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Hardy Spaces on $\mathbb{R}^n$ with Pointwise Variable Anisotropy

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Abstract In this work we develop highly geometric Hardy spaces, for the full range $0 < p \leq 1$. These spaces are constructed over multi-level ellipsoid covers of $\mathbb{R}^n$ that are highly anisotropic in the sense that the ellipsoids can change shape rapidly from point to point and from level to level. This generalizes previous work on anisotropic Hardy spaces where the geometry of the space was 'fixed' over $\mathbb{R}^n$ and extends Hardy spaces over spaces of homogeneous type, where the theory holds for p values that are 'close' to 1.

Keywords Anisotropic function space · Spaces of homogeneous type · Atomic decomposition

Mathematics Subject Classification 42B30 · 42C40 · 41A63

1 Introduction

Anisotropic phenomena appear in various contexts in mathematical analysis and its applications. The formation of shocks results in jump discontinuities of solutions of hyperbolic conservation laws across lower dimensional manifolds and sharp edges often separate areas of little detail in digital images, to name just two examples.

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In Sect. 2 we review a general anisotropic framework on $\mathbb{R}^n$ using the multi-level ellipsoid covers introduced in [8]. Whereas in previous work the anisotropy is fixed and global over $\mathbb{R}^n$, in our settings only mild 'local' conditions are imposed on the ellipsoids which allow them to rapidly change from point to point and in depth, from level to level. The ellipsoid covers induce anisotropic quasi-distances on $\mathbb{R}^n$ and together with the usual Lebesgue measure, form spaces of homogeneous type.

The theory of function spaces defined over spaces of homogeneous type has been extensively studied from the 70s [6, 7, 13] (see [10] for an excellent survey). In this context, the theory of real Hardy spaces in more 'geometric' settings has also received much attention. Coifman and Weiss pioneered this field in the 70s [6, 7]. Then, Folland and Stein in the 80s studied Hardy spaces over homogeneous groups [11]. However, in general settings, such as the setting of spaces of homogeneous type, the Hardy Spaces with p 'close' to zero do not have sufficient structure. Bownik [3] (see also [4, 5, 12, 13]) investigated a special form of Hardy spaces defined over $\mathbb{R}^n$, where the Euclidian balls are replaced by images of the unit ball by powers of a fixed expansion matrix. In this setup, Bownik was able to construct and fully analyze anisotropic Hardy spaces for the full range $0 < p \le 1$.

In this work we generalize Bownik's spaces, by constructing Hardy spaces $H^p(\Theta)$, $0 < p \le 1$, over ellipsoid multi-level covers Θ, where the anisotropy may change rapidly from point to point. In Sect. 3 we define the Hardy spaces using anisotropic maximal functions. In Sect. 4 we introduce the atomic Hardy spaces and prove the equivalence between the two definitions. This section is rather technical but the general framework generalizes Sects. 4–6 in [3]. Finally, in Sect. 5, we show that two anisotropic Hardy spaces $H^p(\Theta_1)$ and $H^p(\Theta_2)$ are equivalent if and only if the quasi distances induced by the covers Θ_1 and Θ_2 are equivalent. In particular, this implies that the class of anisotropic Hardy spaces we construct contains and is strictly bigger than the class in [3].

Throughout the paper, the constants $c > 0$, depend on various fixed constants such as the parameters of our covers, the dimension n as well as other parameters and their value may change from line to line.

2 Anisotropic Ellipsoid Covers of $\mathbb{R}^n$

We recall the definitions of [8]. An ellipsoid is the image of the Euclidian unit ball B^* in $\mathbb{R}^n$ via an affine transform. For a given ellipsoid θ we let A_θ be an affine transform such that $\theta = A_\theta(B^*)$. Denoting by $v_\theta := A_\theta(0)$ the center of θ we have

$$A_\theta(x) = M_\theta x + v_\theta,$$

where M_θ is a nonsingular $n \times n$ matrix.

Definition 2.1 We say that

$$\Theta := \bigcup_{t\in\mathbb{R}} \Theta_t$$

is a continuous multilevel ellipsoid cover of $\mathbb{R}^n$ if it satisfies the following conditions, where $p(\Theta) := \{a_1, \dots, a_6\}$ are positive constants:

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ there exists an ellipsoid $\theta(x,t) \in \Theta_t$ and an affine transform $A_{x,t}(y) = M_{x,t}y + x$ such that $\theta(x,t) = A_{x,t}(B^*)$ and

$$a_1 2^{-t} \le \left|\theta(x,t)\right| \le a_2 2^{-t}. \tag{2.1}$$

(ii) For any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \ge 0$, if $\theta(x,t) \cap \theta(y,t+s) \ne \emptyset$, then

$$a_3 2^{-a_4 s} \le 1/\left\|M_{y,t+s}^{-1} M_{x,t}\right\| \le \left\|M_{x,t}^{-1} M_{y,t+s}\right\| \le a_5 2^{-a_6 s}. \tag{2.2}$$

Let us describe a useful form of covers of $\mathbb{R}^2$. We select all ellipses on levels ≤ 0 to be Euclidian balls. For levels > 0 we allow the ellipses to change from Euclidian balls to ellipses with the 'parabolic scaling' parameters $(a_6, a_4) = (1/3, 2/3)$. This choice of parameters relates to polygonal approximation of a planar curve, with segments of length h and approximation error of $O(h^2)$. Roughly speaking, with this choice we can simulate the performance of polygonal approximation by constructing at the level $t > 0$ 'thin' ellipses of length $\sim 2^{-t/3}$ and width $\sim 2^{-2t/3}$, such they (are aligned with and) cover the function's curve singularities with a 'strip width' of $\sim 2^{-2t/3}$. Away from the curve singularities, the ellipses can be selected to be Euclidian balls (see also the constructions in Sect. 7.1 of [8]).

We will need the following lemmas

Lemma 2.2 *Let Θ be a cover. Then there exists $c > 0$ such that for any $x \in \mathbb{R}^n$, $t > 0$, and $\lambda \ge 1$,*

$$x + \lambda M_{x,t}\left(B^*\right) \subseteq \theta(x, t - c\lambda), \tag{2.3}$$

Proof Fix $x \in \mathbb{R}^n$ and $t > 0$. Note that (2.3) holds if and only if

$$M_{x,t-c\lambda}^{-1} M_{x,t}(B^*) \subseteq \frac{1}{\lambda} B^*, \quad \lambda \ge 1.$$

From (2.2) we have $M_{x,t-c\lambda}^{-1} M_{x,t}(B^*) \subseteq a_5 2^{-a_6 c\lambda} B^*$. Therefore, one should choose large enough c, such that $a_5 2^{-a_6 c\lambda} \le \frac{1}{\lambda}$ for all $\lambda \ge 1$. □

Choosing $\lambda = 2$ in (2.3) gives

Lemma 2.3 *Let Θ be a cover. Then, there exists $J(p(\Theta)) \ge 1$ such that for every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$*

$$\theta(x,t) \subseteq x + (1/2) M_{x,t-J}(B^*) \subset \theta(x, t - J).$$

The following two covering lemmas for ellipsoid covers are versions of classic results on ball coverings in arbitrary spaces of homogeneous type (see e.g. [15]). They are essential for the Calderón-Zygmund decomposition, which we will use later.

Lemma 2.4 (Wiener) *Let Θ be a cover of $\mathbb{R}^n$. There exists a constant $\gamma(p(\Theta)) > 0$ such that for any Ω, a bounded subset of $\mathbb{R}^n$ or open with $|\Omega| < \infty$ and $t : \Omega \to \mathbb{R}$ a function, there exists a sequence of points $\{x_j\} \subset \Omega$ (finite or infinite), such that the ellipsoids $\theta(x_j, t(x_j))$ are mutually disjoint and $\Omega \subset \bigcup_j \theta(x_j, t(x_j) - \gamma)$.*

Proof One chooses the constant γ such that for all $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$, if $\theta(x,t) \cap \theta(y,s) \neq \emptyset$ with $t < s$, then $\theta(y,s) \subset \theta(x, t-\gamma)$ (see Lemma 2.8 in [8]). Using this property, the proof is standard (see e.g. [3] for details). □

Lemma 2.5 (Whitney) *Let Θ be a cover of $\mathbb{R}^n$. There exists a constant $\gamma(p(\Theta)) > 0$, such that for any open $\Omega \subset \mathbb{R}^n$ with $|\Omega| < \infty$ and any $m \geq 0$, there exist a sequence of points $\{x_j\}_{j\in\mathbb{N}} \subset \Omega$ and a sequence $\{t_j\}_{j\in\mathbb{N}}$, so that*

(i) $\Omega = \bigcup_j \theta(x_j, t_j)$,
(ii) $\theta(x_j, t_j + \gamma)$ *are pairwise disjoint,*
(iii) *For every $j \in \mathbb{N}$, $\theta(x_j, t_j - m - \gamma) \cap \Omega^c = \emptyset$, but $\theta(x_j, t_j - m - \gamma - 1) \cap \Omega^c \neq \emptyset$,*
(iv) *If $\theta(x_j, t_j - m) \cap \theta(x_i, t_i - m) \neq \emptyset$ then $|t_i - t_j| < \gamma + 1$,*
(v) *For every $j \in \mathbb{N}$*

$$\#\{i \in \mathbb{N} : \theta(x_i, t_i - m) \cap \theta(x_j, t_j - m) \neq \emptyset\} \leq L,$$

where L depends only on the parameters of the cover and m.

Proof We choose the constant γ as in the Wiener Lemma. For every $x \in \Omega$ define

$$t(x) := \inf_{s\in\mathbb{R}} \big\{\theta(x, s - m - \gamma) \subset \Omega\big\}.$$

Since Ω is open and since for each point $x \in \mathbb{R}^n$, the diameters of the ellipsoids $\theta(x, s - m - \gamma)$ decrease as $s \to \infty$ we get that $t(x)$ is well defined. Also, since Ω has finite volume, $t(x)$ is finite. By the Wiener Lemma, we can find for the function $t(x)$ a sequence $\{x_j\}$, such that $\theta(x_j, t_j + \gamma)$ are disjoint and $\Omega = \bigcup_j \theta(x_j, t_j)$, where $t_j := t(x_j)$. This gives properties (i) and (ii). By construction, $\theta(x_j, t_j - m - \gamma) \cap \Omega^c = \emptyset$ but $\theta(x_j, t_j - m - \gamma - 1) \cap \Omega^c \neq \emptyset$ which implies property (iii). To prove property (iv), assume by contradiction that there exist indices i, j such that $\theta(x_i, t_i - m) \cap \theta(x_j, t_j - m) \neq \emptyset$ with $t_j \leq t_i - \gamma - 1$. This gives that $\theta(x_i, t_i - m - \gamma - 1) \cap \theta(x_j, t_j - m) \neq \emptyset$ with $t_j - m \leq t_i - m - \gamma - 1$. Therefore $\theta(x_i, t_i - m - \gamma - 1) \subset \theta(x_j, t_j - m - \gamma)$ which is a contradiction since

$$\emptyset \neq \theta(x_i, t_i - m - \gamma - 1) \cap \Omega^c \subset \theta(x_j, t_j - m - \gamma) \cap \Omega^c = \emptyset.$$

We now prove property (v). For $j \geq 1$, let $I(j) := \{i : \theta(x_i, t_i - m) \cap \theta(x_j, t_j - m) \neq \emptyset\}$. From property (iv) we derive that $t_j \leq t_i + \gamma + 1$, $\forall i \in I(j)$. Therefore $\bigcup_{i\in I(j)} \theta(x_i, t_i - m) \subset \theta(x_j, t_j - m - 2\gamma - 1)$. On the other hand, since $t_j \geq t_i - \gamma - 1$, we also have that $|\theta(x_j, t_j - m - 2\gamma - 1)| \leq L|\theta(x_i, t_i + \gamma)|$, $\forall i \in I(j)$, for some $L \geq 1$ that depends on the properties of the cover and m. This, coupled with property (ii) gives

$$\begin{aligned}\#I(j) &\leq \frac{1}{\min_{i\in I(j)} |\theta(x_i, t_i + \gamma)|} \sum_{i\in I(j)} \big|\theta(x_i, t_i + \gamma)\big| \\ &\leq \frac{|\theta(x_j, t_j - m - 2\gamma - 1)|}{\min_{i\in I(j)} |\theta(x_i, t_i + \gamma)|} \leq L.\end{aligned}$$

□

The ellipsoid covers induce quasi-distances on $\mathbb{R}^n$. A *quasi-distance* on a set X is a mapping $\rho : X \times X \to [0,\infty)$ that satisfies the following conditions for all $x, y, z \in X$:

(a) $\rho(x,y) = 0 \Leftrightarrow x = y$,
(b) $\rho(x,y) = \rho(y,x)$,
(c) For some $\kappa \geq 1$

$$\rho(x,y) \leq \kappa\big(\rho(x,z) + \rho(z,y)\big). \tag{2.4}$$

Let Θ be a cover. We define $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ by

$$\rho(x,y) = \inf_{\theta\in\Theta}\big\{|\theta| : x, y \in \theta\big\}. \tag{2.5}$$

The following results are proved in [8].

Theorem 2.6 *The function ρ in* (2.5), *induced by an ellipsoid cover, is a quasi-distance on $\mathbb{R}^n$.*

Let Θ be an ellipsoid cover inducing a quasi-distance ρ. We denote

$$B(x,r) := \big\{y \in \mathbb{R}^n : \rho(x,y) < r\big\}. \tag{2.6}$$

Evidently,

$$B(x,r) = \bigcup_{\theta\in\Theta}\big\{\theta : |\theta| < r,\ x \in \theta\big\}.$$

Theorem 2.7 *Let Θ be an ellipsoid cover. For each ball $B(x,r)$, there exist ellipsoids $\theta', \theta'' \in \Theta$, such that $\theta' \subset B(x,r) \subset \theta''$ and $|\theta'| \sim |B(x,r)| \sim |\theta''| \sim r$, where the constants depend on $p(\Theta)$.*

Spaces of homogeneous type were first introduced in [6] (see also [14]) as a means to extend the Calderón-Zygmund theory of singular integral operators to more general settings. Let X be a topological space endowed with a Borel measure μ and a quasi-distance ρ. Assume that the balls $B(x,r) := \{y \in X : \rho(x,y) < r\}$, $x \in X$, $r > 0$, form a basis for the topology in X. The space (X, ρ, μ) is said to be of *homogenous type* if there exists a constant λ such that for all $x \in X$ and $r > 0$,

$$\mu\big(B(x,2r)\big) \leq \lambda\mu\big(B(x,r)\big). \tag{2.7}$$

If (2.7) holds then μ is said to be a *doubling measure* [15]. A space of homogeneous type is said to be *normal*, if the equivalence $\mu(B(x,r)) \sim r$ holds. Theorem 2.7 ensures (2.7) holds for the case of an ellipsoid cover and implies that it induces a normal space of homogeneous type $(\mathbb{R}^n, \rho, dx)$, where ρ is the quasi-distance (2.5) and dx is the Lebesgue measure.

