3.13}$$

and

$$\|f_\theta\|_{H_1} \leq c|\theta|^{1/2}. \tag{3.14}$$

We first prove that each Courant element φ_θ is an Lip ε function with respect to the quasi-distance $d_{\mathcal{T}}$.

Lemma 3.9. *There exist constants $\varepsilon > 0$ and $c > 0$ depending only on the parameters of $\mathcal{T}$ such that for any $\theta \in \Theta$*

$$|\varphi_\theta(x) - \varphi_\theta(y)| \leq c\frac{d_{\mathcal{T}}(x,y)^\varepsilon}{|\theta|^\varepsilon}, \quad x, y \in E. \tag{3.15}$$

Proof. Let $\theta \in \Theta_n$, $n \geq 0$, and assume that $d_{\mathcal{T}}(x,y) > 0$ and $x \in \theta$ or $y \in \theta$. (Otherwise the claim is trivial.) Also, let $d_{\mathcal{T}}(x,y) = |\theta^\diamond|$ with $\theta^\diamond \in \Theta_m$ $(m \geq 0)$ and $\theta^\diamond$ containing x and y.

If $m \leq n$, (3.15) is trivial because there is a constant $c > 0$ such that $|\theta^\diamond|/|\theta| > c$ $(\theta \cap \theta^\diamond \neq \emptyset)$.

Assume that $n + 3N_0k < m \leq n + 3N_0(k+1)$ for some $k \geq 0$.

Case 1: Let $x, y \in \triangle$, where $\triangle \in \mathcal{T}_n$ is one of the triangles forming θ. It is easy to see that estimate (3.15) is invariant under affine transforms. So, without loss of generality we may assume that $\triangle$ is an equilateral triangle with side lengths 1. Then there exist two triangles $\triangle', \triangle'' \in \mathcal{T}_m$ with a common vertex such that $x \in \triangle'$, $y \in \triangle''$ and $\triangle', \triangle'' \subset \triangle$. (It may happen that $\triangle' = \triangle''$.) By Lemma 2.3,

$$\max\{|\text{max edge}\,(\triangle')|, |\text{max edge}\,(\triangle'')|\} \leq \rho_1^k|\text{max edge}\,(\triangle)| \leq \rho_1^k$$

with $0 < \rho_1 < 1$. Choose $\varepsilon_1 > 0$ so that $\rho_1 = r^{\varepsilon_1}$, where $0 < r < 1$ is from (2.1). Then

$$\begin{aligned}
|\phi_\theta(x) - \phi_\theta(y)| &\leq \|\nabla\varphi_\theta\|_{L_\infty(\triangle)}|x-y| \\
&\leq 2\max\{|\text{max edge}\,(\triangle')|, |\text{max edge}\,(\triangle'')|\} \\
&\leq 2\rho_1^k \leq 2(r^{3N_0k})^{\frac{\varepsilon_1}{3N_0}} \leq 2(r^{3N_0k}|\triangle|)^{\frac{\varepsilon_1}{3N_0}} \\
&\leq c|\triangle'|^\varepsilon \leq c|\theta^\diamond|^\varepsilon,
\end{aligned}$$

where $\varepsilon = \varepsilon_1/3N_0$ and we used (2.1). Since $|\theta| \approx |\triangle| \approx 1$, estimate (3.15) follows.

Case 2: Let $x \in \triangle_1$ and $y \in \triangle_2$, where $\triangle_1, \triangle_2 \in \mathcal{T}_n$ are distinct triangles with a common vertex v_θ and $\triangle_1, \triangle_2 \subset \theta$. Since $x, y \in \theta^\diamond$, there are two smaller subtriangles $\triangle', \triangle'' \in \mathcal{T}_m$ of $\theta^\diamond$ containing x and y, respectively, such that $\triangle' \subset \triangle_1$, $\triangle'' \subset \triangle_2$. Choose $z \in \triangle' \cap \triangle''$. Then using that estimate (3.15) holds in Case 1, we obtain

$$\begin{aligned}|\varphi_\theta(x) - \varphi_\theta(y)| &\le |\varphi_\theta(x) - \varphi_\theta(z)| + |\varphi_\theta(z) - \varphi_\theta(y)| \\ &\le c\Big(\frac{d_{\mathcal{T}}(x,z)^\varepsilon}{|\theta|^\varepsilon} + \frac{d_{\mathcal{T}}(z,y)^\varepsilon}{|\theta|^\varepsilon}\Big) \le c\frac{d_{\mathcal{T}}(x,y)^\varepsilon}{|\theta|^\varepsilon},\end{aligned}$$

where we used that $d_{\mathcal{T}}(x,z) \le d_{\mathcal{T}}(x,y)$ since $x, z \in \theta^\diamond$ and similarly $d_{\mathcal{T}}(z,y) \le d_{\mathcal{T}}(x,y)$.

Case 3: Let $x \in \theta$ and $y \notin \theta$ (or $y \in \theta$ and $x \notin \theta$). This case reduces to the first case by introducing an appropriate point z on the boundary of θ such that $d_{\mathcal{T}}(x,z) \le d_{\mathcal{T}}(x,y)$ and taking into account that $\varphi_\theta(y) = \varphi_\theta(z) = 0$. $\square$

Proof of Theorem 3.8. By Lemmas 3.7 and 3.9, we have for $\theta \in \Theta_m^*$,

$$\begin{aligned}|f_\theta(x) - f_\theta(y)| &\le \sum_{\eta\in\Theta_m} |\eta|^{-1/2}|c_{\theta\eta}||\varphi_\eta(x) - \varphi_\eta(y)| \\ &\le \sum_{\eta\in\Theta_m:\, x\in\eta^\circ \text{ or } y\in\eta^\circ} |\eta|^{-1/2-\varepsilon} q^{\rho_m(\theta,\eta)} d_{\mathcal{T}}(x,y)^\varepsilon \\ &\le |\theta|^{-1/2-\varepsilon} d_{\mathcal{T}}(x,y)^\varepsilon \Big(\sum_{\eta\in\Theta_m:\, x\in\eta^\circ} + \sum_{\eta\in\Theta_m:\, y\in\eta^\circ}\Big)(|\theta|/|\eta|)^{1/2+\varepsilon} q^{\rho_m(\theta,\eta)}.\end{aligned}$$

Note first that for any $x \in E$ there are at most 3 cells $\eta \in \Theta_m$ such that $x \in \eta^\circ$. By the definition of θ_x^m, we have $x \in \theta_x^m$ and $\theta_x^m \in \Theta_m$, and hence $|\eta| \approx |\theta_x^m|$ if $\eta \in \Theta_m$ and $x \in \eta$. Also, by (2.3) it follows that $|\theta|/|\theta_x^m| \le c(\rho_m(\theta, \theta_x^m) + 1)^s$ Finally, we choose the constants $c_1 > 0$ and $q < q_1 < 1$, depending only on q, ε, and s, so that $(\nu + 1)^{(1/2+\varepsilon)s} q^\nu \le c_1 q_1^\nu$ for $\nu \ge 1$. We use this preparation to obtain

$$\sum_{\eta\in\Theta_m:\, x\in\eta^\circ} (|\theta|/|\eta|)^{1/2+\varepsilon} q^{\rho_m(\theta,\eta)} \le c(\rho_m(\theta,\theta_x^m) + 1)^{(1/2+\varepsilon)s} q^{\rho_m(\theta,\theta_x^m)} \le c q_1^{\rho_m(\theta,\theta_x^m)}.$$

We similarly estimate the second sum above, and (3.10) follows.

Estimate (3.11) follows in a similar way but it is easier and will be omitted.

To prove (3.12) we need estimate $(\mathcal{M}_{\mathcal{T}}^s \mathbb{1}_\theta)(x)$ ($\theta \in \Theta_m$) from below. By (2.2) it follows that

$$(\mathcal{M}_{\mathcal{T}}^s \mathbb{1}_\theta)(x) \ge c > 0 \quad \text{for} \quad x \in \operatorname{Star}_m^1(\theta). \tag{3.16}$$

Suppose $x \in E \setminus \operatorname{Star}_m^1(\theta)$. Then $\rho_m(\theta, \theta_x^m) \ge 2$. Let $d_{\mathcal{T}}(v_\theta, x) = |\theta^\diamond|$ with $\theta^\diamond \in \Theta_\ell$ and $\theta^\diamond$ containing v_θ and x. Evidently, $\ell < m$ and $|\theta \cap \theta^\diamond| \ge c|\theta|$. Further, by Lemma 2.6, $d_{\mathcal{T}}(v_\theta, x) \le c|\theta|\rho_m(\theta, \theta_x^m)^\beta$. Using all of the above, we obtain

$$(\mathcal{M}_{\mathcal{T}}^s \mathbb{1}_\theta)(x) \ge \Big(\frac{1}{|\theta^\diamond|}\int_{\theta^\diamond} \mathbb{1}_\theta(y)\,dy\Big)^{1/s} = (|\theta \cap \theta^\diamond|/|\theta^\diamond|)^{1/s} \ge c\rho_m(\theta, \theta_x^m)^{-\beta/s}.$$

Combining this estimate with (3.16) and (3.11) yields (3.12).