We conclude this section by relating the quasi-distances induced by ellipsoid covers with the Euclidian distance. To this end we first require the following definition.

Definition 2.8 Let ρ be a quasi-distance on $\mathbb{R}^n$ and let $\mu = (\mu_0, \mu_1)$, $0 < \mu_0 \le \mu_1$. For any $x, y \in \mathbb{R}^n$ and $d > 0$ we define

$$\mu(x, y, d) := \begin{cases} \mu_0 & \rho(x, y) < d, \\ \mu_1 & \rho(x, y) \ge d. \end{cases} \qquad \tilde{\mu}(x, y, d) := \begin{cases} \mu_1 & \rho(x, y) < d, \\ \mu_0 & \rho(x, y) \ge d. \end{cases} \tag{2.8}$$

The following is proved in [9] for a discrete version of covers, but it also holds for the continuous version.

Theorem 2.9 *Let Θ be a cover and ρ the induced quasi-distance (2.5). Denote by $\mu := (\mu_0, \mu_1) = (a_6, a_4)$ where $0 < a_6 \le a_4$ are the parameters from (2.2). Then for each fixed $y \in \mathbb{R}^n$ there exist constants $0 < c_1 < c_2 < \infty$ that depend on y and $p(\Theta)$ such that*

$$c_1 \rho(x, y)^{\tilde{\mu}(x,y,1)} \le |x - y| \le c_2 \rho(x, y)^{\mu(x,y,1)}, \quad \forall x \in \mathbb{R}^n, \tag{2.9}$$

where $|x - y|$ is the usual Euclidian distance between x and y.

In the special case where the ellipsoid cover is composed of Euclidian balls, we have that the parameters in (2.2) satisfy $a_4 = a_6 = 1/n$ and (2.9) is easily verified by

$$\begin{aligned} |x - y| &\sim |\{z : |z - x| \le |y - x|\}|^{1/n} \\ &= \rho(x, y)^{1/n} = \rho(x, y)^{\mu(x,y,1)} = \rho(x, y)^{\tilde{\mu}(x,y,1)}. \end{aligned}$$

3 Anisotropic Hardy Spaces via Maximal Functions

Let $\mathcal{S}$ denote the Schwartz class of rapidly decreasing test functions (in Euclidian sense) and $\mathcal{S}'$ the dual space.

Definition 3.1 Let Θ be an ellipsoid cover. We define the following maximal functions of Hardy-Littlewood type.

$$M_B g(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |g(y)| dy, \tag{3.1}$$

$$M_\Theta g(x) := \sup_{t \in \mathbb{R}} \frac{1}{|\theta(x, t)|} \int_{\theta(x,t)} |g(y)| dy, \tag{3.2}$$

where $B(x, r)$ are the (anisotropic) balls corresponding to the quasi-distance (2.5).

Lemma 3.2 *Let Θ be an ellipsoid cover. Then for $g \in \mathcal{S}'$,*

$$M_B g(x) \sim M_\Theta g(x), \quad \forall x \in \mathbb{R}^n. \tag{3.3}$$

Proof The equivalence (3.3) is a direct consequence of the fact that by Theorem 2.7, for any anisotropic ball $B(x, r)$, there exist ellipsoids $\theta', \theta'' \in \Theta$, such that $\theta' \subseteq B(x, r) \subseteq \theta''$ and $|\theta'| \sim |B(x, r)| \sim |\theta''| \sim r$. This easily implies that

$M_B g(x) \leq c M_\Theta g(x)$. The observation that any ellipsoid $\theta \in \Theta$, with center x_θ, is contained in $B(x_\theta, |\theta|)$ provides the other direction. □

It is a classic result [15] that the Maximal Theorem holds for the Hardy-Littlewood maximal function in the general setup of spaces of homogeneous type. This, combined with Lemma 3.2 yields

Theorem 3.3 *Let Θ be an ellipsoid cover. Then*

(i) *There exists a constant c depending only on the parameters of the cover and n such that for all $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$*

$$\left|\left\{x : M_\Theta f(x) > \alpha\right\}\right| \leq c\alpha^{-1}\|f\|_1. \tag{3.4}$$

(ii) *For $1 < p < \infty$ there exists a constant A_p depending only on c and p such that for all $f \in L^p(\mathbb{R}^n)$*

$$\|M_\Theta f\|_p \leq A_p \|f\|_p. \tag{3.5}$$

It is known [15] that in contrast to the case $p > 1$, the nature of Hardy spaces for $0 < p \leq 1$ involves not only the size of a given distribution, but also some delicate cancellation properties. Therefore, we are required to replace the Hardy-Littlewood type maximal function by convolutions with functions of sufficient smoothness and fast decay.

Definition 3.4 For a function $\psi \in C^N(\mathbb{R}^n)$ and $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq N \leq \tilde{N}$, let

$$\begin{aligned}\|\psi\|_{\alpha,\tilde{N}} &:= \sup_{y\in\mathbb{R}^n} (1+|y|)^{\tilde{N}} |\partial^\alpha \psi(y)|,\\ \|\psi\|_{N,\tilde{N}} &:= \max_{|\alpha|\leq N} \|\psi\|_{\alpha,\tilde{N}},\end{aligned} \tag{3.6}$$

and

$$\mathcal{S}_{N,\tilde{N}} := \{\psi \in \mathcal{S} : \|\psi\|_{N,\tilde{N}} \leq 1\}. \tag{3.7}$$

We also denote $\mathcal{S}_N := \mathcal{S}_{N,N}$.

Let Θ be a cover where $\theta(x,t) = M_{x,t}(B^*) + x$ for each $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Denote

$$\psi_{x,t}(y) := \left|\det\left(M_{x,t}^{-1}\right)\right| \psi\left(M_{x,t}^{-1}(x-y)\right).$$

Definition 3.5 Let $g \in \mathcal{S}'$ and let $\psi \in \mathcal{S}$. We define the radial maximal function as

$$M_\psi^\circ g(x) = \sup_{t\in\mathbb{R}} \left| \int_{\mathbb{R}^n} g(y)\psi_{x,t}(y)dy \right|, \tag{3.8}$$

and for any $0 < N \leq \tilde{N}$, the grand radial maximal function of g as

$$M_{N,\tilde{N}}^\circ g(x) = \sup_{\psi\in\mathcal{S}_{N,\tilde{N}}} M_\psi^\circ g(x). \tag{3.9}$$

Let Θ be a continuous cover of $\mathbb{R}^n$ with parameters $p(\Theta) = (a_1, \dots, a_6)$ and let $0 < p \le 1$. We define $N_p(\Theta)$ as the minimal integer satisfying

$$N_p(\Theta) > \frac{\max(1, a_4)n + 1}{a_6 p}, \tag{3.10}$$

and then $\tilde{N}_p(\Theta)$ as the minimal integer satisfying

$$\tilde{N}_p(\Theta) > \frac{a_4 N_p(\Theta) + 1}{a_6}. \tag{3.11}$$

Definition 3.6 Let Θ be an ellipsoid cover and let $0 < p \le 1$. Denoting $M^\circ := M^\circ_{N_p, \tilde{N}_p}$, we define the anisotropic Hardy space as

$$H^p(\Theta) := \{g \in \mathcal{S}' : M^\circ g \in L^p\},$$

with the quasi-norm $\|g\|_{H^p(\Theta)} := \|M^\circ g\|_p$.

The next Lemma is needed to show that (up to a constant) the grand maximal function can be defined using test functions supported on B^*. The fact that $\tilde{N}_p(\Theta)$ satisfies (3.11) in relation to $N_p(\Theta)$, comes into play here.

Lemma 3.7 *Let Θ be a cover and $N \ge 1$. Denote by $\tilde{N}$ the minimal integer that satisfies $\tilde{N} > (a_4 N + 1)/a_6$. Then there exist constants $c_1, c_2 > 0$, that depend on the parameters of Θ, the dimension n and N such that for any $\psi \in \mathcal{S}_{N,\tilde{N}}$, $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$, there exists a representation $\psi_{x,s} := |\det(M_{x,s}^{-1})| \psi(M_{x,s}^{-1}(x - \cdot)) = \sum_{j=1}^{\infty} \phi^j_{x,s_j}$, where for each j*

(i) $s_j \in \mathbb{R}$,
(ii) $\phi^j \in \mathcal{S}$ *and* $\operatorname{supp}(\phi^j) \subseteq B^*$,
(iii) $\|\phi^j\|_{N,\tilde{N}} \le c_1 2^{-c_2 j}$.

Proof Without loss of generality, by applying an affine transform argument, one may assume that $x = 0$, $s = 0$ and that $\theta(x, s) = B^*$. By Lemma 2.2 there exists a constant $\gamma(p(\Theta))$, such that $2\theta(0, t) \subseteq \theta(0, t - \gamma)$, $\forall t \in \mathbb{R}$. Using classic Sobolev extension principles [1], one can construct $\phi^1 \in \mathcal{S}$ with the following properties:

1. $\operatorname{supp}(\phi^1) \subseteq B^*$,
2. $\phi^1(y) = \psi(y)$ on $\theta(0, \gamma + 1) \subseteq 1/2B^*$,
3. $\|\phi^1\|_{N,\tilde{N}} \le \tilde{c}\|\psi\|_{N,\tilde{N}} \le \tilde{c}$.

Assume by induction that we have constructed for $k \ge 1$ a series $\psi_k := \sum_{j=1}^{k} \phi^j_{0,1-j\gamma}$, with the following properties:

1. $\operatorname{supp}(\phi^j) \subseteq B^*$, $1 \le j \le k$,
2. $\operatorname{supp}(\psi_k) \subseteq \theta(0, 1 - (k-1)\gamma)$,
3. $\psi_k(y) = \psi(y)$ on $\theta(0, 1 - (k-2)\gamma)$,
4. $\|\phi^j\|_{N,\tilde{N}} \le c_1 2^{-jc_2}$, $1 \le j \le k$.

Let

$$g^{k+1}(x) := \begin{cases} (\psi - \psi_k)(x), & x \in \theta(0, 1-(k-1)\gamma), \\ \psi(x), & x \in \theta(0, 1-k\gamma) \backslash \theta(0, 1-(k-1)\gamma), \\ 0, & \text{else.} \end{cases}$$

Notice that $g^{k+1}(x) = 0$ for $x \in \theta(0, 1-(k-2)\gamma)$, since by our induction process $\psi = \psi_k$ on this ellipsoid. Let

$$h^{k+1}(y) := |\det M_{0,1-(k+1)\gamma}| g^{k+1}(M_{0,1-(k+1)\gamma} y).$$

Then

$$\operatorname{supp}\big(h^{k+1}\big) \subseteq M_{0,1-(k+1)\gamma}^{-1} M_{0,1-k\gamma}\big(B^*\big) \subseteq 1/2B^*.$$

Again, by Sobolev extension principles [1], there exists ϕ^{k+1} such that

(i) $\operatorname{supp}(\phi^{k+1}) \subseteq B^*$,
(ii) $\phi^{k+1}(y) = h^{k+1}(y)$ for $y \in M_{0,1-(k+1)\gamma}^{-1} M_{0,1-k\gamma}(B^*)$,
(iii) $\|\phi^{k+1}\|_{N,\tilde{N}} \leq \tilde{c}\|h^{k+1}\|_{N,\tilde{N}}$.

Let

$$y \in M_{0,1-(k+1)\gamma}^{-1} M_{0,1-k\gamma}\big(B^*\big) \backslash M_{0,1-(k+1)\gamma}^{-1} M_{0,1-(k-1)\gamma}\big(B^*\big).$$

Then

$$h^{k+1}(y) = |\det M_{0,1-(k+1)\gamma}| \psi(M_{0,1-(k+1)\gamma} y),$$

and with $c_2 := \gamma(a_6\tilde{N} - a_4 N - 1)/2 > 0$, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq N$,

$$\begin{aligned} |\partial^\alpha h^{k+1}(y)| &= |\det M_{0,1-(k+1)\gamma}| |\partial^\alpha (\psi(M_{0,1-(k+1)\gamma} \cdot))(y)| \\ &\leq c 2^{\gamma k(1+a_4|\alpha|)} |\partial^\alpha \psi(M_{0,1-(k+1)\gamma} y)| \\ &\leq c 2^{\gamma k(1+a_4|\alpha|)} (1 + |M_{0,1-(k+1)\gamma} y|)^{-\tilde{N}} \|\psi\|_{\mathcal{S}_{N,\tilde{N}}} \\ &\leq c 2^{\gamma k(1+a_4 N - a_6\tilde{N})} \leq c_1 2^{-c_2(k+1)}. \end{aligned}$$

Note that $\phi_{0,1-(k+1)\gamma}^{k+1}(y)$ is supported on $\theta(0, 1-(k+1)\gamma) \backslash \theta(0, 1-(k-1)\gamma)$ with $\phi_{0,1-(k+1)\gamma}^{k+1}(y) = \psi - \psi_k$ on $\theta(0, 1-(k-1)\gamma) \backslash \theta(0, 1-(k-2)\gamma)$ and $\phi_{0,1-(k+1)\gamma}^{k+1}(y) = \psi$ on $\theta(0, 1-k\gamma) \backslash \theta(0, 1-(k-1)\gamma)$. Therefore for $\psi_{k+1} := \sum_{j=0}^{k+1} \phi_{0,1-j\gamma}^{j}$ we have that $\psi_{k+1}(y) = \psi(y)$ on $\theta(0, 1-k\gamma)$. □

Theorem 3.8 *For any cover Θ, there exists constants $c_1, c_2 > 0$ depending on the parameters of the cover such that*

$$M^\circ f(x) \leq c_1 \sup_{\psi \in \mathcal{S}_{N,\tilde{N}}, \operatorname{supp}(\psi) \subseteq B^*} M_\psi^\circ f(x), \quad x \in \mathbb{R}^n, \tag{3.12}$$

$$M^\circ f(x) \leq c_2 M_\Theta f(x), \quad x \in \mathbb{R}^n. \tag{3.13}$$

Therefore the Maximal Theorem (see Theorem 3.3) also holds for M°.