We next prove (3.13). If $p < \infty$ and $\theta \in \Theta_m^*$, we have using (3.11),

$$\|f_\theta\|_p^p \le \sum_{\nu=0}^\infty \sum_{\eta\in\mathcal{X}_m^\nu} \|f_\theta\|_{L_p(\eta)}^p \le |\theta|^{-p/2} \sum_{\nu=0}^\infty \sum_{\eta\in\mathcal{X}_m^\nu} |\eta| q_1^{\rho_m(\theta,\eta)}, \tag{3.17}$$

where $\mathcal{X}_m^\nu := \{\eta \in \theta_m : \rho_m(\theta,\eta) = \nu\}$. As above $|\eta|/|\theta| \le c(\rho_m(\theta,\eta)+1)^s = c(\nu+1)^s$ if $\eta \in \mathcal{X}_m^\nu$, and by Lemma 2.4 it follows that $\#\mathcal{X}_m^\nu \le c(\nu+1)^t$. Using these in (3.17), we find

$$\|f_\theta\|_p^p \le c|\theta|^{-p/2+1} \sum_{\nu=0}^\infty (\nu+1)^{s+t} q_1^\nu \le c|\theta|^{p(1/p-1/2)}. \tag{3.18}$$

We now estimate $\|f_\theta\|_p$ from below. From the proof of Lemma 3.7 it follows that $f_\theta = \pm\|g_\theta\|_2^{-1} g_\theta$, where $g_\theta = \sum_{\xi\in\Lambda_\theta} b_{\theta\xi}\tilde{\varphi}_\xi$, and also $\langle g_\theta, \tilde{\varphi}_\theta\rangle = 1$. Exactly as in (3.17)-(3.18) (with f_θ replaced by g_θ) we obtain $\|g_\theta\|_2 \le c$. Therefore,

$$|\langle f_\theta, \tilde{\varphi}_\theta\rangle| = \|g_\theta\|_2^{-1} |\langle g_\theta, \tilde{\varphi}_\theta\rangle| = \|g_\theta\|_2^{-1} \ge c.$$

On the other hand $|\langle f_\theta, \tilde{\varphi}_\theta\rangle| \le |\theta|^{1/2} \|f_\theta\|_{L_\infty(\theta)}$. Consequently, $\|f_\theta\|_{L_\infty(\theta)} \approx |\theta|^{-1/2}$ and using (2.12), we infer $\|f_\theta\|_{L_p(\theta)} \approx |\theta|^{1/p-1/2}$, $0 < p \le \infty$.

It remains to prove that each Franklin function f_θ belongs to H_1 and (3.14) holds true. To this end it suffices to prove that $g_\theta := |\theta|^{-1/2} f_\theta$ is a constant multiple of an ε-molecule centered at v_θ. Evidently $\int_E |g_\theta(x)|^2\,dx = |\theta|^{-1}$. We use Lemma 2.6 and (3.11) to obtain

$$\begin{aligned} \int_E |g_\theta(x)|^2 d_{\mathcal{T}}(x, v_\theta)^{1+\varepsilon}\,dx &\le c|\theta|^{-1+\varepsilon} \int_E q_1^{\rho_m(\theta,\theta_x^m)} (\rho_m(\theta,\theta_x^m)+1)^{\beta(1+\varepsilon)} dx \\ &\le c|\theta|^{-1+\varepsilon} \sum_{\eta\in\Theta_m} |\eta| q_1^{\rho_m(\theta,\eta)} (\rho_m(\theta,\eta)+1)^{\beta(1+\varepsilon)} \\ &\le c|\theta|^{\varepsilon}, \end{aligned} \tag{3.19}$$

where for the latter estimate we proceed exactly as in (3.17)-(3.18). Therefore, according to the definition of a molecule (see Definition 2.12) g_θ is a constant multiple of an ε-molecule and hence $f_\theta \in H_1$ and $\|f_\theta\|_{H_1} \le c|\theta|^{1/2}$. □

3.4 Proof of Theorem 3.2

We first observe that since by Theorem 3.8 each Franklin function f_θ belongs to H_1, then $(f_\theta, f_\theta)_{\theta\in\Theta^*}$ is a biorthogonal system in H_1.

We next prove a technical result which will provide the main step in the proof of Theorem 3.2.

Proposition 3.10. *For any 2-atom $a(x)$,*

$$a = \sum_{\theta\in\Theta^*} \langle a, f_\theta\rangle f_\theta, \tag{3.20}$$

where the series converges unconditionally in H_1. Moreover, there exists a constant $c > 0$ depending only on the parameters of $\mathcal{T}$ such that for any $\mathcal{M} \subset \Theta^*$ and any sequence $\omega = (\omega_\theta)_{\theta\in\mathcal{M}}$ with $\omega_\theta = \pm 1$,

$$\Big\| \sum_{\theta\in\mathcal{M}} \omega_\theta \langle a, f_\theta\rangle f_\theta \Big\|_{H_1} \leq c \tag{3.21}$$

and

$$\Big\| \Big(\sum_{\theta\in\Theta^*} |\langle a, f_\theta\rangle|^2 |f_\theta(x)|^2 \Big)^{1/2} \Big\|_{L_1} \leq c. \tag{3.22}$$

For given $\mathcal{M} \subset \Theta^*$ and $\omega = (\omega_\theta)_{\theta\in\mathcal{M}}$ with $\omega_\theta = \pm 1$, consider the linear operator

$$(T_{\mathcal{M},\omega} f)(x) := \int_E K_{\mathcal{M},\omega}(x,y) f(y)\, dy \tag{3.23}$$

with kernel

$$K_{\mathcal{M},\omega}(x,y) := \sum_{\theta\in\mathcal{M}} \omega_\theta f_\theta(x) f_\theta(y). \tag{3.24}$$

With the next lemma we show that the kernel $K_{\mathcal{M},\omega}$ satisfies condition (2.28) of Proposition 2.13

Lemma 3.11. *There exists a constant $c > 0$ such that if $d_\mathcal{T}(x, y_0) > 2K d_\mathcal{T}(y, y_0)$, where K is the constant from the definition of the quasi-distance $d_\mathcal{T}(\cdot,\cdot)$, and $\min\{l(\theta) : \theta \in \mathcal{M}\} \geq \mu \geq 0$, then*

$$|K_{\mathcal{M},\omega}(x,y) - K_{\mathcal{M},\omega}(x,y_0)| \leq \gamma_\mu \Big(\frac{d_\mathcal{T}(y,y_0)}{d_\mathcal{T}(x,y_0)}\Big)^\varepsilon \frac{1}{d_\mathcal{T}(x,y_0)}, \tag{3.25}$$

where $\varepsilon > 0$ is from Theorem 3.8, $0 < \gamma_\mu \leq c$, and $\gamma_\mu \to 0$ as $\mu \to \infty$.

Proof. Denote $K_m(x,y) := \sum_{\theta\in\mathcal{M}\cap\Theta_m^*} \omega_\theta f_\theta(x) f_\theta(y)$. We use (3.10)-(3.11) to obtain

$$\begin{aligned} |K_m(x,y) - K_m(x,y_0)| &\leq \sum_{\theta\in\Theta_m} |f_\theta(x)||f_\theta(y) - f_\theta(y_0)| \\ &\leq \; c d_\mathcal{T}(y,y_0)^\varepsilon \sum_{\theta\in\Theta_m} |\theta|^{-1-\varepsilon} q_1^{\rho_m(\theta,\theta_x^m)} \Big(q_1^{\rho_m(\theta,\theta_y^m)} + q_1^{\rho_m(\theta,\theta_{y_0}^m)} \Big). \end{aligned} \tag{3.26}$$

We now claim that there exist constants $c > 0$ and $0 < q_2 < 1$ such that

$$\sum_{\theta\in\Theta_m} |\theta|^{-1-\varepsilon} q_1^{\rho_m(\theta,\theta_x^m)+\rho_m(\theta,\theta_y^m)} \leq c|\theta_y^m|^{-1-\varepsilon} q_2^{\rho_m(\theta_x^m,\theta_y^m)}. \tag{3.27}$$

To see this set $n := \rho_m(\theta_x^m, \theta_y^m)$ and define

$$A_k := \{\theta \in \Theta_m : \rho_m(\theta,\theta_x^m) + \rho_m(\theta,\theta_y^m) = k + n\}.$$

By Lemma 2.4, $\#A_k \le c(n+k)^t$, and by Lemma 2.1, $|\theta_y^m|/|\theta| \le c\rho_m(\theta,\theta_y^m)^s$ if $\theta\in\Theta_m$. We use the above to obtain

$$\begin{aligned}
\sum_{\theta\in\Theta_m}\frac{q_1^{\rho_m(\theta,\theta_x^m)+\rho_m(\theta,\theta_y^m)}}{|\theta|^{1+\varepsilon}} &= |\theta_y^m|^{-1-\varepsilon}\sum_{k\ge0}\sum_{\theta\in A_k}(|\theta_y^m|/|\theta|)^{1+\varepsilon}q_1^{n+k}\\
&\le c|\theta_y^m|^{-1-\varepsilon}\sum_{k\ge0}\sum_{\theta\in A_k}\rho_m(\theta,\theta_y^m)^{s(1+\varepsilon)}q_1^{n+k}\\
&\le c|\theta_y^m|^{-1-\varepsilon}\sum_{k\ge0}(n+k)^{t+s(1+\varepsilon)}q_1^{n+k}\\
&\le c|\theta_y^m|^{-1-\varepsilon}q_2^n = c|\theta_y^m|^{-1-\varepsilon}q_2^{\rho_m(\theta_x^m,\theta_y^m)},
\end{aligned}$$

where $q_1<q_2<1$. Thus (3.27) is established.

Applying (3.27) in (3.26) we get

$$|K_m(x,y)-K_m(x,y_0)| \le cd_{\mathcal{T}}(y,y_0)^\varepsilon\left(\frac{q_2^{\rho_m(\theta_x^m,\theta_y^m)}}{|\theta_y^m|^{1+\varepsilon}}+\frac{q_2^{\rho_m(\theta_x^m,\theta_{y_0}^m)}}{|\theta_{y_0}^m|^{1+\varepsilon}}\right)$$

and hence

$$\begin{aligned}
|K_{\mathcal{M},\omega}(x,y)-K_{\mathcal{M},\omega}(x,y_0)| &\le \sum_{m=0}^{\infty}|K_m(x,y)-K_m(x,y_0)|\\
&\le cd_{\mathcal{T}}(y,y_0)^\varepsilon\sum_{m=\mu}^{\infty}\left(\frac{q_2^{\rho_m(\theta_x^m,\theta_y^m)}}{|\theta_y^m|^{1+\varepsilon}}+\frac{q_2^{\rho_m(\theta_x^m,\theta_{y_0}^m)}}{|\theta_{y_0}^m|^{1+\varepsilon}}\right). \qquad (3.28)
\end{aligned}$$

To estimate the right-hand-side of (3.28) let us assume that $d_{\mathcal{T}}(x,y)=|\theta^*|$ with $\theta^*\in\Theta_{m^*}$. Then

$$\sum_{m=\mu}^{\infty}\frac{q_2^{\rho_m(\theta_x^m,\theta_y^m)}}{|\theta_y^m|^{1+\varepsilon}} = \Big(\sum_{m=\mu}^{m^*}+\sum_{m=m^*+1}^{\infty}\Big)\frac{q_2^{\rho_m(\theta_x^m,\theta_y^m)}}{|\theta_y^m|^{1+\varepsilon}} =: \sigma_1+\sigma_2 \qquad (3.29)$$

For σ_1, using (2.1)-(2.2), we have

$$\begin{aligned}
\sigma_1 &\le \frac{c}{|\theta^*|^{1+\varepsilon}}\sum_{m=\mu}^{m^*}(|\theta^*|/|\theta_y^m|)^{1+\varepsilon}q_2^{\rho_m(\theta_x^m,\theta_y^m)}\\
&\le \frac{c}{|\theta^*|^{1+\varepsilon}}\sum_{m=\mu}^{m^*}\rho^{(m^*-m)(1+\varepsilon)} \le \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}}. \qquad (3.30)
\end{aligned}$$

We now estimate σ_2. Denote $\mathcal{I}_k := [m^*+2N_0k, m^*+2N_0(k+1))$. Note that by (2.9)

$$\rho_\ell(\theta_x^\ell,\theta_y^\ell)\ge 2^{k-2}\rho_{m^*+2N_0}(\theta_x^{m^*+2N_0},\theta_y^{m^*+2N_0})\ge 2^{k-2},$$

whenever $\ell\ge m^*+2N_0k$. We use this and (2.1)-(2.2) to obtain

$$\sigma_2 \le \frac{c}{|\theta^*|^{1+\varepsilon}}\sum_{m=m^*+1}^{\infty}(|\theta^*|/|\theta_y^m|)^{1+\varepsilon}q_2^{\rho_m(\theta_x^m,\theta_y^m)}$$

$$\leq \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}} \sum_{m=m^*+1}^{\infty} r^{(m^*-m)(1+\varepsilon)} q_2^{\rho_m(\theta_x^m,\theta_y^m)}$$
$$\leq \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}} \sum_{k=0}^{\infty} \sum_{m\in\mathcal{I}_k} r^{(m^*-m)(1+\varepsilon)} q_2^{2^k}$$
$$\leq \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}} \sum_{k=0}^{\infty} r^{-2N_0(k+1)(1+\varepsilon)} q_2^{2^k} \leq \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}}. \tag{3.31}$$

Estimates (3.30)-(3.31) yield

$$\sum_{m=0}^{\infty} \frac{q^{\rho_m(\theta_x^m,\theta_y^m)}}{|\theta_y^m|^{1+\varepsilon}} \leq \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}}.$$

Using this inequality twice in (3.28) (for y and y_0) we obtain

$$|K_{\mathcal{M},\omega}(x,y) - K_{\mathcal{M},\omega}(x,y_0)| \leq c d_{\mathcal{T}}(y,y_0)^{\varepsilon} \Big(\frac{1}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}} + \frac{1}{d_{\mathcal{T}}(x,y_0)^{1+\varepsilon}} \Big). \tag{3.32}$$

Finally, since $d_{\mathcal{T}}(x,y_0) > 2K d_{\mathcal{T}}(y,y_0)$, then $d(x,y) \approx d(x,y_0)$ and (3.32) implies (3.25) with $\gamma_\mu = c$.

If $m^* + 2N_0 k^\natural \leq \mu < m^* + 2N_0(k^\natural + 1)$ for some $k^\natural \geq 1$, then $\sigma_1 = 0$ and as above

$$\sigma_2 \leq \frac{c}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}} \sum_{k=k^\natural}^{\infty} r^{-2N_0(k+1)(1+\varepsilon)} q_2^{2^k} \leq \frac{c q_3^{2^{(\mu-m^*)/2N_0}}}{d_{\mathcal{T}}(x,y)^{1+\varepsilon}}$$

for some $0 < q_3 < 1$. This estimate yields (3.25) with $\gamma_\mu := c q_3^{2^{(\mu-m^*)/2N_0}}$. Evidently, $\gamma_\mu \to 0$ as $\mu \to \infty$. The proof of Lemma 3.11 is complete. □

Proof of Proposition 3.10. Suppose first that $a(x) = \pm|E|^{-1}\mathbb{1}_E$. If $\theta_0 \in \mathcal{M}$, then using the orthogonality of the Franklin functions and that $f_{\theta_0} = \pm|E|^{-1/2}\mathbb{1}_E$ it follows that $T_{\mathcal{M},\omega} a = \pm a$, while if $\theta_0 \notin \mathcal{M}$ then $T_{\mathcal{M},\omega} a = 0$. In both cases the result obviously holds.

Let now $a(x)$ be a 2-atom and $a \neq \pm|E|^{-1}\mathbb{1}_E$. Then $\int_E a = 0$. As above, we see that $\int_E T_{\mathcal{M},\omega} a = 0$. Then by Proposition 2.13 and Lemma 3.11 with $\mathcal{M} := \{\theta \in \Theta^* : l(\theta) \geq \mu\}$ and an arbitrary sequence $\omega = (\omega_\theta)_{\theta\in\Theta^*}$ with $\omega_\theta = \pm 1$, it follows that the function

$$h_\mu := \sum_{\theta\in\Theta^*,\, l(\theta)\geq\mu} \omega_\theta \langle a, f_\theta\rangle f_\theta$$

belongs to H_1 and $\|h_\mu\|_{H_1} \leq \gamma_\mu$, where $\gamma_\mu \to 0$ as $\mu \to \infty$. Therefore, the series in (3.20) converges unconditionally in H_1. We also know that (3.20) holds in L_2. Since both spaces H_1 and L_2 are continuously embedded in L_1, it follows that (3.20) holds in H_1.

Estimate (3.21) follows by Proposition 2.13 and Lemma 3.11.

It remains to prove (3.22). From above for an arbitrary sequence $\omega = (\omega_\theta)_{\theta\in\Theta^*}$ with $\omega_\theta = \pm 1$, we have

$$\Big\| \sum_{\theta\in\Theta^*} \omega_\theta \langle a, f_\theta\rangle f_\theta \Big\|_{L_1} \leq c \Big\| \sum_{\theta\in\Theta^*} \omega_\theta \langle a, f_\theta\rangle f_\theta \Big\|_{H_1} \leq c.$$

Applying now the usual trick with Khintchine's inequality, we infer

$$\Big\| \sum_{\theta\in\Theta^*} |\langle a, f_\theta\rangle|^2 |f_\theta(x)|^2 \Big\|_{L_1} \le c \Big\| \sum_{\theta\in\Theta^*} \omega_\theta \langle a, f_\theta\rangle f_\theta \Big\|_{L_1} \le c.$$

□

Proof of Theorem 3.2. We begin by proving that the Franklin system $\mathcal{F}_\mathcal{T}$ is a unconditional basis for $H_1 := H_1(E,\mathcal{T})$. As we have already mentioned $f_\theta \in H_1$ for all $\theta \in \Theta^*$, and hence $(f_\theta, f_\theta)_{\theta\in\Theta^*}$ is a biorthogonal system in H_1.

By Proposition 3.10 for any atom $a(x)$, $a = \sum_{\theta\in\Theta^*}\langle a, f_\theta\rangle f_\theta$ in H_1. This along with the definition of H_1 yields that $\mathcal{F}_\mathcal{T}$ is dense in H_1.