Proof To prove (3.12), denote by M_C° the restriction of M°, defined by only using functions in $\mathcal{S}_{N,\tilde{N}}$ with support in B^*. For any $\psi \in \mathcal{S}_{N,\tilde{N}}$, $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$, let $\psi_{x,s} = \sum_{j=1}^\infty \phi_{x,s_j}^j$, be the representation of Lemma 3.7, where ϕ^j are supported on B^*. Thus,

$$M_\psi^\circ f(x) = \left|\int_{\mathbb{R}^n} f(y)\psi_{x,s}(y)dy\right| \le \sum_{j=1}^\infty \left|\int_{\mathbb{R}^n} f(y)\phi_{x,s_j}^j(y)dy\right|$$

$$\le M_C^\circ f(x) \sum_{j=1}^\infty \|\phi^j\|_{N,\tilde{N}} \le c_1 M_C^\circ f(x).$$

Inequality (3.13) is a simple consequence of (3.12) and The Maximal Theorem for M° is a direct application of (3.13) and Theorem 3.3. □

Using classical arguments as in Sect. III of [15], one can show that for any cover Θ and $1 < p \le \infty$, $H^p(\Theta) \sim L^p(\mathbb{R}^n)$. Therefore, for the rest of the paper, we focus our attention on the range $0 < p \le 1$. In particular, we show in Sect. 5 that for this range of p, anisotropic Hardy spaces are equivalent if and only if the underlying covers induce equivalent quasi-distances.

4 Atomic Decompositions

As in the classical case, the anisotropic Hardy spaces can be characterized and then investigated through atomic decompositions.

Definition 4.1 For a cover Θ, we say that (p,q,l) is admissible if $0 \le p \le 1$, $1 \le q \le \infty$, $p < q$, and $l \in \mathbb{N}$, such that $l \ge N_p(\Theta)$ (see (3.10)). An (p,q,l)-atom is a function $a : \mathbb{R}^n \to \mathbb{R}$ such that

(i) $\operatorname{supp}(a) \subseteq \theta(x,t)$ for some $\theta(x,t) \in \Theta$, where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,
(ii) $\|a\|_q \le |\theta(x,t)|^{1/q-1/p}$,
(iii) $\int_{\mathbb{R}^n} a(y)y^\alpha dy = 0$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \le l$.

It follows from property (2.1) that an (p,q,l)-atom a supported on an ellipsoid at the level t satisfies

$$\|a\|_q \le c2^{-t(1/q-1/p)}.$$

Definition 4.2 Let Θ be an ellipsoid cover, and let (p,q,l) be an admissible triple. We define the atomic Hardy space $H_{q,l}^p(\Theta)$ associated with Θ as the set of all tempered distribution $f \in \mathcal{S}'$ of the form $\sum_{i=1}^\infty \lambda_i a_i$, where $\sum_{i=1}^\infty |\lambda_i|^p < \infty$ and $a_i \in (p,q,l)$ for every $i \in \mathbb{N}$. The quasi norm of f is defined as

$$\|f\|_{H_{q,l}^p(\Theta)} := \inf\left\{\left(\sum_{i=1}^\infty |\lambda_i|^p\right)^{1/p} : f = \sum_{i=1}^\infty \lambda_i a_i,\ a_i \in (p,q,l)\ \forall i \in \mathbb{N}\right\}.$$

Our goal is to prove that $H^p_{q,l}(\Theta) \sim H^p(\Theta)$, for every admissible triple (p,q,l), where $H^p(\Theta)$ is defined with the maximal function M° (see Definition 3.6).

4.1 The Inclusion $H^p_{q,l}(\Theta) \subseteq H^p(\Theta)$

First we prove that each admissible atom is in $H^p(\Theta)$.

Theorem 4.3 *Suppose (p,q,l) is admissible for a cover Θ. Then there exist c such that*

$$\|M^\circ a\|_p \le c,$$

for any (p,q,l)-atom a, where c depends on p,q,n and $p(\Theta)$.

Proof Let $\theta(z,t)$ be the ellipsoid associated with an atom a, where $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We estimate the integral of the function $(M^\circ a)^p$ separately on $\theta(z,t-J)$ and on $\theta(z,t-J)^c$, where J is from Lemma 2.3.

We begin with the estimate of $\int_{\theta(z,t-J)}(M^\circ a(x))^p dx$. There are two cases: $q>1$, and $q=1$. We start with $1<q<\infty$. Since $p \le 1$ we have $q/p>1$ and by Hölder inequality we have

$$\int_{\theta(z,t-J)} \big(M^\circ a(x)\big)^p dx \le \left(\int_{\theta(z,t-J)} \big(M^\circ a(x)\big)^q dx\right)^{p/q} |\theta(z,t-J)|^{1-p/q}. \quad (4.1)$$

Applying Theorem 3.8 and then property (ii) in Definition 4.1, gives

$$\begin{aligned}\left(\int_{\theta(z,t-J)} \big(M^\circ a(x)\big)^q dx\right)^{p/q} &\le \left(\int_{\mathbb{R}^n} \big(M^\circ a(x)\big)^q dx\right)^{p/q}\\ &\le \|M^\circ a\|_q^p \le c\|a\|_q^p \le c|\theta(z,t-J)|^{p/q-1}.\end{aligned}$$

When combined with (4.1) we conclude $\int_{\theta(z,t-J)}(M^\circ a(x))^p dx \le c$. The case $q=\infty$ is simpler.

The second case is $q=1$. Since $p<q$, we have $p<1$. By the Maximal Theorem, for any $\lambda>0$, we have that $|\omega_\lambda| \le c\|a\|_1/\lambda$ for the set $\omega_\lambda := \{x \in \mathbb{R}^n : M^\circ a(x) > \lambda\}$. Combined with property (ii) in Definition 4.1 gives

$$|\omega_\lambda \cap \theta(z,t-J)| \le (c/\lambda)|\theta(z,t-J)|^{1-1/p}.$$

We proceed with

$$\begin{aligned}\int_{\theta(z,t-J)} (M^\circ a(x))^p dx &= \int_0^\infty |\omega_\lambda \cap \theta(z,t-J)| p\lambda^{p-1} d\lambda\\ &\le \int_0^{|\theta(z,t-J)|^{-1/p}} |\theta(z,t-J)| p\lambda^{p-1} d\lambda\\ &\quad + c\int_{|\theta(z,t-J)|^{-1/p}}^\infty |\theta(z,t-J)|^{1-1/p} p\lambda^{p-2} d\lambda = \tilde{c},\end{aligned}$$

where $\tilde{c} < \infty$, since $p<1$.

We now estimate $\int_{\theta(z,t-J)^c}(M^\circ a(x))^p dx$. From Lemma 2.3 we have $\theta(z,t-kJ+J)\subset\theta(z,t-kJ)$, for every $k\in\mathbb{N}$. We write

$$\int_{\theta(z,t-J)^c}(M^\circ a(x))^p dx=\sum_{k=2}^{\infty}\int_{\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)}(M^\circ a(x))^p dx$$
$$\leq c\sum_{k=2}^{\infty}2^{-t}2^{kJ}\sup_{x\in\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)}(M^\circ a(x))^p.$$

Therefore, to prove the lemma, it is sufficient to show that

$$\sup_{x\in\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)}\big(M^\circ a(x)\big)^p\leq c_1 2^t 2^{-c_2 k}, \tag{4.2}$$

for every $k\geq 3$, where $c_2>J$.

To this end, by (3.12), we may estimate $|\int_{\mathbb{R}^n}a(y)\psi_{x,s}(y)dy|^p$, where $\psi\in\mathcal{S}_N$ with support in B^*, $s\in\mathbb{R}$ and $x\in\theta(z,t-kJ)\setminus\theta(z,t-kJ+J)$. It is easy see that if $\theta(z,t)\cap\theta(x,s)=\emptyset$ then $\int_{\mathbb{R}^n}a(y)\psi_{x,s}(y)dy=0$. Thus, we may assume

$$\theta(z,t)\cap\theta(x,s)\neq\emptyset. \tag{4.3}$$

Suppose P is a polynomial (to be chosen later) of degree $N-1$, where $N\geq N_p(\Theta)$ (see (3.10)). Applying (2.1), the zero moment property of atoms (Definition 4.1) and the Hölder inequality we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}a(y)\psi_{x,s}(y)dy\right|\\
&\quad\leq c2^s\left|\int_{\mathbb{R}^n}a(y)\psi(M_{x,s}^{-1}(x-y))dy\right|\\
&\quad\leq c2^s\left|\int_{\mathbb{R}^n}a(y)(\psi(M_{x,s}^{-1}(x-y))-P(M_{x,s}^{-1}(x-y)))dy\right|\\
&\quad\leq c2^s\int_{\theta(z,t)}|a(y)||\psi(M_{x,s}^{-1}(x-y))-P(M_{x,s}^{-1}(x-y))|dy\\
&\quad\leq c2^s\|a\|_q\left(\int_{\theta(z,t)}|\psi(M_{x,s}^{-1}(x-y))-P(M_{x,s}^{-1}(x-y))|^{q'}dy\right)^{1/q'}\\
&\quad\leq c2^s\|a\|_q 2^{-s/q'}\left(\int_{F(\theta(z,t))}|\psi(y)-P(y)|^{q'}dy\right)^{1/q'},
\end{aligned}$$

where $1/q+1/q'=1$ and

$$F\big(\theta(z,t)\big):=M_{x,s}^{-1}\big(x-\big[M_{z,t}\big(B^*\big)+z\big]\big)=M_{x,s}^{-1}(x-z)-M_{x,s}^{-1}M_{z,t}\big(B^*\big).$$

Therefore

$$\left|\int_{\mathbb{R}^n}a(y)\psi_{x,s}(y)dy\right|\leq c2^{s/q}\|a\|_q\big|F\big(\theta(z,t)\big)\big|^{1/q'}\sup_{y\in F(\theta(z,t))}|\psi(y)-P(y)|.$$

Since $1 - 1/q' = 1/q$ and $\|a\|_q \le c2^{-t(1/q-1/p)}$, we get

$$\left|\int_{\mathbb{R}^n} a(y)\psi_{x,s}(y)dy\right|^p \le c2^t 2^{(s-t)(p/q)} |M_{x,s}^{-1} M_{z,t}(B^*)|^{p/q'} \sup_{y\in F(\theta(z,t))} |\psi(y) - P(y)|^p. \tag{4.4}$$

We now analyze the set $F(\theta(z,t))$. We know that

$$F(\theta(z,t)) = M_{x,s}^{-1}(x - z) - M_{x,s}^{-1} M_{z,t}(B^*),$$

where

$$x \in \theta(z, t - kJ) \setminus \theta(z, t - kJ + J) = M_{z,t-kJ}(B^*) \setminus M_{z,t-kJ+J}(B^*) + z,$$

which implies that

$$x - z \in M_{z,t-kJ}(B^*) \setminus M_{z,t-kJ+J}(B^*).$$

Therefore

$$F\big(\theta(z,t)\big) \subset \big[M_{x,s}^{-1} M_{z,t-kJ}\big(B^*\big) \setminus M_{x,s}^{-1} M_{z,t-kJ+J}\big(B^*\big)\big] - M_{x,s}^{-1} M_{z,t}\big(B^*\big). \tag{4.5}$$

Since Lemma 2.3 gives

$$M_{x,s}^{-1} M_{z,t}(B^*) \subseteq (1/2) M_{x,s}^{-1} M_{z,t-kJ+J}(B^*),$$

this yields

$$F\big(\theta(z,t)\big) \subseteq \big((1/2) M_{x,s}^{-1} M_{z,t-kJ+J}\big(B^*\big)\big)^c. \tag{4.6}$$

Case 1. $t \le s$. We choose $P = 0$ and estimate the term $|M_{x,s}^{-1} M_{z,t}(B^*)|^{p/q'}$. From (2.2) and (4.3) we induce that

$$M_{x,s}^{-1} M_{z,t}\big(B^*\big) \subset a_3^{-1} 2^{a_4(s-t)} B^*, \tag{4.7}$$

which implies that

$$|M_{x,s}^{-1} M_{z,t}(B^*)|^{p/q'} \le c2^{a_4(s-t)np/q'}.$$

Since $\psi \in \mathcal{S}_N$, where $N \ge N_p(\Theta)$ is defined in (3.10), we may apply (4.4) and (3.7) to obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^n} a(y)\psi_{x,s}(y)dy\right|^p &\le c2^t 2^{(s-t)(p/q+na_4p/q')} \sup_{y\in F(\theta(z,t))} |\psi(y)|^p \\ &\le c2^t 2^{(s-t)(p/q+na_4p/q')} \sup_{y\in F(\theta(z,t))} (1 + |y|)^{-pN}. \end{aligned} \tag{4.8}$$

We now estimate the term

$$\sup_{y\in F(\theta(z,t))} (1 + |y|)^{-pN}.$$

Since $2 \le k$ and $t \le s$, we have $t - kJ + J \le s$, which implies by (4.6) that for $y \in F(\theta(z,t))$

$$|y| \ge (2a_5)^{-1} 2^{a_6(s-t)} 2^{a_6 Jk} 2^{-a_6 J},$$

which lead to

$$(1+|y|)^{-pN} \le c 2^{-a_6(s-t)pN} 2^{-a_6 JpNk},$$

where c depends only on p and the parameters of Θ. From (4.8) we conclude that

$$\left| \int_{\mathbb{R}^n} a(y) \psi_{x,s}(y) dy \right|^p \le c 2^t 2^{(s-t)[p/q + pna_4/q' - a_6 pN]} 2^{-(a_6 JpN)k}. \tag{4.9}$$

Since $s - t \ge 0$ and $N \ge N_p(\Theta)$ satisfies (3.10) we obtain the desired estimate (4.2).
Case 2. We now assume $s \le t$. From (4.3) and (2.2) we have

$$|M_{x,s}^{-1} M_{z,t}(B^*)|^{p/q'} \le c 2^{(s-t)a_6 pn/q'}.$$

Hence (4.4) yields

$$\left| \int_{\mathbb{R}^n} a(y) \psi_{x,s}(y) dy \right|^p \le c 2^t 2^{(s-t)(p/q + a_6 pn/q')} \sup_{y \in F(\theta(z,t))} |\psi(y) - P(y)|^p, \tag{4.10}$$

which implies that

$$\left| \int_{\mathbb{R}^n} a(y) \psi_{x,s}(y) dy \right|^p \le c 2^t \sup_{y \in F(\theta(z,t))} |\psi(y) - P(y)|^p. \tag{4.11}$$