By estimate (3.21) from Proposition 3.10, it readily follows that for any $\mathcal{M} \subset \Theta^*$,

$$\Big\| \sum_{\theta\in\mathcal{M}} \langle f, f_\theta\rangle f_\theta \Big\|_{H_1} \le c\|f\|_{H_1} \quad \text{for} \quad f \in H_1. \tag{3.33}$$

Therefore, $\mathcal{F}_\mathcal{T}$ is a unconditional basis for H_1.

We now turn to L_p $(1 < p < \infty)$. Taking into account Lemma 2.3, it is obvious that $\mathcal{F}_\mathcal{T}$ is dense in $L_p(E)$. For a given $\mathcal{M} \subset \Theta^*$, consider the operator $T_\mathcal{M} := T_{\mathcal{M},\omega}$ with $\omega = (1)_{\theta\in\mathcal{M}}$, where $T_{\mathcal{M},\omega}$ is defined in (3.23). By (3.33) $T_\mathcal{M}$ is bounded on H_1 and, since $\mathcal{F}_\mathcal{T}$ is an orthogonal basis for L_2, $T_\mathcal{M}$ is bounded on L_2 as well. Then by interpolation it follows that $T_\mathcal{M}$ is bounded on L_p for $1 < p \le 2$. Finally, by a standard duality argument, it easily follows that $T_\mathcal{M}$ is bounded on $L_p, 2 < p < \infty$, as well. Consequently, $\mathcal{F}_\mathcal{T}$ is a unconditional basis for $L_p(E)$, $1 < p < \infty$. □

3.5 Proof of Theorem 3.3

We first note that the implication $(a) \Longrightarrow (b)$ is immediate from the fact that $\mathcal{F}_\mathcal{T}$ is a unconditional basis for H_1 (see Theorem 3.2), since H_1 is embedded in L_1.

One applies Khintchine's inequality as usual to show that $(b) \Longleftrightarrow (c)$ (see e.g. [11, 17]).

We now show that $(c) \Longleftrightarrow (d)$. We know from Theorem 3.8 that for any $s > 0$, $|f_\theta(x)| \le c_s(\mathcal{M}_\mathcal{T}^s \tilde{\mathbb{1}}_\theta)(x)$, $x \in E$. Then choosing $0 < s < 1$ and applying Proposition 2.9 we obtain $\|S_f\|_{L_1} \le c\|F_f\|_{L_1}$; thus $(d) \Longrightarrow (c)$.

For the other direction, we first note that by Theorem 3.8, $\|f_\theta\|_{L_\infty(\theta)} \ge c|\theta|^{-1/2}$. Then since f_θ is linear on each triangle of θ, there exist a set $G_\theta \subset \theta$ with $|G_\theta| \ge \alpha|\theta|$ such that $|f_\theta(x)| \ge c|G_\theta|^{-1/2}$ for $x \in G_\theta$, where $c > 0$ and $0 < \alpha < 1$ are constants depending only on the parameters of $\mathcal{T}$. Therefore, $\tilde{\mathbb{1}}_{G_\theta}(x) \le c|f_\theta(x)|$ for $x \in E$. Denote

$$\Gamma_f(x) := \Big(\sum_{\theta\in\Theta^*} |\langle f, f_\theta\rangle|^2 |\tilde{\mathbb{1}}_{G_\theta}(x)|^2 \Big)^{1/2}$$

Then by the above estimate $\|\Gamma_f\|_{L_1} \le c\|S_f\|_{L_1}$. On the other hand, $\tilde{\mathbb{1}}_\theta(x) \le c_s(\mathcal{M}_\mathcal{T}^s \tilde{\mathbb{1}}_{G_\theta})(x)$, since $|G_\theta| \ge \alpha|\theta|$. Applying again Proposition 2.9 we infer $\|F_f\|_{L_1} \le c\|\Gamma_f\|_{L_1} \le c\|S_f\|_{L_1}$. Consequently, $(c) \Longrightarrow (d)$.

It remains to show that $(d) \Longrightarrow (a)$ which is the main step in the proof of Theorem 3.3. We give it in the following proposition.

Proposition 3.12. *Suppose that for a collection of numbers $(a_\theta)_{\theta\in\Theta^*}$ we have*

$$F(x) := \Big(\sum_{\theta\in\Theta^*} |a_\theta|^2|\tilde{\mathbb{1}}_\theta(x)|^2\Big)^{1/2} \in L_1 \tag{3.34}$$

which is equivalent to

$$S(x) := \Big(\sum_{\theta\in\Theta^*} |a_\theta|^2|f_\theta(x)|^2\Big)^{1/2} \in L_1. \tag{3.35}$$

Then $f := \sum_{\theta\in\Theta^*} a_\theta f_\theta$ belongs to H_1 and $\|f\|_{H_1} \le c\|F\|_{L_1}$.

Proof. We shall use the idea of the proof in the wavelet case (see e.g. [14, 17]). Note first that the equivalence $\|F\|_{L_1} \approx \|S\|_{L_1}$ follows by the argument which we used above to show that $(c) \Longleftrightarrow (d)$.

Let us denote

$$G_k := \{x \in E : F(x) > 2^k\}, \quad k \in \mathbb{Z}.$$

It is easy to see that

$$\sum_{k\in\mathbb{Z}} 2^k|G_k| \le 2\|F\|_{L_1(E)} \tag{3.36}$$

(see e.g. [17], Proposition 8.15).

We introduce the collections of cells

$$\mathcal{C}_k := \{\theta \in \Theta^* : |\theta \cap G_k| > |\theta|/2\}.$$

Since $G_{k+1} \subset G_k$, then $\mathcal{C}_{k+1} \subset \mathcal{C}_k$. It is easy to see that

$$\Theta^* = \cup_{j\in\mathbb{Z}}\mathcal{C}_j.$$

Indeed, if $a_\theta \ne 0$, then $|a_\theta||\theta|^{-1/2} > 2^j$ for some $j \in \mathbb{Z}$, and hence $F(x) \ge |a_\theta|\tilde{\mathbb{1}}(x) > 2^j$ on θ. Therefore, $\theta \in \mathcal{C}_j$.

Denote now

$$G_k^* := \cup_{\theta\in\mathcal{C}_k}\theta.$$

It is not hard to see that the Lebesgue differentiation theorem holds with the cells from Θ. (For its proof one can use the maximal operator $\mathcal{M}^1_{\mathcal{T}}$ introduced in §2.2.) Consequently, $G_k \subset G_k^*$ modulo a set of measure zero.

By the coloring lemma in [10] (Lemma 3.2), Θ can be represented as a finite disjoint union of subsets $(\Theta^\nu)_{\nu=1}^K$ with $K = K(N_0, M_0)$ such that each Θ^ν has a tree structure with respect to the inclusion relation, i.e., if $\theta', \theta'' \in \Theta^\nu$, then $(\theta')^\circ \cap (\theta'')^\circ = \emptyset$ or $\theta' \subset \theta''$ or $\theta'' \subset \theta'$.

Further, denote by $\mathcal{M}_{k\nu}$ the set of all maximal cells in $\mathcal{C}_k \cap \Theta^\nu$, i.e. $\theta \in \mathcal{M}_{k\nu}$ if $\theta \in \mathcal{C}_k \cap \Theta^\nu$ and θ is not contained in any other cell from $\mathcal{C}_k \cap \Theta^\nu$. Clearly, $G_k^* = \cup_{\nu=1}^K \cup_{\theta\in\mathcal{M}_{k\nu}} \theta$ and

$$\begin{aligned} |G_k^*| &\le \sum_{\nu=1}^K \sum_{\theta\in\mathcal{M}_{k\nu}} |\theta| \le 2\sum_{\nu=1}^K \sum_{\theta\in\mathcal{M}_{k\nu}} |G_k \cap \theta| \\ &\le 2\sum_{\nu=1}^K |G_k \cap \cup_{\theta\in\mathcal{M}_{k\nu}}\theta| \le 2K|G_k|. \end{aligned} \tag{3.37}$$

Denote $\mathcal{D}_k := \mathcal{C}_k \setminus \mathcal{C}_{k+1}$. Now for any $\theta \in \mathcal{M}_{k\nu}$ we define $\mathcal{D}_\theta := \{\eta \in \mathcal{D}_k \cap \Theta^\nu : \eta \subset \theta\}$. Clearly, the sets $\mathcal{D}_\theta$ are disjoint and

$$\Theta^* = \bigcup_{k\in\mathbb{Z}} \bigcup_{\nu=1}^{K} \bigcup_{\theta\in\mathcal{M}_{k\nu}} \mathcal{D}_\theta.$$

We also define

$$\mathcal{A}_\theta := \sum_{\eta\in\mathcal{D}_\theta} a_\theta f_\theta, \quad \theta \in \mathcal{M}_{k\nu}.$$

We next show that $m_\theta := 2^{-k}|\theta|^{-1}\mathcal{A}_\theta$ is a constant multiple of an ε-molecule, which will imply $\|m_\theta\|_{H_1} \le c$ and as a consequence $\|\mathcal{A}_\theta\|_{H_1} \le 2^k|\theta|$. Note first that, for $\theta \in \mathcal{M}_{k\nu}$,

$$\int_{\theta\setminus G_{k+1}} F^2(x)\,dx \ge \sum_{\eta\in\mathcal{D}_\theta} |a_\eta|^2 \int_{\theta\setminus G_{k+1}} \tilde{\mathbb{1}}_\eta^2(x)\,dx \ge \sum_{\eta\in\mathcal{D}_\theta} |a_\eta|^2|\eta|^{-1}|\eta\setminus G_{k+1}|.$$

Since $\eta \notin \mathcal{C}_{k+1}$, $|G_{k+1}\cap\eta| \le |\eta|/2$ and hence $|\eta \setminus G_{k+1}| \ge |\eta|/2$. Therefore,

$$\int_{\theta\setminus G_{k+1}} F^2(x)\,dx \ge \frac{1}{2}\sum_{\eta\in\mathcal{D}_\theta} |a_\eta|^2, \quad \theta \in \mathcal{M}_{k\nu}. \tag{3.38}$$

On the other hand,

$$\int_{\theta\setminus G_{k+1}} F^2(x)\,dx \le 2^{2(k+1)}|\theta\setminus G_{k+1}| \le 4\cdot 2^{2k}|\theta|.$$

Combining this with (3.38) we arrive at

$$\|\mathcal{A}_\theta\|_2^2 = \sum_{\eta\in\mathcal{D}_\theta} |a_\eta|^2 \le c2^{2k}|\theta|. \tag{3.39}$$

To prove that m_θ is a constant multiple of an ε-molecule it suffices to show that m_θ satisfies (2.26) with the 1 in the write-hand-side of (2.26) replaced by a constant $c > 0$ (for some $\varepsilon > 0$). This is apparently equivalent to

$$\Big(\int_E |\mathcal{A}_\theta(x)|^2\,dx\Big)\Big(\int_E |\mathcal{A}_\theta(x)|^2 d_{\mathcal{T}}(x, v_\theta)^{1+\varepsilon}dx\Big)^{1/\varepsilon} \le c(2^{2k}|\theta|^2)^{1+1/\varepsilon}.$$

We chose an arbitrary $\varepsilon > 0$, e.g. $\varepsilon = 1$ and fix it. Taking into account (3.39) it suffices to show that

$$\int_E |\mathcal{A}_\theta(x)|^2 d_{\mathcal{T}}(x, v_\theta)^{1+\varepsilon}dx \le c2^{2k}|\theta|^{2+\varepsilon}. \tag{3.40}$$

To prove this estimate we need the following lemma:

Lemma 3.13. *For $\theta \in \Theta_m$ $(m \ge 0)$ and $x \in E$ with $\rho_m(\theta, \theta_x^m) \ge 3$, we have*

$$R(x) := \sum_{\eta\in\Theta^*,\,\eta\subset\theta} |f_\eta(x)| \le c|\theta|^{-1/2}q_*^{\rho_m(\theta,\theta_x^m)}, \tag{3.41}$$

where the constants $c > 0$ and $0 < q_* < 1$ depend only on the parameters of $\mathcal{T}$.

Proof. Write briefly $\mathcal{N} := \rho_m(\theta, \theta_x^m)$ and $\mathcal{I}_k := [2N_0k, 2N_0(k+1))$, where N_0 is from condition (d) on LR-triangulations (§2.1). If $\nu \in \mathcal{I}_k$ $(k \geq 0)$, then using (2.9) we obtain $\rho_{m+\nu}(v_\theta, x) \geq 2^{k-2}\rho_m(v_\theta, x) = 2^{k-2}\mathcal{N}$. Then by (3.11) we have for $\nu \in \mathcal{I}_k$,

$$\begin{aligned}
\sum_{\eta \in \Theta^*_{m+\nu},\, \eta \subset \theta} |f_\eta(x)| &\leq c \sum_{\eta \in \Theta^*_{m+\nu},\, \eta \subset \theta} |\eta|^{-1/2} q_1^{\rho_{m+\nu}(\eta, \theta_x^{m+\nu})} \\
&\leq c \sum_{\mu \geq 2^{k-2}\mathcal{N}} \sum_{\eta \in \mathcal{X}^\mu_{m+\nu}} |\eta|^{-1/2} q_1^\mu, \qquad (3.42)
\end{aligned}$$

where

$$\mathcal{X}^\mu_{m+\nu} := \{\eta \in \Theta_{m+\nu} : \rho_{m+\nu}(\eta, \theta_x^{m+\nu}) = \mu \text{ and } \eta \subset \theta\}.$$

By Lemma 2.4, $\#\mathcal{X}^\mu_{m+\nu} \leq c\mu^t$ and by (2.1)-(2.2) we have for $\eta \in \mathcal{X}^\mu_{m+\nu}$, $|\eta| \geq cr^\nu|\theta| \geq cr^{2N_0k}|\theta|$, since $\eta \subset \theta$. Using these in (3.42), we obtain

$$\begin{aligned}
\sum_{\eta \in \Theta^*_{m+\nu},\, \eta \subset \theta} |f_\eta(x)| &\leq c|\theta|^{-1/2} r^{-N_0k} \sum_{\mu \geq 2^{k-1}\mathcal{N}} q_1^\mu \#\mathcal{X}^\mu_{m+\nu} \\
&\leq c|\theta|^{-1/2} r^{-N_0k} \sum_{\mu \geq 2^{k-1}\mathcal{N}} q_1^\mu \mu^t \\
&\leq c|\theta|^{-1/2} q_*^{2^k\mathcal{N}},
\end{aligned}$$

where $0 < q_* < 1$ and we used that $0 < q_1 < 1$. Summing up the above estimates we get

$$R(x) \leq c|\theta|^{-1/2} \sum_{k=0}^\infty 2N_0 q_*^{2^k\mathcal{N}} \leq c|\theta|^{-1/2} q_*^{\mathcal{N}} = c|\theta|^{-1/2} q_*^{\rho_m(\theta, \theta_x^m)}.$$

□

We are now prepared to prove (3.40). Assuming that $\theta \in \Theta_m$, $m \geq 0$, we can write

$$\int_E |\mathcal{A}_\theta(x)|^2 d_\mathcal{T}(x, v_\theta)^{1+\varepsilon} dx = \int_{\text{Star}^3_m(v_\theta)} + \int_{E \setminus \text{Star}^3_m(v_\theta)} =: J_0 + J_1.$$

To estimate the first integral we note that $d_\mathcal{T}(v_\theta, x) \leq c|\theta|$ if $x \in \text{Star}^3_m(v_\theta)$ and using (3.39) we obtain

$$J_0 \leq c|\theta|^{1+\varepsilon} \|\mathcal{A}_\theta\|_2^2 \leq c2^{2k}|\theta|^{2+\varepsilon}. \qquad (3.43)$$

To estimate J_1 we first observe that by (3.39) it follows that $|a_\eta| \leq c2^k|\theta|^{1/2}$ for $\eta \in \mathcal{D}_\theta$. Using this and Lemma 2.6, we obtain

$$\begin{aligned}
J_1 &= \int_{E \setminus \text{Star}^3_m(v_\theta)} |\mathcal{A}_\theta(x)|^2 d_\mathcal{T}(x, v_\theta)^{1+\varepsilon} dx \\
&\leq \int_{E \setminus \text{Star}^3_m(v_\theta)} \Big(\sum_{\eta \in \mathcal{D}_\theta} |a_\eta||f_\eta(x)| \Big)^2 d_\mathcal{T}(x, v_\theta)^{1+\varepsilon} dx \\
&\leq c2^{2k}|\theta|^{2+\varepsilon} \int_{E \setminus \text{Star}^3_m(v_\theta)} \Big(\sum_{\eta \in \Theta^*,\, \eta \subset \theta} |f_\eta(x)| \Big)^2 \rho_m(\theta, \theta_x^m)^{\beta(1+\varepsilon)} dx.
\end{aligned}$$

Applying Lemma 3.13 we obtain

$$J_1 \le c2^{2k}|\theta|^{1+\varepsilon} \int_{E\setminus \mathrm{Star}\,^3_m(v_\theta)} q_*^{\rho_m(\theta,\theta_x^m)} \rho_m(\theta,\theta_x^m)^{\beta(1+\varepsilon)} dx$$

and exactly as in (3.19) (see also (3.17)-(3.18)) we obtain $J_1 \le c2^{2k}|\theta|^{2+\varepsilon}$. This and (3.43) yield (3.40). Consequently, $\|\mathcal{A}_\theta\|_{H_1} \le c2^k|\theta|$, which implies

$$\|f\|_{H_1} \le \sum_{k\in\mathbb{Z}} \sum_{\nu=1}^{K} \sum_{\theta\in\mathcal{D}_{k\nu}} \|\mathcal{A}_\theta\|_{H_1} \le c\sum_{k\in\mathbb{Z}} 2^k \sum_{\nu=1}^{K} \sum_{\theta\in\mathcal{M}_{k\nu}} |\theta| \le c\sum_{k\in\mathbb{Z}} 2^k|G_k| \le c\|F\|_{L_1},$$

where we used (3.36). This completes the proof of Proposition 3.12. □

It remains to prove equivalences (3.2). The estimate $\|f\|_{H_1} \le c\|S_f\|_{L_1}$ and equivalence $\|S_f\|_{H_1} \approx \|F_f\|_{L_1}$ are immediate from Proposition 3.12. The estimate $\|S_f\|_{L_1} \le c\|f\|_{H_1}$ holds since by Proposition 3.10 it is true for each individual atom. The proof of Theorem 3.3 is complete.