We now choose P to be the Taylor expansion of ψ at point $M_{x,s}^{-1}(x - z)$ of order $N \ge N_p(\Theta)$ and estimate $\sup_{y \in F(\theta(z,t))} |\psi(y) - P(y)|^p$. From (2.2) we have $M_{x,s}^{-1} M_{z,t}(B^*) \subset a_5 2^{-a_6(t-s)} B^*$. The Taylor Remainder Theorem gives

$$\begin{aligned}
&\sup_{y \in M_{x,s}^{-1}(x-z) + M_{x,s}^{-1} M_{z,t}(B^*)} |\psi(y) - P(y)| \\
&\quad \le c \sup_{u \in M_{x,s}^{-1} M_{z,t}(B^*)} \sup_{|\alpha| = N} |\partial^\alpha \psi(M_{x,s}^{-1}(x-z) + u)| |u|^N \\
&\quad \le c 2^{-a_6(t-s)N} \sup_{y \in M_{x,s}^{-1}(x-z) + M_{x,s}^{-1} M_{z,t}(B^*)} \sup_{|\alpha| = N} |\partial^\alpha \psi(y)| \\
&\quad \le c 2^{-a_6(t-s)N} \sup_{y \in M_{x,s}^{-1}(x-z) + M_{x,s}^{-1} M_{z,t}(B^*)} (1+|y|)^{-N}.
\end{aligned}$$

From (4.11) we get that

$$\left| \int_{\mathbb{R}^n} a(y) \psi_{x,s}(y) dy \right|^p \le c 2^t 2^{-a_6(t-s)pN} \sup_{y \in M_{x,s}^{-1}(x-z) + M_{x,s}^{-1} M_{z,t}(B^*)} (1+|y|)^{-pN}. \tag{4.12}$$

We have two cases to consider. The first one is when $t-kJ+J\le s$, and the second one is when $s\le t-kJ+J$. We start with the first case. From (2.2) we have

$$c2^{-a_6J}2^{a_6(s-t)}2^{a_6kJ}B^*\subset M_{x,s}^{-1}M_{z,t-kJ+J}\big(B^*\big), \tag{4.13}$$

which combined with (4.6) leads to

$$(1+|y|)^{-pN}\le c2^{a_6pN(t-s)}2^{-a_6pNkJ}. \tag{4.14}$$

From (4.12) and (3.10) we conclude that

$$\left|\int_{\mathbb{R}^n}a(y)\psi_{x,s}(y)dy\right|^p\le c2^t2^{-(n+1)Jk}.$$

For the case when $s\le t-kJ+J$ we proceed from (4.12) using the estimate $(1+|y|)^{-pN}\le c$, $y\in\mathbb{R}^n$, the fact that $J(k-1)\le t-s$ and the assumption (3.10) to obtain for $k\ge 3$

$$\begin{aligned}\left|\int_{\mathbb{R}^n}a(y)\psi_{x,s}(y)dy\right|^p&\le c2^t2^{-a_6(t-s)pN}\\&\le c2^t2^{-a_6J(k-1)pN}\le c2^t2^{-(n+1)(k-1)J}\le c2^t2^{-(2(n+1)/3)kJ}.\end{aligned}$$

Thus, we may conclude (4.2) for the case $s\le t$ which completes the proof. □

Theorem 4.4 *Let Θ be a cover and suppose (p,q,l) is admissible (see Definitions 3.6, 4.1 and 4.2). Then*

$$H_{q,l}^p(\Theta)\subseteq H^p(\Theta).$$

Proof Let $f\in H_{q,l}^p$. For $\epsilon>0$, assume that $f=\sum_{i=1}^\infty\lambda_ia_i$, where $\sum_{i=1}^\infty|\lambda_i|^p\le\|f\|_{H_{q,l}^p}^p+\epsilon$. Then, from Theorem 4.3

$$\begin{aligned}\|f\|_{H^p(\Theta)}^p&=\int_{\mathbb{R}^n}\left[M^\circ\left(\sum_{i=1}^\infty\lambda_ia_i\right)(x)\right]^pdx\\&\le\sum_{i=1}^\infty|\lambda_i|^p\int_{\mathbb{R}^n}[M^\circ(a_i)(x)]^pdx\le c(\|f\|_{H_{q,l}^p(\Theta)}^p+\epsilon).\end{aligned}$$

□

4.2 The Calderón-Zygmund Decomposition

To show the converse inclusion $H^p(\Theta)\subseteq H_{q,l}^p(\Theta)$ we need to carefully construct, for each given distribution, an appropriate atomic decomposition. This is achieved by using the Calderón-Zygmund decomposition. Throughout this section for a given cover Θ, we consider a tempered distribution f such that for every $\lambda>0$, $|\{x:M^\circ f(x)>\lambda\}|<\infty$. For fixed $\lambda>0$ we define

$$\Omega:=\{x:M^\circ f(x)>\lambda\}.$$

Recall that there exists a constant γ such that for all $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$, if $\theta(x,t) \cap \theta(y,s) \neq \emptyset$ with $t < s$, then $\theta(y,s) \subset \theta(x, t-\gamma)$. Applying the Whitney Lemma 2.5 on Ω with $m := J + \gamma$, where the constants J and γ are defined in Sect. 2, yields sequences $(x_i)_{i\in\mathbb{N}} \subset \Omega$ and $(t_i)_{i\in\mathbb{N}}$, such that

$$\Omega = \bigcup_{j\in\mathbb{N}} \theta(x_i, t_i), \tag{4.15}$$

$$\theta(x_i, t_i + \gamma) \cap \theta(x_j, t_j + \gamma) = \emptyset, \quad \forall i \neq j, \tag{4.16}$$

$$\begin{aligned} &\theta(x_i, t_i - J - 2\gamma) \cap \Omega^c = \emptyset, \\ &\theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c \neq \emptyset, \quad \forall i \in \mathbb{N}, \end{aligned} \tag{4.17}$$

$$\theta(x_i, t_i - J - \gamma) \cap \theta(x_j, t_j - J - \gamma) \neq \emptyset \quad \text{then } |t_i - t_j| \leq \gamma + 1, \tag{4.18}$$

$$\#\bigl\{ j \in \mathbb{N} : \theta(x_j, t_j - J - \gamma) \cap \theta(x_i, t_i - J - \gamma) \neq \emptyset \bigr\} \leq L, \quad \forall i \in \mathbb{N}. \tag{4.19}$$

Fix $\phi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\phi) \subset 2B^*$, $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on B^*. For every $i \in \mathbb{N}$ we define

$$\tilde{\phi}_i(x) := \phi\bigl(M_{x_i,t_i}^{-1}(x - x_i)\bigr). \tag{4.20}$$

We have that $\tilde{\phi}_i \equiv 1$ on $\theta(x_i, t_i)$ and also by Lemma 2.3

$$\operatorname{supp}(\tilde{\phi}_i) \subseteq x_i + 2M_{x_i,t_i}\bigl(B^*\bigr) \subseteq \theta(x_i, t_i - J).$$

We define

$$\phi_i(x) := \begin{cases} \frac{\tilde{\phi}_i(x)}{\sum_j \tilde{\phi}_j(x)}, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases} \tag{4.21}$$

Observe that ϕ_i is well defined since by (4.15) and (4.19), $1 \leq \sum_i \tilde{\phi}_i(x) \leq L$, for every $x \in \Omega$. Also $\phi_i \in C^\infty(\mathbb{R}^n)$, and $\operatorname{supp}(\phi_i) \subseteq \theta(x_i, t_i - J)$. From (4.15) and (4.17), we conclude that for every $x \in \mathbb{R}^n$

$$\sum_i \phi_i(x) = \mathbb{1}_\Omega(x),$$

which implies that the family $\{\phi_i\}$ forms a smooth partition of unitary subordinate to the covering of Ω by the ellipsoids $\{\theta(x_i, t_i - J)\}$.

Let $\mathcal{P}_l$ denote the space of polynomials of n variables with degree $\leq l$, where $N_p(\Theta) \leq N \leq l$ (see Definition 4.1). For each $i \in \mathbb{N}$ we introduce an Hilbert space structure on the space $\mathcal{P}_l$ by setting

$$\langle P, Q\rangle_i := \frac{1}{\int \phi_i} \int_{\mathbb{R}^n} P(x)Q(x)\phi_i(x)dx, \quad \forall P, Q \in \mathcal{P}_l. \tag{4.22}$$

The distribution $f \in \mathcal{S}'$ induces a linear functional on $\mathcal{P}_l$ by

$$Q \to \langle f, Q\rangle_i, \quad \forall Q \in \mathcal{P}_l,$$

which by the Riesz Lemma is represented by a unique polynomial $P_i \in \mathcal{P}_l$ such that

$$\langle f, Q\rangle_i = \langle P_i, Q\rangle_i, \quad \forall Q \in \mathcal{P}_l. \tag{4.23}$$

Obviously P_i is the orthogonal projection of f with respect to the norm induced by (4.22).

For every $i \in \mathbb{N}$ we define the locally 'bad part' $b_i = (f - P_i)\phi_i$. We will show that with $N := N_p(\Theta)$, $\tilde{N} := \tilde{N}_p(\Theta)$ and $l \geq N$, the series $\sum_i b_i$ converges in $\mathcal{S}'$, which will allow us to define the 'good part' $g := f - \sum_i b_i$.

Definition 4.5 The representation $f = g + \sum_i b_i$, where g and b_i as above, is a Calderón-Zygmund decomposition of degree l and height λ associated with M°.

Lemma 4.6 *For any $i \in \mathbb{N}$, let $z_i \in \theta(x_i, t_i - K_1)$ and $s_i \in \mathbb{R}$ such that $t_i \leq s_i + K_2$, where $K_1, K_2 > 0$. Then, there exist a constant $c > 0$ depending on the parameters of the cover, N, K_1, K_2 and choice of ϕ, such that*

$$\sup_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} |\partial^\alpha \hat{\phi}_i(y)| \leq c, \quad \textit{where } \hat{\phi}_i(y) := \phi_i\big(M_{z_i,s_i}(y)\big).$$

Proof Recall that for $i \in \mathbb{N}$, $\operatorname{supp}(\phi_i) \subseteq \theta(x_i, t_i - J)$. Also, by (4.19), for $U := \{j \in \mathbb{N} : \theta(x_j, t_j - J) \cap \theta(x_i, t_i - J) \neq \emptyset\}$, we have that $\#U \leq L$. Thus, we may write

$$\begin{aligned}\hat{\phi}_i(y) = \phi_i((M_{z_i,s_i}(y)) &= \frac{\tilde{\phi}_i((M_{z_i,s_i}(y))}{\sum_{j\in\mathbb{N}} \tilde{\phi}_j((M_{z_i,s_i}(y))} \\ &= \frac{\phi(M^{-1}_{x_i,t_i} M_{z_i,s_i}(y) - M^{-1}_{x_i,t_i}(x_i))}{\sum_{j\in U} \phi(M^{-1}_{x_j,t_j} M_{z_i,s_i}(y) - M^{-1}_{x_j,t_j}(x_j))}.\end{aligned}$$

The desired estimate follows from iterative application of quotient rule combined with

$$\sup_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} \big|\partial^\alpha \phi\big(M^{-1}_{x_j,t_j} M_{z_i,s_i}(\cdot)\big)(y)\big| \leq c, \tag{4.24}$$

where $c > 0$ depends on the parameters of the cover, N, K_1, K_2 and choice of ϕ. Indeed, (4.24) holds, since by (2.2), for every $j \in U$, we have that $\|M^{-1}_{x_j,t_j} M_{x_i,t_i}\| \leq c_1$ and $\|M^{-1}_{x_i,t_i} M_{z_i,s_i}\| \leq c_2$ for some constants $c_1, c_2 > 0$. Thus we also have $\|M^{-1}_{x_j,t_j} M_{z_i,s_i}\| \leq c_3$ for some constant $c_3 > 0$. □

For a fixed $i \in \mathbb{N}$, let $\{\pi_\beta : \beta \in \mathbb{N}^n_+, |\beta| \leq l\}$, be an orthonormal basis for $\mathcal{P}_l$ with respect to the Hilbert space structure (4.22). For $|\beta| \leq l$ and a point $z \in \theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c$ (whose existence is guaranteed by (4.17)) we define

$$\Phi_\beta(y) := \frac{|\det(M_{z,t_i})|}{\int \phi_i} \pi_\beta\big(z - M_{z,t_i}(y)\big)\phi_i\big(z - M_{z,t_i}(y)\big), \tag{4.25}$$

Lemma 4.7 *There exists $c > 0$ such that*

$$\|\Phi_\beta\|_{N,\tilde{N}} \leq c, \quad \textit{for all } |\beta| \leq l.$$

Proof We begin with the estimate of the first term in (4.25). We know that

$$\int_{\mathbb{R}^n} \phi_i(y)dy = \int_{\theta(x_i,t_i-J)} \phi_i(y)dy \geq \int_{\theta(x_i,t_i)} \frac{1}{L}dy = \frac{1}{L}|\theta(x_i,t_i)|.$$

Applying (2.1) gives

$$\frac{|\det(M_{z,t_i})|}{\int \phi_i} \leq cL\frac{|\theta(z,t_i)|}{|\theta(x_i,t_i)|} \leq cLa_1^{-1}a_2 \leq c. \tag{4.26}$$

For the third term in (4.25), we get from Lemma 4.6

$$\sup_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} |\partial^\alpha \phi_i(z - M_{z,t_i}(\cdot))(y)| \leq c.$$

We now deal with the second term, $\pi_\beta(z - M_{z,t_i}(y))$. We have

$$\begin{aligned}\operatorname{supp}(\Phi_\beta) &= \operatorname{supp}(\phi_i(z - M_{z,t_i}(\cdot))) \\ &\subseteq \{y \in \mathbb{R}^n : y \in M_{z,t_i}^{-1}(z - x_i) + M_{z,t_i}^{-1}M_{x_i,t_i-J}(B^*)\}.\end{aligned}$$

Since $z \in \theta(x_i, t_i - J - 2\gamma - 1) = x_i + M_{x_i,t_i-J-2\gamma-1}(B^*)$, there exist constants $c_1, c_2 > 0$ that depend only on $p(\Theta)$ such that

$$M_{x_i.t_i}^{-1}(z - x_i) \in M_{x_i,t_i}^{-1}M_{x_i,t_i-J-2\gamma-1}\left(B^*\right) \subseteq c_1B^*,$$
$$M_{z,t_i}^{-1}M_{x_i,t_i-J}\left(B^*\right) \subset M_{z,t_i}^{-1}M_{x_i,t_i-J-2\gamma-1}\left(B^*\right) \subseteq c_2B^*.$$

Therefore we conclude that for some $c_3 > 0$, $\operatorname{supp}(\Phi_\beta) \subset c_3B^*$. Thus, to prove the Lemma, it remains to show that the partial derivatives of Φ_β up to the order N are bounded.