3.6 Proof of Theorem 3.4

We shall follow the scheme of the proof of Wojtaszczyk [16] combined with our techniques of this article. We begin with one technical lemma.

Lemma 3.14. *For any $\theta \in \Theta_m$ $(m \ge 0)$ and $1 \le q < \infty$, we have*

$$\Big\| \sum_{\eta\in\Theta^*,\,\eta\subset\theta} |\eta|^{1/2}|f_\eta| \Big\|_{L_q(E\setminus\theta)} \le c|\theta|^{1/q} \tag{3.44}$$

and

$$\Big\| \sum_{\eta\in\Theta^*,\,\eta\not\subset\theta,\,l(\eta)>m} |\eta|^{1/2}|f_\eta| \Big\|_{L_q(\theta)} \le c|\theta|^{1/q}. \tag{3.45}$$

Proof. We shall prove only (3.44) since the proof of (3.45) is the same. We first observe that by (3.11) and Lemma 2.4,

$$\sum_{\eta\in\Theta_n^*} |\eta|^{1/2}|f_\eta(x)| \le c, \quad x \in E, \quad n \ge 0. \tag{3.46}$$

For $x \in E\setminus\theta$ and $\nu \ge 1$, we define $\rho_{m+\nu}(\theta,\theta_x^{m+\nu}) := \inf_{y\in\theta} \rho_{m+\nu}(\theta_y^{m+\nu},\theta_x^{m+\nu})$. Exactly as in the proof of Lemma 3.13 one shows that there exist constants $0 < q_* < 1$ and $c > 0$ such that

$$\sum_{\eta\in\Theta^*,\,\eta\subset\theta,\,l(\eta)\ge m+\nu} |\eta|^{1/2}|f_\eta(x)| \le cq_*^{\rho_{m+\nu}(\theta,\theta_x^{m+\nu})}, \quad x \in E\setminus \mathrm{Star}\,^2_{m+\nu}(\theta). \tag{3.47}$$

and, in particular,

$$\sum_{\eta\in\Theta^*,\,\eta\subset\theta} |\eta|^{1/2}|f_\eta(x)| \le cq_*^{\rho_m(\theta,\theta_x^m)}, \quad x \in E\setminus \mathrm{Star}\,^2_m(\theta). \tag{3.48}$$

Taking into account (2.5), estimates (3.46)-(3.47) yield

$$\sum_{\eta\in\Theta^*,\,\eta\subset\theta} |\eta|^{1/2}|f_\eta(x)| \le c(\nu+1), \quad x\in E\setminus \mathrm{Star}^{\,2}_{m+\nu}(\theta), \quad \nu\ge 0. \tag{3.49}$$

For the proof of (3.44) we also need the estimate:

$$|\mathrm{Star}^{\,2}_{m+\nu}(\theta)\setminus\theta| \le c\rho_2^\nu|\theta|, \quad \nu\ge 0, \tag{3.50}$$

where $0<\rho_2<1$ and $c>0$ are constants depending only on the parameters of $\mathcal{T}$. Estimate (3.50) is an immediate consequence of estimate (2.4) from Lemma 2.3 and properties (2.1)-(2.2) of LR-triangulations.

Denote briefly $F(x) := \sum_{\eta\in\Theta^*,\,\eta\subset\theta}|\eta|^{1/2}|f_\eta(x)|$. We have

$$\int_{E\setminus\theta}|F(x)|^q dx = \int_{E\setminus \mathrm{Star}^{\,2}_m(\theta)}|F(x)|^q dx + \sum_{\nu=0}^\infty \int_{\mathrm{Star}^{\,2}_{m+\nu}(\theta)\setminus \mathrm{Star}^{\,2}_{m+\nu+1}(\theta)}|F(x)|^q dx. \tag{3.51}$$

Using (3.48) we obtain

$$\int_{E\setminus \mathrm{Star}^{\,2}_m(\theta)}|F(x)|^q dx \le c\sum_{\omega\in\Theta_m:\,\omega^\circ\cap \mathrm{Star}^{\,2}_m(\theta)=\emptyset}|\omega| q_*^{q\rho_m(\theta,\omega)} \le c|\theta|, \tag{3.52}$$

where for the latter estimate we proceed exactly as in (3.17)-(3.18).

By (3.49)-(3.50) we get

$$\sum_{\nu=0}^\infty \int_{\mathrm{Star}^{\,2}_{m+\nu}(\theta)\setminus \mathrm{Star}^{\,2}_{m+\nu+1}(\theta)}|F(x)|^q dx \le c\sum_{\nu=0}^\infty(\nu+1)\rho_2^\nu|\theta| \le c|\theta|.$$

This and (3.52) yield (3.44). □

We are now in a position to prove Theorem 3.4.

(a) Assume that (3.3) holds and denote by A the quantity in (3.3). We shall prove that $f\in BMO(E,\mathcal{T})$ and $\|f\|_{BMO}\le cA$.

Let $\theta\in\Theta_m$ $(m\ge 0)$ (the case $\theta=E$ is trivial). We write

$$f = \Big(\sum_{\eta\in\Theta^*,\,\eta\subset\theta} + \sum_{\eta\in\Theta^*,\,\eta\not\subset\theta,\,l(\eta)>m} + \sum_{\eta\in\Theta^*,\,l(\eta)\le m}\Big)\langle f, f_\eta\rangle f_\eta =: F_1+F_2+F_3.$$

Using (3.3) we have

$$\|F_1\|^2_{L_2(E)} = \sum_{\eta\subset\theta}|\langle f,f_\eta\rangle|^2 \le A^2|\theta|. \tag{3.53}$$

From (3.3), $|\langle f,f_\eta\rangle|\le A|\eta|^{1/2}$ and using (3.45) with $q=2$, we obtain

$$\|F_2\|_{L_2(\theta)} \le cA|\theta|^{1/2}. \tag{3.54}$$

We use (3.10) and that $|\langle f, f_\eta\rangle| \le A|\eta|^{1/2}$ to obtain, for $x \in \theta$,

$$\begin{aligned} |F_3(x) - F_3(v_\theta)| &\le cA \sum_{\eta\in\Theta^*, l(\eta)\le m} |\eta|^{1/2}|f_\eta(x) - f_\eta(v_\theta)| \\ &\le cA\sum_{\nu=0}^{m}\sum_{\eta\in\Theta_\nu^*} d_{\mathcal{T}}(v_\theta, x)^\varepsilon|\eta|^{-\varepsilon}\Big(q_1^{\rho_\nu(\eta,\theta_x^\nu)} + q_1^{\rho_\nu(\eta,\theta_{v_\theta}^\nu)}\Big) \\ &\le cA|\theta|^\varepsilon\sum_{\nu=0}^{m}\sum_{\eta\in\Theta_\nu^*}|\eta|^{-\varepsilon}q_1^{\rho_\nu(\eta,\theta_x^\nu)}. \end{aligned} \tag{3.55}$$

Now, exactly as in the proof of Lemma 3.11, we have

$$\sum_{\eta\in\Theta_\nu^*}|\eta|^{-\varepsilon}q_1^{\rho_\nu(\eta,\theta_x^\nu)} \le c|\theta|^{-\varepsilon}\rho^{(m-\nu)\varepsilon}\sum_{\eta\in\Theta_\nu^*}\rho_\nu(\eta,\theta_x^\nu)^{s\varepsilon}q_1^{\rho_\nu(\eta,\theta_x^\nu)} \le c|\theta|^{-\varepsilon}\rho^{(m-\nu)\varepsilon}.$$

We use this in (3.55) to obtain

$$|F_3(x) - F_3(v_\theta)| \le cA, \quad x \in \theta. \tag{3.56}$$

By (3.53)-(3.54) and (3.56) it readily follows that

$$\Big(\frac{1}{|\theta|}\int_\theta |f(x) - f(v_\theta)|^2dx\Big)^{1/2} \le cA.$$

Consequently, $\|f\|_{BMO} \le cA$.

(b) Assume that $f \in BMO$ and $\|f\|_{BMO} = B$. Fix $\theta \in \Theta_m$ $(m \ge 0)$ (the case $\theta = E$ is easier). We shall prove that

$$\sum_{\eta\in\Theta^*, \eta\subset\theta}|\langle f, f_\eta\rangle|^2 \le cB^2|\theta|, \tag{3.57}$$

which implies (3.3).