Observe that $z - M_{z,t_i}(B^*) \subset \theta(x_i, t_i - J - 2\gamma - 1)$. Since $\mathcal{P}_l$ is finite vector space, all the norms are equivalent and there exists a constant $c_4 > 0$ such that for every $P \in \mathcal{P}_l$

$$\sup_{|\alpha|\leq N} \sup_{y\in c_3B^*} |\partial^\alpha P(y)| \leq c_4 \int_{B^*} |P(y)|^2dy.$$

For the same reason, since $\theta(x_i, t_i - J - 2\gamma - 1)$ and $\theta(x_i, t_i + \gamma)$ have similar shape and volume, we also have that

$$\int_{\theta(x_i,t_i-J-2\gamma-1)} |P(y)|^2dy \leq c \int_{\theta(x_i,t_i+\gamma)} |P(y)|^2dy.$$

Applying the last two estimates together with $\phi_i(y) = 1$ for $y \in \theta(x_i, t_i)$ yields

$$\begin{aligned}&\sup_{|\alpha|\leq N} \sup_{y\in c_3B^*} |\partial^\alpha(\pi_\beta(z - M_{z,t_i}(\cdot)))(y)| \\ &\quad\leq c_4 \int_{B^*} |\pi_\beta(z - M_{z,t_i}(y))|^2dy\end{aligned}$$

$$\leq \frac{c}{|\det(M_{z,t_i})|} \int_{z-M_{z,t_i}(B^*)} |\pi_\beta(y)|^2 dy \leq \frac{c}{|\det(M_{z,t_i})|} \int_{\theta(x_i,t_i-J-2\gamma-1)} |\pi_\beta(y)|^2 dy$$

$$\leq \frac{c}{|\det(M_{z,t_i})|} \int_{\theta(x_i,t_i+\gamma)} |\pi_\beta(y)|^2 \phi_i(y) dy \leq \frac{c}{\int \phi_i} \int_{\mathbb{R}^n} |\pi_\beta(y)|^2 \phi_i(y) dy \leq c.$$

Now, since Φ_β is supported on $c_3 B^*$ and we have bounded the $\mathcal{S}_{N,\tilde{N}}$ norm of the three terms in (4.25) by absolute constants we can apply the product rule to conclude the Lemma. □

We can now estimate 'local' good parts of f.

Lemma 4.8 *There exist a constant $c > 0$ such that*

$$\sup_{y\in\mathbb{R}^n} |P_i(y)\phi_i(y)| \leq c\lambda,$$

where ϕ_i is defined in (4.21) *and P_i is defined by* (4.23).

Proof Combining $\mathrm{supp}(\phi_i) \subseteq \theta(x_i, t_i - J)$ and $|\phi_i(y)| \leq 1$, we have

$$\sup_{y\in\mathbb{R}^n} |P_i(y)\phi_i(y)| \leq \sup_{y\in\theta(x_i,t_i-J)} |P_i(y)|.$$

For the function Φ_β defined in (4.25) and the point $z \in \Omega^c$, Lemma 4.7 yields

$$\left| \int f(y)(\Phi_\beta)_{z,t_i}(y) dy \right| \leq \|\Phi_\beta\|_{N,\tilde{N}} M^\circ f(z) \leq c\lambda.$$

Also, using definitions (4.25) and then (4.22)

$$\left| \int f(y)(\Phi_\beta)_{z,t_i}(y) dy \right| = \left| \int f(y) |\det(M_{z,t_i}^{-1})| \Phi_\beta(M_{z,t_i}^{-1}(z-y)) dy \right|$$

$$= c \left| \frac{1}{\int \phi_i} \int_{\mathbb{R}^n} f(y)\pi_\beta(y)\phi_i(y) dy \right| = c|\langle f, \pi_\beta\rangle_i|.$$

Therefore for all $|\beta| \leq l$

$$|\langle f, \pi_\beta\rangle_i| \leq c\lambda. \tag{4.27}$$

Since $\{\pi_\beta\}$ are polynomials of degree $\leq l$ and an orthonormal system in the Hilbert space defined by (4.22), one can show using the equivalence of norms of finite dimensional Banach spaces that

$$\|\pi_\beta\|_{L_\infty(\theta(x_i,t_i-J))} \leq c|\theta(x_i, t_i - J)|^{-1/2} \|\pi_\beta\|_{L_2(\theta(x_i,t_i-J))} \leq c. \tag{4.28}$$

Recall that by (4.23) we have that

$$P_i = \sum_{|\beta|\leq l} \langle f, \pi_\beta\rangle_i \pi_\beta. \tag{4.29}$$

Therefore, combining (4.27) and (4.28) we have

$$\sup_{y\in\theta(x_i,t_i-J)} |P_i(y)| \le \sum_{|\beta|\le l} |\langle f,\pi_\beta\rangle_i||\pi_\beta(y)| \le c\lambda,$$

which completes the proof. □

Lemma 4.9 *There exist a constant* $c>0$ *such that*

$$M^\circ b_i(x) \le cM^\circ f(x) \quad \textit{for all } x\in\theta(x_i,t_i-J). \tag{4.30}$$

Proof Take $\psi\in\mathcal{S}_{N,\tilde N}$, $x\in\theta(x_i,t_i-J)$ and $s\in\mathbb{R}$. We have

$$\begin{aligned}\int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy &= \int_{\mathbb{R}^n}(f(y)-P_i(y))\phi_i(y)\psi_{x,s}(y)dy\\ &= \int_{\mathbb{R}^n} f(y)\phi_i(y)\psi_{x,s}(y)dy - \int_{\mathbb{R}^n} P_i(y)\phi_i(y)\psi_{x,s}(y)dy,\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy\right|\\ &\quad\le \left|\int_{\mathbb{R}^n} f(y)\phi_i(y)\psi_{x,s}(y)dy\right| + \left|\int_{\mathbb{R}^n} P_i(y)\phi_i(y)\psi_{x,s}(y)dy\right|\\ &\quad=: I_1+I_2.\end{aligned} \tag{4.31}$$

First we estimate I_2. Since $\psi\in\mathcal{S}_{N,\tilde N}$ and $\tilde N>n$, we have that $\|\psi\|_{L^1}\le c$. For $x\in\Omega$, we have $M^\circ f(x)>\lambda$ and thus combined with Lemma 4.8 we have

$$I_2 = \left|\int_{\mathbb{R}^n} P_i(y)\phi_i(y)\psi_{x,s}(y)dy\right| \le c\lambda\|\psi\|_{L^1} \le c\lambda \le cM^\circ f(x).$$

For the estimate of I_1 in (4.31), there are two cases.
Case 1: $t_i\le s$. Define $\Phi(y) := \phi_i(x-M_{x,s}(y))\psi(y)$. Since $I_1 = |\int_{\mathbb{R}^n} f(y)\times \Phi_{x,s}(y)dy|$ and $\Phi\in\mathcal{S}$ we have

$$\left|\int_{\mathbb{R}^n} f(y)\Phi_{x,s}(y)dy\right| \le \|\Phi\|_{N,\tilde N} M^\circ f(x).$$

Now we estimate the term $\|\Phi\|_{N,\tilde N}$. Since $t_i\le s$ and $x\in\theta(x_i,t_i-J)$, Lemma 4.6 yields

$$\sup_{y\in\mathbb{R}^n,|\alpha|\le N} |\partial^\alpha\phi_i(x-M_{x,s}(\cdot))(y)| \le c.$$

By the product rule

$$\|\Phi\|_{N,\tilde N} = \sup_{y\in\mathbb{R}^n,|\alpha|\le N}(1+|y|)^{\tilde N}|\partial^\alpha(\phi_i(x-M_{x,s}(\cdot))\psi(\cdot))(y)|$$

$$\leq c\Big(\sup_{y\in\mathbb{R}^n,|\alpha|\leq N}(1+|y|)^{\tilde{N}}|\partial^\alpha\psi(y)|\Big)\Big(\sup_{y\in\mathbb{R}^n,|\alpha|\leq N}|\partial^\alpha(\phi_i(x-M_{x,s}(\cdot))(y)|\Big)$$

$$\leq c\|\psi\|_{N,\tilde{N}}\leq c.$$

Hence

$$I_1=\left|\int_{\mathbb{R}^n}f(y)\phi_i(y)\psi_{x,s}(y)dy\right|\leq cM^\circ f(x).$$

Case 2: $s<t_i$ We define

$$\tilde{\Phi}(y):=|\det(M_{x,s}^{-1}M_{x,t_i})|\phi_i(x-M_{x,t_i}(y))\psi(M_{x,s}^{-1}M_{x,t_i}(y)).$$

First we note that

$$I_1=\int_{\mathbb{R}^n}f(y)\phi_i(y)\psi_{x,s}(y)dy=\int_{\mathbb{R}^n}f(y)\tilde{\Phi}_{x,t_i}(y)dy,$$

which implies that

$$I_1=\left|\int_{\mathbb{R}^n}f(y)\tilde{\Phi}_{x,t_i}(y)dy\right|\leq\|\tilde{\Phi}\|_{N,\tilde{N}}M^\circ f(x). \tag{4.32}$$

Therefore, it suffices to show that $\|\tilde{\Phi}\|_{N,\tilde{N}}\leq c$. Because $x\in\theta(x_i,t_i-J)$, we get

$$\operatorname{supp}(\tilde{\Phi})\subset\operatorname{supp}(\phi_i(x-M_{x,t_i}(\cdot)))\subset cB^*.$$

From Lemma 4.6 we conclude that

$$\sup_{y\in\mathbb{R}^n,|\alpha|\leq N}|\partial^\alpha(\phi_i(x-M_{x,t_i}(\cdot)))(y)|\leq c.$$

Since $\psi\in\mathcal{S}_{N,\tilde{N}}$ and $\|M_{x,s}^{-1}M_{x,t_i}\|\leq c$ for $s\leq t_i$, we have

$$\sup_{y\in\mathbb{R}^n,|\alpha|\leq N}(1+|y|)^{\tilde{N}}|\partial^\alpha(\psi(M_{x,s}^{-1}M_{x,t_i}(\cdot)))(y)|\leq c.$$

From the last two estimates and the chain rule we conclude that

$$\|\tilde{\Phi}\|_{N,\tilde{N}}=\sup_{y\in\mathbb{R}^n,|\alpha|\leq N}(1+|y|)^{\tilde{N}}|\tilde{\Phi}(y)|\leq c.$$

Therefore, by (4.32) we have

$$I_1\leq\|\tilde{\Phi}\|_{N,\tilde{N}}M^\circ f(x)\leq cM^\circ f(x). \qquad \square$$

Lemma 4.10 *There exists a constant $c>0$ such that for all $i\in\mathbb{N}$ and $k\geq 0$*

$$M^\circ b_i(x)\leq c\lambda\nu^{-k} \tag{4.33}$$

for all $x\in\theta(x_i,t_i-J(k+2))\setminus\theta(x_i,t_i-J(k+1))$, where $\nu:=2^{a_6JN}$.

Proof Since $M^\circ b_i(x) = \sup_{\psi\in\mathcal{S}_{N,\tilde{N}}} \sup_{s\in\mathbb{R}} \int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy$, choose any $\psi \in \mathcal{S}_{N,\tilde{N}}$ and $s \in \mathbb{R}$. We then consider two cases.
Case 1: $s \geq t_i$. From the definition of b_i

$$\begin{aligned}\left|\int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy\right| &\leq \left|\int_{\mathbb{R}^n} f(y)\phi_i(y)\psi_{x,s}(y)dy\right| + \left|\int_{\mathbb{R}^n} P_i(y)\phi_i(y)\psi_{x,s}(y)dy\right| \\ &=: I_1 + I_2.\end{aligned}$$

We begin with the estimation of I_1. For $w \in \theta(x_i, t_i - J - 2\gamma - 1) \cap \Omega^c$ define

$$\Phi(z) := \frac{|\det(M_{x,s}^{-1})|}{|\det(M_{w,s}^{-1})|}\phi_i\big(w - M_{w,s}(z)\big). \tag{4.34}$$

Since $w \in \Omega^c$, we get

$$\begin{aligned}I_1 &= \left|\int_{\mathbb{R}^n} f(y)\Phi_{w,s}(y)\psi\big(M_{x,s}^{-1}(x-y)\big)dy\right| \\ &\leq \max_{y\in\theta(x_i,t_i-J)} \left|\psi\big(M_{x,s}^{-1}(x-y)\big)\right| \|\Phi\|_{N,\tilde{N}}\lambda.\end{aligned}$$

First we estimate $\|\Phi\|_{N,\tilde{N}}$ by estimating each of the two terms in (4.34). From property (2.1) of covers

$$\frac{|\det(M_{x,s}^{-1})|}{|\det(M_{w,s}^{-1})|} \leq c(n)a_1^{-1}a_2.$$

Since $w \in \theta(x_i, t_i - 2\gamma - J - 1)$ and $s \geq t_i$, Lemma 4.6 yields

$$\sup_{y\in\mathbb{R}^n, |\alpha|\leq N} \left|\partial^\alpha\phi_i\big(w - M_{w,s}(\cdot)\big)(y)\right| \leq c. \tag{4.35}$$

Now we estimate the term $\max_{y\in\theta(x_i,t_i-J)} |\psi(M_{x,s}^{-1}(x-y))|$. Since $x \notin \theta(x_i, t_i - J(k+1))$ and $y \in \theta(x_i, t_i - J)$, there exists a constant c' such that $y \notin \theta(x, t_i + c' - Jk)$. Thus $y \notin x + M_{x,t_i+c'-Jk}B^*$ which implies that for some constant $c'' > 0$, $M_{x,s}^{-1}(x-y) \notin c'' M_{x,s}^{-1} M_{x,t_i-Jk}B^*$. This gives

$$\left|M_{x,s}^{-1}(x-y)\right| \geq c2^{a_6(s-t_i+Jk)}. \tag{4.36}$$

Therefore, since $\psi \in \mathcal{S}_{N,\tilde{N}}$

$$\begin{aligned}\max_{y\in\theta(x_i,t_i-J)} |\psi(M_{x,s}^{-1}(x-y))| &\leq \max_{y\in\theta(x_i,t_i-J)} (1 + |M_{x,s}^{-1}(x-y)|)^{-\tilde{N}} \\ &\leq c2^{-a_6(s-t_i+Jk)\tilde{N}} \leq c2^{-\nu k}.\end{aligned}$$

This concludes the estimate for I_1 and we now proceed with the estimate of I_2. From Lemma 4.8, and from the fact that $\psi \in \mathcal{S}_{N,\tilde{N}}$ we get

$$\begin{aligned}
I_2 &= \left| \int_{\mathbb{R}^n} P_i(y)\phi_i(y)\psi_{x,s}(y)dy \right| \\
&\le \int_{\theta(x_i,t_i-J)} |P_i(y)\phi_i(y)||\det(M_{x,s}^{-1})\|\psi(M_{x,s}^{-1}(x-y))|dy \\
&\le c\lambda|\det(M_{x,s}^{-1})| \int_{\theta(x_i,t_i-J)} |\psi(M_{x,s}^{-1}(x-y))|dy \\
&\le c\lambda|\det(M_{x,s}^{-1})| \int_{\theta(x_i,t_i-J)} (1+|M_{x,s}^{-1}(x-y))|)^{-\tilde{N}}dy.
\end{aligned}$$

Since $|\det(M_{x,s}^{-1})||\theta(x_i,t_i-J)| \le c2^{s-t_i}$, we have using (4.36)

$$I_2 \le c\lambda 2^{s-t_i} 2^{-(s-t_i+Jk)a_6\tilde{N}} \le c\lambda 2^{-\nu k},$$

which complete the first case.

Case 2: $s < t_i$. From Theorem 3.8 we know that

$$M^\circ b_i(x) \le c \sup_{\psi\in\mathcal{S}_{N,\tilde{N}},\mathrm{supp}(\psi)\subseteq B^*} \left| \int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy \right|.$$

Thus, let $\psi \in \mathcal{S}_{N,\tilde{N}}$ such that $\mathrm{supp}(\psi) \subseteq B^*$. Since $\mathrm{supp}(b_i) \subset \theta(x_i,t_i-J)$, if $\theta(x_i,t_i-J) \cap \theta(x,s) = \emptyset$ then

$$\int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy = 0. \tag{4.37}$$

Hence we assume

$$\theta(x_i,t_i-J) \cap \theta(x,s) \neq \emptyset. \tag{4.38}$$

The Taylor expansion of ψ about $z := M_{x,s}^{-1}(x-x_i)$, of order N is

$$\begin{aligned}
\psi(M_{x,s}^{-1}(x-y)) &= \psi(z+M_{x,s}^{-1}(x_i-y)) \\
&= \sum_{|\alpha|\le N} \frac{\partial^\alpha\psi(z)}{\alpha!}(M_{x,s}^{-1}(x_i-y))^\alpha + R_z(z+M_{x,s}^{-1}(x_i-y)),
\end{aligned}$$

where R_z is the residue.

Let $w \in \theta(x_i,t_i-J-\gamma-1) \cap \Omega^c$.

$$\Phi(y) := \frac{|\det(M_{x,s}^{-1})|}{|\det(M_{w,t_i}^{-1})|}\phi_i(w-M_{w,t_i}(y)).$$

From Lemma 4.6 and the fact that $s < t_i$ we get $\|\Phi\|_{N,\tilde{N}} \le c$. By (4.23), the local 'bad' part b_i has $l \ge N$ zero moments and therefore

$$\begin{aligned}\left|\int_{\mathbb{R}^n} b_i(y)\psi_{x,s}(y)dy\right| &= \left|\int_{\mathbb{R}^n} b_i(y)|\det(M_{x,s}^{-1})|R_z(z+M_{x,s}^{-1}(x_i-y))dy\right| \\ &\le \left|\int_{\theta(x_i,t_i-J)} f(y)\Phi_{w,t_i}(y)R_z(z+M_{x,s}^{-1}(x_i-y))dy\right| \\ &\quad + |\det(M_{x,s}^{-1})|\left|\int_{\theta(x_i,t_i)} P_i(y)\phi_i(y)R_z(z+M_{x,s}^{-1}(x_i-y))dy\right| \\ &=: I_1+I_2.\end{aligned}$$

Since $w \in \Omega^c$ we have

$$\begin{aligned}I_1 &\le \|\Phi\|_{N,\tilde{N}} M^\circ f(w) \sup_{y\in\theta(x_i,t_i-J)} |R_z(z+M_{x,s}^{-1}(x_i-y))| \\ &\le c\lambda \sup_{y\in\theta(x_i,t_i-J)} |R_z(z+M_{x,s}^{-1}(x_i-y))|.\end{aligned}$$

Thus, to complete the estimate of I_1 it is sufficient to show that

$$\sup_{y\in\theta(x_i,t_i-J)} \left|R_z\big(z+M_{x,s}^{-1}(x_i-y)\big)\right| \le c\big(2^{a_6JN}\big)^{-k}. \tag{4.39}$$

Assuming $s \le t_i - J$ (the case $t_i - J \le s < t_i$ is easier), using (4.38) and applying the Taylor remainder theorem gives

$$\begin{aligned}&\sup_{y\in\theta(x_i,t_i-J)} |R_z(z+M_{x,s}^{-1}(x_i-y))| \\ &\le c \sup_{u\in M_{x,s}^{-1}M_{x_i,t_i-J}(B^*)} \sup_{|\alpha|=N} |\partial^\alpha\psi(z+u)||u|^N \\ &\le c2^{-a_6(t_i-s)N} \sup_{v\in M_{x,s}^{-1}(x-x_i)+M_{x,s}^{-1}M_{x_i,t_i-J}(B^*)} (1+|v|)^{-N}.\end{aligned}$$

Since $x \notin \theta(x_i, t_i - J(k+1))$, we get that $M_{x,s}^{-1}(x-x_i) \notin M_{x,s}^{-1}M_{x_i,t_i-J(k+1)}B^*$, and therefore $v \notin M_{x,s}^{-1}M_{x_i,t_i-Jk}B^*$.

We need to consider two subcases. The first one $t_i - Jk \le s < t_i$, and the other is $s < t_i - Jk$. We begin we the first subcase.

Since we assumed (4.38), we may apply (2.2) to obtain

$$M_{x,s}^{-1}M_{x_i,t_i-Jk}(B^*) \supseteq c2^{a_6(s-t_i)}2^{a_6Jk}B^*,$$

which implies that

$$\sup_{v\in M_{x,s}^{-1}(x-x_i)+M_{x,s}^{-1}M_{x_i,t_i-J}(B^*)} (1+|v|)^{-N} \le c2^{a_6(t_i-s)N}(2^{a_6JN})^{-k}.$$

Therefore, (4.39) holds and we have

$$I_1 \le c\lambda(2^{a_6JN})^{-k}.$$

To show (4.39) holds also for the other subcase, $s < t_i - Jk$, we simply proceed by

$$\sup_{y\in\theta(x_i,t_i-J)} \left|R_z\big(z + M_{x,s}^{-1}(x_i - y)\big)\right| \le c2^{-a_6(t_i-s)N} \le c2^{-a_6JNk}.$$

To conclude the proof, we estimate I_2 by combining Lemma 4.8 and (4.39)

$$\begin{aligned}
I_2 &\le \left|\det(M_{x,s}^{-1}) \int_{\theta(x_i.t_i-J)} |P_i(y)\phi_i(y)||R_z(z + M_{x,s}^{-1}(x_i - y))|\right| dy \\
&\le c\lambda|\det(M_{x,s}^{-1})||\theta(x_i.t_i - J)| \sup_{y\in\theta(x_i,t_i-J)} |R_z(z + M_{x,s}^{-1}(x_i - y))| \\
&\le c\lambda 2^{s-t_i}(2^{a_6JN})^{-k} \le c\lambda(2^{a_6JN})^{-k}.
\end{aligned}$$

□

Lemma 4.11 *Suppose $f \in H^p(\Theta)$, $0 < p \le 1$. Then there exists a constants $c_1, c_2 > 0$, independent of f, $i \in \mathbb{N}$, and $\lambda > 0$ such that*

(i) $\int_{\mathbb{R}^n} (M^\circ b_i)(x)^p dx \le c_1 \int_{\theta(x_i,t_i-J)} (M^\circ f)(x)^p dx.$

Moreover, the series $\sum_i b_i$ converges in $H^p(\Theta)$, and

(ii) $\int_{\mathbb{R}^n} M^\circ(\sum_i b_i)(x)^p dx \le c_2 \int_\Omega (M^\circ f)(x)^p dx.$

Proof We apply Lemma 4.9, Lemma 4.10, the facts that $\theta(x_i, t_i - J) \subset \Omega$ and that $\nu^{-p}2^J = 2^{(1-a_6pN)J} < 1$ (recall assumption (3.10)), to obtain (i)

$$\begin{aligned}
&\int_{\mathbb{R}^n} (M^\circ b_i(x))^p dx \\
&\quad = \int_{\theta(x_i,t_i-J)} (M^\circ b_i(x))^p dx + \sum_{k=0}^\infty \int_{\theta(x_i,t_i-J(k+2))\setminus\theta(x_i,t_i-J(k+1))} (M^\circ b_i(x))^p dx \\
&\quad \le c\int_{\theta(x_i,t_i-J)} (M^\circ f(x))^p dx + c\lambda^p \sum_{k=0}^\infty |\theta(x_i, t_i - J(k+2))|\nu^{-kp} \\
&\quad \le c\sum_{k=0}^\infty (\nu^{-p}2^J)^k \int_{\theta(x_i,t_i-J)} (M^\circ f(x))^p dx \le c\int_{\theta(x_i,t_i-J)} (M^\circ f(x))^p dx.
\end{aligned}$$

Since $H^p(\Theta)$ is complete, from (i) and (4.19) we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(M^\circ\left(\sum_i b_i\right)(x)\right)^p dx &\le \sum_i \int_{\mathbb{R}^n} (M^\circ(b_i)(x))^p dx \\
&\le c\sum_i \int_{\theta(x_i,t_i-J)} (M^\circ(f)(x))^p dx \le c\int_\Omega (M^\circ(f)(x))^p dx.
\end{aligned}$$

□

Lemma 4.12 *If $f \in L^1(\mathbb{R}^n)$, then the series $\sum_{i\in\mathbb{N}} b_i$ converges in $L^1(\mathbb{R}^n)$. Moreover there exist a constant $c > 0$ independent of f, i, λ, such that*

$$\int_{\mathbb{R}^n} \sum_{i\in\mathbb{N}} |b_i(x)|dx \le c \int_{\mathbb{R}^n} |f(x)|dx \tag{4.40}$$

Proof From the definition of $\{b_i\}$ and Lemma 4.8

$$\begin{aligned}\int_{\mathbb{R}^n} |b_i(x)|dx &= \int_{\mathbb{R}^n} |(f(x) - p_i(x))\phi_i(x)|dx \\ &\le \int_{\theta(x_i,t_i-J)} |f(x)\phi_i(x)|dx + \int_{\theta(x_i,t_i-J)} |P_i(x)\phi_i(x)|dx \\ &\le \int_{\theta(x_i,t_i-J)} |f(x)|dx + c\lambda|\theta(x_i,t_i-J)|.\end{aligned}$$

Therefore from (4.15), (4.19), and the Maximal Theorem (see Theorem 3.8), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_i |b_i(x)|dx &\le \sum_i \int_{\mathbb{R}^n} |b_i(x)|dx \\ &\le \sum_i \left(\int_{\theta(x_i,t_i-J)} |f(x)|dx + c\lambda|\theta(x_i,t_i-J)|\right) \\ &\le L\left(\int_{\Omega} |f(x)|dx + c\lambda|\Omega|\right) \\ &\le c\int_{\mathbb{R}^n} |f(x)|dx.\end{aligned}$$

□

Lemma 4.13 *Suppose $\sum_i b_i$ converges in $\mathcal{S}'$. Then there exist a constant c, independent of $f \in \mathcal{S}'$ and $\lambda > 0$, such that*

$$M^\circ g(x) \le c\lambda \sum_i \nu^{-k_i(x)} + M^\circ f(x)\mathbb{1}_{\Omega^c}(x), \tag{4.41}$$

where

$$k_i(x) = \begin{cases} k, & \text{if for } k \ge 0,\ x \in \theta(x_i,t_i - J(k+2)) \setminus \theta(x_i,t_i - J(k+1)), \\ 0, & x \in \theta(x_i,t_i-J). \end{cases} \tag{4.42}$$

Proof If $x \in \Omega^c$, we know from Lemma 4.10

$$M^\circ g(x) \le M^\circ f(x) + \sum_i M^\circ b_i(x) \le M^\circ f(x)\mathbb{1}_{\Omega^c}(x) + c\lambda \sum_i \nu^{-k_i(x)}.$$

For any $x \in \Omega$, there exists $j \in \mathbb{N}$, such that $x \in \theta(x_j,t_j - J)$. Recall from (4.19) that

$$I(j) := \{i \in \mathbb{N} : \theta(x_i,t_i-J) \cap \theta(x_j,t_j-J) \ne \emptyset\},$$

with $\#I(j) \le L$. We have that

$$M^\circ g(x) \le M^\circ\Big(f - \sum_{i\in I(j)} b_i\Big)(x) + M^\circ\Big(\sum_{i\notin I(j)} b_i\Big)(x). \tag{4.43}$$

By Lemma 4.10

$$M^\circ\Big(\sum_{i\notin I(j)} b_i\Big)(x) \le c \sum_{i\notin I(j)} \nu^{-k_i(x)},$$

so to prove (4.41), it suffices to bound $M^\circ(f - \sum_{i\in I(j)} b_i)(x)$. Let $\psi \in \mathcal{S}_{N,\tilde{N}}$, and $s \in \mathbb{R}$. Defining $\eta := 1 - \sum_{i\in I(j)} \phi_i$, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\Big(f - \sum_{i\in I(j)} b_i\Big)(y)\psi_{x,s}(y)dy\right| \\
&\quad \le \left|\int_{\mathbb{R}^n} f(y)\eta(y)\psi_{x,s}(y)dy\right| + \left|\int_{\mathbb{R}^n}\Big(\sum_{i\in I(j)} P_i(y)\phi_i(y)\Big)\psi_{x,s}(y)dy\right| \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Since $\psi \in \mathcal{S}_{N,\tilde{N}}$ and $\#I(j) \le L$, from Lemma 4.8,

$$I_2 \le \sum_{i\in I(j)} \int_{\mathbb{R}^n} |P_i(y)\phi_i(y)||\psi_{x,s}(y)|dy \le c\lambda \sum_{i\in I(j)} \|\psi\|_1 \le c\lambda = c\lambda\nu^{k_j(x)}.$$