We write

$$f = \Big(\sum_{\eta\in\Theta^*, \eta\subset\theta} + \sum_{\eta\in\Theta^*, \eta\not\subset\theta, l(\eta)>m} + \sum_{\eta\in\Theta^*, l(\eta)\le m}\Big)\langle f, f_\eta\rangle f_\eta =: F_1 + F_2 + F_3.$$

As was shown in the proof of Theorem 3.2 $\|f_\eta\|_{H_1} \le c|\eta|^{1/2}$. Then by (2.25) it follows that

$$|\langle f, f_\eta\rangle| \le \|f\|_{BMO}\|f_\eta\|_{H_1} \le cB|\eta|^{1/2}. \tag{3.58}$$

We use this and (3.44) to obtain

$$\begin{aligned} \Big|\int_\theta F_1(x)dx\Big| &\le \sum_{\eta\in\Theta^*, \eta\subset\theta}|\langle f, f_\eta\rangle|\Big|\int_\theta f_\eta(x)dx\Big| \\ &\le cB\sum_{\eta\in\Theta^*, \eta\subset\theta}|\eta|^{1/2}\Big|\int_{E\setminus\theta} f_\eta(x)dx\Big| \\ &\le cB\Big\|\sum_{\eta\in\Theta^*, \eta\subset\theta}|\eta|^{1/2}|f_\eta|\Big\|_{L_1(E\setminus\theta)} \\ &\le cB|\theta|^{1/2} \end{aligned} \tag{3.59}$$

Using (3.45) we have

$$\|F_2\|_{L_2(\theta)} \le cB \Big\| \sum_{\eta\in\Theta^*,\, \eta\not\subset\theta,\, l(\eta)>m} |\eta|^{1/2}|f_\eta| \Big\|_{L_2(\theta)} \le cB|\theta|^{1/2}. \tag{3.60}$$

Finally, exactly as in (3.56) (using (3.58) instead of $|\langle f, f_\eta\rangle| \le cA|\eta|^{1/2}$), we obtain

$$|F_3(x) - F_3(v_\theta)| \le cB, \quad x\in\theta. \tag{3.61}$$

Now, (3.59)-(3.61) readily imply

$$|F_3(v_\theta) - f_\theta| = \Big|F_3(v_\theta) - \frac{1}{|\theta|}\int_\theta f(x)dx\Big| \le cB.$$

This combined with the definition of BMO yields

$$\Big(\frac{1}{|\theta|}\int_\theta |f(x) - F_3(v_\theta)|^2 dx\Big)^{1/2} \le cB.$$

This in turn along with (3.60)-(3.61) implies $\|F_1\|_{L_2(\theta)} \le cB|\theta|^{1/2}$. On the other hand using (3.44) we have

$$\|F_1\|_{L_2(E\setminus\theta)} \le cB \Big\| \sum_{\eta\in\Theta^*,\, \eta\subset\theta} |\eta|^{1/2}|f_\eta(x)| \Big\|_{L_2(E\setminus\theta)} \le cB|\theta|^{1/2}.$$

Therefore, $\|F_1\|_{L_2(E)} \le cB|\theta|^{1/2}$. Consequently,

$$\sum_{\eta\in\Theta^*,\, \eta\subset\theta} |\langle f, f_\eta\rangle|^2 = \|F_1\|^2_{L_2(E)} \le cB^2|\theta|,$$

which is (3.57). The proof of Theorem 3.4 is complete. □

4 Appendix

Example of a space $H_1(E,\mathcal{T}) \neq H_1(E)$. Here $H_1(E)$ denotes the regular H_1-space on E. We consider the case $E := [-1,1]^2$.

Denote

$$g(x) = g(x_1,x_2) := \begin{cases} |\ln|x_1||^{1/2}|\ln|x_2||^{1/2} & \text{if } x\in[-1,1], \\ 0 & \text{if } x\in\mathbb{R}^2\setminus[-1,1]. \end{cases} \tag{4.1}$$

In the following we denote by $BMO(E)$ the regular BMO space on E.

Lemma 4.1. *The above defined function $g(x)$ belongs to $BMO(\mathbb{R}^2)$.*

Proof. Let $I := [a, b] \times [c, d]$ with $b - a = d - a = h > 0$. We shall prove that there is a constant C such that

$$\frac{1}{|I|}\int_I |g(x) - C|dx \le c < \infty, \tag{4.2}$$

where $c > 0$ is an absolute constant.

We first consider the important situation when $I \subset [-\frac{1}{4}, \frac{1}{4}] \times [-1, 1] \cup [-1, 1] \times [-\frac{1}{4}, \frac{1}{4}]$ and hence $0 < h \le 1/2$. Note that $g(-x_1, x_2) = g(x_1, -x_2) = g(x_1, x_2)$. Therefore, all possibilities for I are covered by considering the following three cases:

Case 1: $a < 0 < b$ and $c < 0 < d$. Because of the symmetry of $g(x)$ we may assume that $|a| \le b$ and $|c| \le d$. By integration by parts we get

$$\int_0^u |\ln t|^{1/2}dt = u|\ln u|^{1/2} + \frac{1}{2}\int_0^u |\ln t|^{-1/2}dt, \quad 0 < u < 1/2,$$

and hence

$$\frac{1}{u}\int_0^u |\ln t|^{1/2}dt = |\ln u|^{1/2} + R_u, \quad \text{where} \quad 0 < R_u \le \frac{1}{2|\ln u|^{1/2}}. \tag{4.3}$$

Denote $I_1 := [0, b] \times [0, d]$, $I_2 := [a, 0] \times [0, d]$, $I_3 := [a, 0] \times [c, 0]$, and $I_4 := [0, b] \times [c, 0]$. From the definition of $g(x)$ it follows that $0 < g(b, d) = \min_{x \in I} g(x)$ and hence

$$\frac{1}{|I|}\int_I |g(x) - g(b,d)|dx = \frac{1}{|I|}\int_I (g(x) - g(b,d))dx = \sum_{j=1}^4 \frac{1}{|I|}\int_{I_j} (g(x) - g(b,d))dx.$$

Using the assumptions and the symmetry of $g(x)$, each integral $\int_{I_j} g(x)dx$ can be written in the form

$$\int_{I_j} g(x)dx = \int_0^u \int_0^v |\ln x_1|^{1/2}|\ln x_2|^{1/2}dx_1dx_2$$

for some u, v satisfying $0 \le u \le b$ and $0 \le v \le d$. Then using (4.3), we have

$$\begin{aligned}
\frac{1}{|I|}\int_{I_j} (g(x) - g(b,d))dx &= uv|I|^{-1}\Big[(|\ln u|^{1/2} + R_u)(|\ln v|^{1/2} + R_v) - |\ln b|^{1/2}|\ln d|^{1/2}\Big] \\
&\le uv(bd)^{-1}\Big[|\ln u|^{1/2}|\ln v|^{1/2} - |\ln b|^{1/2}|\ln d|^{1/2}\Big] \\
&\quad + uv(bd)^{-1}|\ln u|^{1/2}R_v + uv(bd)^{-1}|\ln v|^{1/2}R_u + uv(bd)^{-1}R_uR_v \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

To estimate A_1 we substitute $u = sb$ and $v = td$, $0 < s, t \le 1$, and obtain

$$\begin{aligned}
A_1 &= st\Big[(|\ln s| + |\ln b|)^{1/2}(|\ln t| + |\ln d|)^{1/2} - |\ln b|^{1/2}|\ln d|^{1/2}\Big] \\
&= \frac{st\Big[(|\ln s| + |\ln b|)(|\ln t| + |\ln d|) - |\ln b||\ln d|\Big]}{(|\ln s| + |\ln b|)^{1/2}(|\ln t| + |\ln d|)^{1/2} + |\ln b|^{1/2}|\ln d|^{1/2}} \\
&\le \frac{st(|\ln s||\ln t| + B|\ln t| + D|\ln s|)}{(BD)^{1/2}},
\end{aligned}$$

where $B := |\ln b|$ and $D := |\ln d|$. Since $b - a = d - c = h > 0$ and $|a| \le b$, $|c| \le d$, then $\ln 2 \le |\ln h| \le B, D \le |\ln(h/2)| \le 2|\ln h|$. We use this above to obtain

$$A_1 \le (\ln 2)^{-1} s|\ln s| \cdot t|\ln t| + 2t|\ln t| + 2s|\ln s| \le c < \infty.$$

To estimate A_2 we use (4.3) and again replace u, v by $u = sb$, $v = td$ $(0 < s, t \le 1)$. We have

$$A_2 \le \frac{uv|\ln u|^{1/2}}{bd|\ln v|^{1/2}} = st\left(\frac{|\ln s| + |\ln b|}{|\ln t| + |\ln d|}\right)^{1/2} \le 2st\left(\frac{|\ln s| + |\ln h|}{|\ln t| + |\ln h|}\right)^{1/2}.$$

Since $1 < |\ln h| < \infty$ $(0 < h \le 1/2)$, it is readily seen that

$$\frac{|\ln s| + |\ln h|}{|\ln t| + |\ln h|} \le \frac{|\ln s| + 1}{|\ln t| + 1} + 1 \le |\ln s| + 2.$$

Consequently,

$$A_2 \le 2s(|\ln s| + 2)^{1/2} \le c < \infty.$$

Exactly in the same way we get $A_3 \le c < \infty$. Also, by (4.3),

$$A_4 \le \frac{uv}{4bd(|\ln u||\ln v|)^{1/2}} \le \frac{1}{4\ln 2}.$$

The above estimates for A_1, A_2, A_3, A_4 imply (4.2) with $C := g(b, d)$ and $c > 0$ an absolute constant.