The proof of the estimate $I_1 \le c\lambda\nu^{k_j(x)}$ is similar to the estimates in Lemma 4.10 of I_1 (in both cases, $s < t_j$ and $s \ge t_j$), where η replaces ϕ_j . □

Lemma 4.14 *If $M^\circ f \in L^p$, $0 < p \le 1$, then $M^\circ g \in L^1$, and there exist a constant $c_1 > 0$, independent of f, λ such that*

$$\int_{\mathbb{R}^n} M^\circ g(x)dx \le c_1\lambda^{1-p} \int_{\mathbb{R}^n} \big(M^\circ f(x)\big)^p dx. \tag{4.44}$$

If $f \in L^1$, then $g \in L^\infty$, and there exist $c_2 > 0$, independent of f, λ, such that

$$\|g\|_\infty \le c_2\lambda. \tag{4.45}$$

Proof By Lemma 4.13 we have

$$\int_{\mathbb{R}^n} M^\circ g(x)dx \le c\lambda \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} \nu^{-k_i(x)}dx + \int_{\Omega^c} M^\circ f(x)dx, \tag{4.46}$$

where $k_i(x)$ are defined in (4.42). Recalling $\nu := 2^{a_6 JN}$ and that $N > a_6^{-1}$, we get for a fixed $i \in \mathbb{N}$

$$\int_{\mathbb{R}^n} \nu^{-k_i(x)} dx = \int_{\theta(x_i, t_i - J)} dx + \sum_{k=0}^{\infty} \int_{\theta(x_i, t_i - J(k+2)) \setminus \theta(x_i, t_i - J(k+1))} \nu^{-k_i(x)} dx$$

$$\leq |\theta(x_i, t_i - J)| + \sum_{k=0}^{\infty} |\theta(x_i, t_i - J(k+2))| \nu^{-k}$$

$$\leq c 2^{-t_i} \left(1 + \sum_{k=0}^{\infty} 2^{Jk} \nu^{-k} \right) \leq c |\theta(x_i, t_i)|.$$

Therefore, we may derive (4.44) from (4.46) by

$$\int_{\mathbb{R}^n} M^\circ g(x) dx \leq c\lambda \sum_{i \in \mathbb{N}} |\theta(x_i, t_i)| + \int_{\Omega^c} M^\circ f(x) dx$$

$$\leq c\lambda |\Omega| + \int_{\Omega^c} M^\circ f(x) dx \leq c\lambda^{1-p} \int_{\mathbb{R}^n} (M^\circ f(x))^p dx.$$

Since $f \in L^1$, by Lemma 4.12 we have that g and b_i, $i \in \mathbb{N}$, are functions and $\sum_{i \in \mathbb{N}} b_i$ converges in L^1. Thus

$$g = f - \sum_i b_i = f \mathbb{1}_{\Omega^c} + \sum_i P_i \phi_i.$$

By Lemma 4.8, and (4.19), for every $x \in \Omega$ we have $|g(x)| \leq c\lambda$. Also, for a.e. $x \in \Omega^c$, $|g(x)| = |f(x)| \leq M^\circ f(x) \leq \lambda$. Therefore $\|g\|_\infty \leq c\lambda$. □

Corollary 4.15 *$H^p(\Theta) \cap L^1$ is dense in $H^p(\Theta)$.*

Proof Let $f \in H^p(\Theta)$ and $\lambda > 0$. Consider the Calderón-Zygmund decomposition of f of degree $l \geq N_p(\Theta)$ and height λ.

$$f = g^\lambda + \sum_{i \in \mathbb{N}} b_i^\lambda.$$

By Lemma 4.11 we have

$$\|f - g^\lambda\|_{H^p(\Theta)} = \left\| \sum_{i \in \mathbb{N}} b_i^\lambda \right\|_{H^p(\Theta)} \to 0 \quad \text{as } \lambda \to \infty,$$

which implies that $g^\lambda \to f$ in $H^p(\Theta)$. Now, Lemma 4.14 gives that $M^\circ g^\lambda \in L^1(\mathbb{R}^n)$. To conclude that $g^\lambda \in L^1(\mathbb{R}^n)$, one can apply the same arguments as in the proof of Theorem 3.9 in [3], since our covers satisfy the following essential property: we have that $\text{diam}(\theta(x,t)) \to 0$, as $t \to \infty$, uniformly on compact sets. In fact, (2.2) implies that for any compact set $K \subset \mathbb{R}^n$, $\text{diam}(\theta(x,t)) \leq c(K) 2^{-a_6 t}$, for $t \geq 0$. □

4.3 The Inclusion $H^p(\Theta) \subseteq H^p_{q,l}(\Theta)$

Let Θ be a cover of $\mathbb{R}^n$ and let f be a tempered distribution such that $M^\circ f \in L_p(\mathbb{R}^n)$ for some $0 < p \le 1$, where M° is associated with Θ. For each $k \in \mathbb{Z}$ we consider the Calderón-Zygmund decomposition of f of degree $l \ge N_p(\Theta)$ at height 2^k associated with M°

$$f = g^k + \sum_i b_i^k, \tag{4.47}$$

where

$$\Omega^k := \{x : M^\circ f > 2^k\}, \qquad b_i^k := (f - P_i^k)\phi_i^k, \qquad \theta_i^k := \theta(x_i^k, t_i^k). \tag{4.48}$$

As before, for every fixed $k \in \mathbb{Z}$, $\{x_i^k\}_{i\in\mathbb{N}}$ is a sequence in Ω^k and $\{t_i^k\}_{i\in\mathbb{R}}$ satisfy (4.15)–(4.19) for Ω^k. Also, ϕ_i^k are defined as in (4.21) and P_i^k are the projection of f onto $\mathcal{P}_l$ with respect to the inner product given by (4.22).

We now define P_{ij}^{k+1} as the orthogonal projection of $(f - P_j^{k+1})\phi_i^k$ with respect to the inner product

$$\langle P, Q\rangle := \frac{1}{\int \phi_j^{k+1}} \int_{\mathbb{R}^n} P(x)Q(x)\phi_j^{k+1}(x)dx \quad \text{for } P, Q \in \mathcal{P}_l. \tag{4.49}$$

That is, if $\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \ne \emptyset$, then P_{ij}^{k+1} is the unique polynomial in $\mathcal{P}_l$ such that

$$\begin{aligned} &\int_{\mathbb{R}^n} \big(f(y) - P_j^{k+1}(y)\big)\phi_i^k(y)Q(y)\phi_j^{k+1}(y)dy \\ &\quad = \int_{\mathbb{R}^n} P_{ij}^{k+1}(y)Q(y)\phi_j^{k+1}(y)dy, \quad \forall Q \in \mathcal{P}_l, \end{aligned}$$

else we may take $P_{ij}^{k+1} = 0$.

Lemma 4.16 *Suppose $\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \ne \emptyset$. Then*

(i) $t_j^{k+1} \ge t_i^k - 2\gamma - 1$,
(ii) $\theta(x_j^{k+1}, t_j^{k+1} - J) \subset \theta(x_i^k, t_i^k - J - 3\gamma - 1)$,
(iii) *There exists $L' > 0$ such that for every $j \in \mathbb{N}$, $\#I(j) \le L'$ with*

$$I(j) := \{i \in \mathbb{N} : \theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) \ne \emptyset\}.$$

Proof To prove (i), assume by contradiction that $t_j^{k+1} < t_i^k - 2\gamma - 1$. Therefore, $\theta(x_i^k, t_i^k - J) \cap \theta(x_j^{k+1}, t_j^{k+1} - J) \ne \emptyset$, implies

$$\theta(x_i^k, t_i^k - J - 2\gamma - 1) \subseteq \theta(x_j^{k+1}, t_j^{k+1} - J - \gamma).$$

Since $\Omega^{k+1} \subset \Omega^k$, we have $(\Omega^{k+1})^c \supset (\Omega^k)^c$. Hence from (4.17) we have

$$\emptyset \neq \left(\Omega^k\right)^c \cap \theta\left(x_i^k, t_i^k - J - 2\gamma - 1\right) \subset \left(\Omega^{k+1}\right)^c \cap \theta\left(x_j^{k+1}, t_j^{k+1} - J - \gamma\right) = \emptyset,$$

which is contradiction. Property (ii) is a consequence of (i). We continue with (iii). For a fixed i, let $I_1(j) := \{i \in I(j) : t_i^k \le t_j^{k+1}\}$. Then for each such i, $\theta(x_j^{k+1}, t_j^{k+1} - J) \subseteq \theta(x_i^k, t_i^k - J - \gamma)$. Since x_j^{k+1} is contained in each $\theta(x_i^k, t_i^k - J - \gamma)$, $i \in I_1(j)$ we obtain by (4.19) that $|I_1(j)| \le L$. Now denote $I_2(j) := \{i \in I(j) : t_i^k > t_j^{k+1}\}$. Observe that

$$\theta(x_i^k, t_i^k + \gamma) \subseteq \theta(x_i^k, t_i^k - J) \subseteq \theta(x_j^{k+1}, t_j^{k+1} - J - \gamma).$$

At the same time, by (i), we have that $t_i^k - 2\gamma - 1 \le t_j^{k+1}$, and therefore all of the ellipsoids $\theta(x_i^k, t_i^k + \gamma)$, $i \in I_2(j)$, are pairwise disjoint, are all contained in the ellipsoid $\theta(x_j^{k+1}, t_j^{k+1} - J - \gamma)$ but also have their volume proportional to it by a multiple constant. Therefore, $|I_2(j)| \le L''$. □

Lemma 4.17 *There exist a constant $c > 0$, independent of $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, such that*

$$\sup_{x \in \mathbb{R}^n} |P_{ij}^{k+1}(x)\phi_j^{k+1}(x)| \le c2^{k+1} \tag{4.50}$$

Proof Let $\{\pi_\beta : \beta \in \mathbb{N}_+^n, |\beta| \le l\}$ be an orthonormal basis with respect to the inner product (4.49). Since P_{ij}^{k+1} is the orthogonal projection of $(f - P_j^{k+1})\phi_i^k$, we have

$$\begin{aligned}
&|P_{ij}^{k+1}(x)\phi_j^{k+1}(x)| \\
&\quad \le |P_{ij}^{k+1}(x)| \\
&\quad = \left| \sum_{|\beta| \le l} \left(\frac{1}{\int \phi_j^{k+1}} \int_{\mathbb{R}^n} (f(y) - P_j^{k+1}(y))\phi_i^k(y)\pi_\beta(y)\phi_j^{k+1}(y)dy \right) \pi_\beta(x) \right| \\
&\quad \le \left| \sum_{|\beta| \le l} \left(\frac{1}{\int \phi_j^{k+1}} \int_{\mathbb{R}^n} f(y)\phi_i^k(y)\pi_\beta(y)\phi_j^{k+1}(y)dy \right) \pi_\beta(x) \right| \\
&\qquad + \left| \sum_{|\beta| \le l} \left(\frac{1}{\int \phi_j^{k+1}} \int_{\mathbb{R}^n} P_j^{k+1}(y)\phi_i^k(y)\pi_\beta(y)\phi_j^{k+1}(y)dy \right) \pi_\beta(x) \right| =: I_1 + I_2.
\end{aligned}$$

We begin by estimating I_2. Since $\mathrm{supp}(\phi_j^{k+1}) \subset \theta(x_j^{k+1}, t_j^{k+1} - J)$ we have for each $|\beta| \le l$,

$$\begin{aligned}
&\frac{1}{\int \phi_j^{k+1}} \int_{\mathbb{R}^n} P_j^{k+1}(y)\phi_i^k(y)\pi_\beta(y)\phi_j^{k+1}(y)dy \\
&\quad = \frac{1}{\int \phi_j^{k+1}} \int_{\theta(x_j^{k+1}, t_j^{k+1} - J)} P_j^{k+1}(y)\phi_i^k(y)\pi_\beta(y)\phi_j^{k+1}(y)dy.
\end{aligned}$$

From Lemma 4.8 and (4.28) we have

$$\sup_{y\in\theta(x_j^{k+1},t_j^{k+1}-J)} |P_j^{k+1}(y)| \le c2^{k+1},$$

and

$$\sup_{y\in\theta(x_j^{k+1},t_j^{k+1}-J)} |\pi_\beta(y)| \le c,$$

which leads to

$$I_2 \le c2^{k+1}.$$

We continue with I_1. Let $w \in (\Omega^{k+1})^c \cap \theta(x_j^{k+1}, t_j^{k+1} - J - 2\gamma - 1)$, and define for each $|\beta| \le l$

$$\Phi(y) := \frac{|\det(M_{w,t_j^{k+1}})|}{\int \phi_j^{k+1}} (\phi_i^k \cdot \pi_\beta \cdot \phi_j^{k+1})(w - M_{w,t_j^{k+1}}(y)).$$

To bound I_1, it is sufficient to bound for each β, $|\int_{\mathbb{R}^n} f(y)\Phi_{w,t_j^{k+1}}(y)dy|$. Since $w \in (\Omega^{k+1})^c$, we get

$$\left|\int_{\mathbb{R}^n} f(y)\Phi_{w,t_j^{k+1}}(y)dy\right| \|\Phi\|_{N,\tilde{N}} 2^{k+1}.$$

Thus, it suffices to show $\|\Phi\|_{N,\tilde{N}} \le c$. We define

$$\hat{\phi}_j^{k+1}(y) := \phi_j^{k+1}(x_j^{k+1} + M_{w,t_j^{k+1}}(y)),$$

$$\hat{\pi}_\beta(y) := \pi_\beta(x_j^{k+1} + M_{w,t_j^{k+1}}(y)),$$

$$\hat{\phi}_i^k(y) := \phi_i^k(w - M_{w,t_j^{k+1}}(y)).$$

Hence the function Φ can be written as

$$\frac{|\det(M_{w,t_j^{k+1}})|}{\int \phi_j^{k+1}} \hat{\phi}_i^k(y)\big(\hat{\phi}_j^{k+1} \cdot \hat{\pi}_\beta\big)\big(M_{w,t_j^{k+1}}^{-1}\big(w - x_j^{k+1}\big) - y\big).$$