Case 2: $0 \le a < b$ and $0 \le c < d$. In this case we shall make use of the following simple identity

$$\int_a^b |\ln t|^{1/2} dt = (b - a)|\ln b|^{1/2} + \frac{1}{2}\int_a^b \frac{t - a}{t|\ln t|^{1/2}}\, dt, \tag{4.4}$$

which can be verified by differentiating both sides with respect to b. We next use (4.4) to prove the following $(h := b - a)$:

$$\frac{1}{h}\int_a^b |\ln t|^{1/2} dt = |\ln b|^{1/2} + R_{ab}, \quad \text{where} \quad 0 < R_{ab} \le \frac{2}{|\ln h|^{1/2}}. \tag{4.5}$$

Note first that from our assumptions it follows that $h \le 1/4$. If $0 \le a \le h$, then by (4.4)

$$R_{ab} \le \frac{1}{2h}\int_a^b \frac{1}{|\ln t|^{1/2}}\, dt \le \frac{1}{2|\ln b|^{1/2}} \le \frac{1}{2|\ln 2h|^{1/2}} \le \frac{1}{|\ln h|^{1/2}}.$$

If $h < a \le 1/4$, then (4.4) implies

$$R_{ab} \le \frac{1}{2}\int_a^b \frac{1}{t|\ln t|^{1/2}}\, dt \le \frac{h}{2h|\ln h|^{1/2}} = \frac{1}{2|\ln h|^{1/2}},$$

where we used that the function $t|\ln t|^{1/2}$ is increasing on $(0, 1/2]$ and $b = a + h \le 1/2$. Finally, (4.5) is trivial if $a > 1/4$. Thus (4.5) holds true.

Evidently, $0 < g(b,d) = \min_{x\in I} g(x)$ and hence

$$\begin{aligned} \frac{1}{|I|}\int_I |g(x)-g(b,d)|dx &= \frac{1}{|I|}\int_I (g(x)-g(b,d))dx \\ &= \frac{1}{h^2}\int_a^b |\ln x_1|^{1/2}dx_1 \cdot \int_c^d |\ln x_2|^{1/2}dx_2 - |\ln b|^{1/2}|\ln d|^{1/2}. \end{aligned}$$

Now, employing (4.5), we obtain

$$\begin{aligned} \frac{1}{|I|}\int_I |g(x)-g(b,d)|dx &= (|\ln b|^{1/2}+R_{ab})(|\ln d|^{1/2}+R_{cd}) - |\ln b|^{1/2}|\ln d|^{1/2} \\ &= |\ln b|^{1/2}R_{cd} + |\ln d|^{1/2}R_{ab} + R_{ab}R_{cd} \\ &\le 2|\ln b|^{1/2}/|\ln h|^{1/2} + 2|\ln d|^{1/2}/|\ln h|^{1/2} + 4/|\ln h| \\ &\le c < \infty, \end{aligned}$$

where we used that $b,d \ge h$ and $0 < h \le 1/4$. Thus (4.2) holds true.

Case 3: $0 \le a < b$ and $c \le 0 < d$. In this case one proceeds as above using (4.3) and (4.5). We omit the details.

It remains to show that (4.2) holds whenever $I \not\subset [-\frac{1}{4},\frac{1}{4}]\times[-1,1]\cup[-1,1]\times[-\frac{1}{4},\frac{1}{4}]$. If in this case $h < 1/8$ and $I \subset [-1,1]^2$, then (4.2) is obvious since $\|g\|_{L_\infty(I)} \le g(1/8,1/8) = \ln 8$.

If $h > 1/8$, then $\int_I g(x)dx \le \int_{[-1,1]^2} g(x)dx = c < \infty$ and (4.2) follows.

If $I^\circ \cap [-1,1]^2 = \emptyset$, then (4.2) is again obvious.

Finally, suppose that $h \le 1/8$, $I \not\subset [-1,1]^2$, and $I\cap[-1,1]^2 \ne \emptyset$. Then evidently there is a square $J = [\alpha,\beta]\times[\gamma,\delta]$ of the same size as I such that $J \subset [-1,1]^2$ and $I\cap[-1,1]^2 \subset J$. Then by the monotonicity of $g(x)$, we have

$$\frac{1}{|I|}\int_I g(x)dx \le \frac{1}{|J|}\int_J g(x)dx \le c < \infty,$$

where we used the results of Case 2 or Case 3 above. This yields (4.2) with $C = 0$. The prove of the lemma is complete. □

Armed with this lemma, we proceed to showing that there is an LR-triangulation $\mathcal{T}$ of $E := [-1,1]^2$ such that $H_1(E,\mathcal{T}) \ne H_1(E)$.

From [9] (see the construction in the beginning of §2.1) it follows that there exists an LR-triangulation $\mathcal{T}$ of E with the property: There is a sequence of cells in $\mathcal{T}$: $\theta_1 \supset \theta_2 \supset \cdots$ such that

$$[-\lambda_\nu/2,\lambda_\nu/2]\times[-\varepsilon_\nu/2,\varepsilon_\nu/2] \subset \theta_\nu \subset [-\lambda_\nu,\lambda_\nu]\times[-\varepsilon_\nu,\varepsilon_\nu],$$

where $1/4 \ge \lambda_1 > \lambda_2 > \cdots > 0$, $1/4 \ge \varepsilon_1 > \varepsilon_2 > \cdots > 0$, $\lim_{\nu\to\infty}\lambda_\nu = 0$, $\lim_{\nu\to\infty}\varepsilon_\nu = 0$, and $\lim_{\nu\to\infty}\varepsilon_\nu/\lambda_\nu = 0$. In addition to this each θ_ν is convex and symmetric with respect to the x_1-axis and the x_2-axis.

Let $\theta_\nu^+ := \{(x_1,x_2)\in\theta_\nu : x_1 \ge 0\}$ and $\theta_\nu^- := \{(x_1,x_2)\in\theta_\nu : x_1 < 0\}$. The functions $a_\nu := |\theta_\nu|^{-1}(\mathbb{1}_{\theta_\nu^+} - \mathbb{1}_{\theta_\nu^-})$ are obviously atoms in $H_1(E,\mathcal{T})$.

Now, fix $\nu \ge 1$ and denote briefly $\theta := \theta_\nu$, $\lambda := \lambda_\nu$, $\varepsilon := \varepsilon_\nu$, $a := a_\nu$, etc. We next estimate from below $\|a\|_{H_1(E)}$, where $H_1(E)$ is the regular H_1-space on E. By Lemma 4.1

the function g from (4.1) is in $BMO(\mathbb{R}^2)$ and hence all functions obtain from g by dilations and shifts also belong to $BMO(\mathbb{R}^2)$ and have the same BMO norm. In particular, $g_\lambda(x) := g(x_1/\lambda+1/2, x_2/\lambda)$ belongs to $BMO(\mathbb{R}^2)$ and $\|g_\lambda\|_{BMO} = \|g\|_{BMO}$. Therefore, the restriction of g_λ on $E := [-1,1]$ (that we denote again by g_λ) belongs to $BMO(E)$ and $\|g_\lambda\|_{BMO(E)} \le \|g\|_{BMO(\mathbb{R}^2)}$.

Clearly,

$$\|a\|_{H_1(E)} = \sup_{\varphi\in BMO(E)} \frac{\langle a,\varphi\rangle}{\|\varphi\|_{BMO}} \ge \frac{\langle a, g_\lambda\rangle}{\|g_\lambda\|_{BMO}} \ge c\int_\theta a(x)g_\lambda(x)dx.$$

Since $g(x_1,-x_2) = g(x_1,x_2)$ and $g(x_1,x_2)$ is monotone decreasing with respect to x_1 on $(0,1)$, we have

$$\|a\|_{H_1(E)} \ge \frac{c}{\lambda\varepsilon}\int_0^{\varepsilon/2}\Big[\int_{-\lambda/2}^0 g\Big(\frac{x_1}{\lambda}+\frac{1}{2},\frac{x_2}{\lambda}\Big)dx_1 - \int_0^{\lambda/2} g\Big(\frac{x_1}{\lambda}+\frac{1}{2},\frac{x_2}{\lambda}\Big)dx_1\Big]dx_2.$$

Substituting $y_1 := x_1/\lambda + 1/2$ and $y_2 := x_2/\lambda$ we infer

$$\begin{aligned}\|a\|_{H_1(E)} &\ge \frac{c\lambda}{\varepsilon}\int_0^{\varepsilon/\lambda}\Big(\int_0^{1/2} g(y_1,y_2)dy_1 - \int_{1/2}^1 g(y_1,y_2)dy_1\Big)dy_2\\ &= \frac{c\lambda}{\varepsilon}\int_0^{\varepsilon/\lambda}|\ln y_2|^{1/2}dy_2\Big(\int_0^{1/2}|\ln y_1|^{1/2}dy_1 - \int_{1/2}^1|\ln y_1|^{1/2}dy_1\Big)\\ &\ge c_1(\ln\lambda/\varepsilon)^{1/2},\end{aligned}$$

where $c_1 > 0$ is an absolute constant and for the last estimate we used (4.3). Thus there is a sequence of atoms $(a_\nu)_{\nu=1}^\infty$ in $H_1(E,\mathcal{T})$ such that $\|a_\nu\|_{H_1(E)} \to \infty$ as $\nu\to\infty$, which leads to the conclusion that $H_1(E,\mathcal{T}) \ne H_1(E)$.

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