Calculation similar to (4.26) yields

$$\frac{|\det(M_{w,t_j^{k+1}})|}{\int \phi_j^{k+1}} \le C.$$

From Lemma 4.6, all the partial derivatives of order $\le N$ of the function $\hat{\phi}_i^k$ are bounded by universal constant. Since $\operatorname{supp}(\Phi) \subset CB^*$, by Lemma 4.6 and the product rule, the partial derivatives of $\hat{\phi}_j^{k+1} \cdot \hat{\pi}_i$ of order $\le N$ are also bounded by universal constant. Now, since $\operatorname{supp}(\Phi) \subset CB^*$ we conclude that $\|\Phi\|_{N,\tilde{N}} \le C$, which completes the proof. $\square$

Lemma 4.18 *Let $k \in \mathbb{Z}$. Then $\sum_{i\in\mathbb{N}}(\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1})=0$, where the series converges pointwise and in $\mathcal{S}'$.*

Proof By (4.19) we have $\#\{j \in \mathbb{N} : \phi_j^{k+1}(x) \neq 0\} \leq L$. Also from definition of P_{ij}^{k+1} we have $P_{ij}^{k+1} = 0$ if $\theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) = \emptyset$. By Lemma 4.16, $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} P_{ij}^{k+1}(x)\phi_j^{k+1}(x)$ contains at most L' nonzero items. Combining that with Lemma 4.17 gives

$$\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} |P_{ij}^{k+1}(x)\phi_j^{k+1}(x)| \leq c2^{k+1}. \tag{4.51}$$

By the Lebesgue Dominated Convergence Theorem $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}$ converges unconditionally in $\mathcal{S}'$. We let $I = \{i \in \mathbb{N} : \theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) \neq \emptyset\}$, and in order to conclude the proof it is suffice to show that

$$\sum_{i\in\mathbb{N}} P_{ij}^{k+1} = \sum_{i\in I} P_{ij}^{k+1} = 0 \quad \text{for every } j \in \mathbb{N}.$$

Indeed for fixed $j \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} P_{ij}^{k+1}$ is an orthogonal projection of $(f - P_j^{k+1}) \times \sum_{i\in I}\phi_i^k$ onto $\mathcal{P}_l$ with respect to the inner product (4.49). Since $\sum_{i\in I}\phi_i^k(x) = 1$ for $x \in \theta(x_j^{k+1}, t_j^{k+1} - J)$, $\sum_{i\in\mathbb{N}} P_{ij}^{k+1}$ is an orthogonal projection of $(f - P_j^{k+1})$ onto $\mathcal{P}_l$ with respect to the inner product (4.49), which is zero by the definition of P_j^{k+1}. □

Theorem 4.19 (Atomic decomposition) *For any cover Θ and $0 < p \leq 1$, $H^p(\Theta) \subseteq H^p_{\infty,l}(\Theta)$.*

Proof Let $f \in H^p(\Theta) \cap L^1$. Consider the Calerón-Zygmund decomposition of f of degree l at height 2^k associated with M°, $f = g^k + \sum_i b^k$. By Lemma 4.11, $g^k \to f$, as $k \to \infty$, in $H^p(\Theta)$, and by (4.45), $\|g^k\|_\infty \to 0$, as $k \to -\infty$. Therefore

$$f = \sum_{k\in\mathbb{Z}} \left(g^{k+1} - g^k\right) \quad \text{in } \mathcal{S}'.$$

From Lemma 4.18 and the fact that $\sum_{i\in\mathbb{N}}\phi_i^k b_j^{k+1} = \mathbb{1}_{\Omega^k} b_j^{k+1} = b_j^{k+1}$,

$$\begin{aligned} g^{k+1} - g^k &= \left(f - \sum_{j\in\mathbb{N}} b_j^{k+1}\right) - \left(f - \sum_{j\in\mathbb{N}} b_j^k\right) \\ &= \sum_{j\in\mathbb{N}} b_j^k - \sum_{j\in\mathbb{N}} b_j^{k+1} + \sum_{i\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}\right) \\ &= \sum_{i\in\mathbb{N}} b_i^k - \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \phi_i^k b_j^{k+1} + \sum_{i\in\mathbb{N}}\left(\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}\right) \end{aligned}$$

$$= \sum_{i\in\mathbb{N}}\Big(b_i^k - \Big[\sum_{j\in\mathbb{N}} \phi_i^k b_j^{k+1} - \sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}\Big]\Big)$$

$$= \sum_{i\in\mathbb{N}}\Big(b_i^k - \sum_{j\in\mathbb{N}}[\phi_i^k b_j^{k+1} - P_{ij}^{k+1}\phi_j^{k+1}]\Big) =: \sum_{i\in\mathbb{N}} h_i^k.$$

Since $b_i^k = (f - P_i^k)\phi_i^k$, one has

$$h_i^k = (f - P_i^k)\phi_i^k - \sum_{j\in\mathbb{N}}[\phi_i^k(f - P_j^{k+1}) - P_{ij}^{k+1}]\phi_j^{k+1}.$$

By the choice of P_i^k, P_{ij}^{k+1}

$$\int_{\mathbb{R}^n} h_i^k(y)Q(y)dy = 0 \quad \text{for all } \; Q \in \mathcal{P}_l. \tag{4.52}$$

Moreover since $\sum_{j\in\mathbb{N}}\phi_j^{k+1} = \mathbb{1}_{\Omega^{k+1}}$, we can write

$$h_i^k = f\mathbb{1}_{(\Omega^{k+1})^c}\phi_i^k - P_i^k\phi_i^k + \sum_{j\in\mathbb{N}} P_j^{k+1}\phi_j^{k+1}\phi_i^k + \sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}.$$

From definition of $P_{i,j}^{k+1}$ we know $P_{i,j}^{k+1} \neq 0$ implies $\theta(x_j^{k+1}, t_j^{k+1} - J) \cap \theta(x_i^k, t_i^k - J) \neq \emptyset$, also we know $\operatorname{supp}(\phi_j^{k+1}) \subset \theta(x_j^{k+1}, t_j^{k+1} - J)$, hence form Lemma 4.16 we come to the conclusion that $\operatorname{supp}(\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}) \subset \theta(x_i^k, t_i^k - J - 3\gamma - 1)$, which implies that

$$\operatorname{supp}(h_i^k) \subset \theta(x_i^k, t_i^k - J - 3\gamma - 1). \tag{4.53}$$

Obviously we have

$$\|h_i^k\|_\infty \le \|f\mathbb{1}_{(\Omega^{k+1})^c}\phi_i^k\|_\infty + \|P_i^k\phi_i^k\|_\infty + \Big\|\sum_{j\in\mathbb{N}} P_j^{k+1}\phi_i^k\phi_j^{k+1}\Big\|_\infty$$
$$+ \Big\|\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}\Big\|_\infty.$$

We know that $|f(x)| \le cM^\circ f(x) \le c2^{k+1}$ for almost every $x \in (\Omega^{k+1})^c$. Also from Lemma 4.8 we have $\|P_i^k\phi_i^k\|_\infty \le c2^k$, and from Lemmas 4.16, 4.17 we conclude

$$\Big\|\sum_{j\in\mathbb{N}} P_j^{k+1}\phi_i^k\phi_j^{k+1}\Big\|_\infty \le c2^{k+1}, \quad \text{and} \quad \Big\|\sum_{j\in\mathbb{N}} P_{ij}^{k+1}\phi_j^{k+1}\Big\|_\infty \le c2^{k+1}.$$

Therefore we get

$$\|h_i^k\|_\infty \le c2^k. \tag{4.54}$$

From (4.52), (4.53) and (4.54) h_i^k is a multiple of a (p,∞,l) atom a_i^k, meaning

$$h_i^k = \lambda_i^k a_i^k,$$

where $\lambda_i^k \sim 2^k 2^{-t_i^k/p}$. From (4.16)

$$\begin{aligned}
\sum_{k=-\infty}^{\infty}\sum_{i\in\mathbb{N}}|\lambda_i^k|^p &\le c\sum_{k=-\infty}^{\infty}2^{kp}\sum_{i\in\mathbb{N}}|\theta(x_i^k,t_i^k+\gamma)| \\
&\le c\sum_{k=-\infty}^{\infty}2^{kp}|\Omega^k| \le c\sum_{k=-\infty}^{\infty}p(2^k)^{p-1}|\Omega^k|2^{k-1} \\
&\le c\int_0^{\infty}p\lambda^{p-1}|\{x\in\mathbb{R}^n : M^\circ f(x)>\lambda\}|d\lambda = c\|M^\circ f\|_p^p \\
&= c\|f\|_{H^p(\Theta)}^p.
\end{aligned}$$

Therefore $f=\sum_{k=-\infty}^{\infty}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k$ defines an atomic decomposition of $f\in H^p(\Theta)$ $\cap L^1$. Applying the density of $H^p(\Theta)\cap L^1$ in H^p (Corollary 4.15), we complete the proof. $\square$

5 Classification of Anisotropic Hardy Spaces

Denote by $H^p(\mathbb{R}^n)$ the classic isotropic Hardy spaces. Let A be a fixed *expansion* matrix, i.e., a matrix whose eigenvalues>1. Thus $A^{-j}\to 0$ as $j\to\infty$. The anisotropic Hardy spaces of [3] are in fact Hardy spaces constructed over a semi-continuous cover, where the ellipsoids $\theta(x,j)$ are determined by setting $M_{x,j} := A^{-j}$. Let us denote these spaces as $H^p(A)$. It is obvious that for $1<p<\infty$, any dilation matrix A and any cover Θ, we have the equivalence $H^p(\mathbb{R}^n)\sim H^p(A)\sim H^p(\Theta)\sim L^p(\mathbb{R}^n)$, where the embedding constants depend on the parameters of A and Θ. Therefore an important question is to what extent are the various Hardy spaces different for the range $0<p\le 1$. Theorem 5.8, which is the main result of this section, shows that for the range $0<p\le 1$, two Hardy spaces are equivalent if and only if the covers they are associated with induce an equivalent quasi-distance.

5.1 Properties of Anisotropic Hardy Spaces

It is straight forward to show that our anisotropic function spaces are invariant under affine transforms

Lemma 5.1 *let Θ be a cover,A be a non-singular affine transform and (p,q,l) an admissible triplet. Then*

(i) *a is a (p,q,l) atom in $H^p(\Theta)$ iff $|\det A|^{-1/p}a(A^{-1}\cdot)$ is a (p,q,l) atom in $H^p(A(\Theta))$.*
(ii) *For any $f\in\mathcal{S}'$, $f\in H^p(\Theta)$ iff $f(A^{-1}\cdot)\in H^p(A(\Theta))$.*

Proof To prove (i), let a be a (p,q,l) atom in $H^p(\Theta)$ and denote $\tilde{a} := |\det A|^{-1/p} a(A^{-1}\cdot)$. We verify that $\tilde{a}$ satisfies the three properties of an atom in $H^p(A(\Theta))$:

1. It is obvious that the support of $\tilde{a}$ is contained in $A(\theta)$, where $\operatorname{supp}(a) \subset \theta$.
2. If $q=\infty$ then $\|\tilde{a}\|_\infty = |\det A|^{-1/p}\|a\|_\infty \le |\det A|^{-1/p}|\theta|^{-1/p} = |A(\theta)|^{-1/p}$. Similar for $q<\infty$.
3. For any $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \le l$, we have the zero moment property by

$$\int_{\mathbb{R}^n} \tilde{a} x^\alpha dx = |\det A|^{-1/p} \int_{\mathbb{R}^n} a\big(A^{-1}x\big) x^\alpha dx = |\det A|^{1-1/p} \int_{\mathbb{R}^n} a(y)(Ay)^\alpha dy = 0.$$

Claim (ii) follows directly from the atomic decomposition. If $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j|^p < \infty$ then $f(A^{-1}\cdot) = \sum_j \tilde{\lambda}_j \tilde{a}_j$, where $\tilde{a}_j := |\det A|^{-1/p} a_j(A^{-1}\cdot)$ are (p,q,l) atoms in $H^p(A(\Theta))$ and $\tilde{\lambda}_j := |\det A|^{1/p}\lambda_j$. Thus,

$$\begin{aligned}\|f(A^{-1}\cdot)\|_{H^p(A(\Theta))} &\sim \inf_{f(A^{-1}\cdot)=\sum_j \tilde{\lambda}_j \tilde{a}_j} \Big(\sum_j |\tilde{\lambda}_j|^p\Big)^{1/p} \\ &= |\det A|^{-1/p} \inf_{f=\sum_j \lambda_j a_j} \Big(\sum_j |\lambda_j|^p\Big)^{1/p} \\ &= |\det A|^{-1/p}\|f\|_{H^p(\Theta)}. \end{aligned}$$ □

5.2 BMO(Θ)

Definition 5.2 Let Θ be a cover and let $f:\mathbb{R}^n \to \mathbb{R}$. Denote the means over the ellipsoids by

$$f_\theta := \frac{1}{|\theta|}\int_\theta f(x)dx, \quad \theta \in \Theta.$$

Then, f is said to belong to the space of *Bounded Mean Oscillation* BMO(Θ) if there exists a constant $0 < M < \infty$ such that

$$\sup_{\theta\in\Theta} \frac{1}{|\theta|}\int_\theta \big|f(x) - f_\theta\big| dx \le M.$$

We denote by $\|f\|_{\mathrm{BMO}(\Theta)}$ the infimum over all such constants.

Recall that the above definition could be extended to allow arbitrary constants c_θ in place of the means f_θ, $\theta \in \Theta$. Indeed, if for given $\{c_\theta\}_{\theta\in\Theta}$, we have

$$\sup_{\theta\in\Theta} \frac{1}{|\theta|}\int_\theta \big|f(x) - c_\theta\big| dx \le M',$$

then $|c_\theta - f_\theta| \le M'$, $\forall \theta \in \Theta$, and $\|f\|_{\mathrm{BMO}(\Theta)} \le 2M'$. This observation along with Theorem 2.7 implies that the Definition 5.2 is equivalent to the classical definition [7] using means over the (anisotropic) balls induced by the quasi-distance. Thus we have,

Theorem 5.3 [7] *The dual space of* $H^1(\Theta)$ *is* $\mathrm{BMO}(\Theta)$.

Remarks

1. For a proof of Theorem 5.3 one can consult the proof in Sect. IV.1.2 in [15] for the isotropic case, since the proof is identical for the anisotropic case.
2. So as to limit the scope of the paper in some reasonable sense, we will address the issue of the Campanato dual spaces of $H^p(\Theta)$, $0 < p < 1$ (see Sect. 8 in [3]), in a follow-up work.

It is obvious that $L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}(\Theta)$ for any cover. The following is a typical example for a non-bounded function in $\mathrm{BMO}(\Theta)$

Lemma 5.4 *For any cover* Θ *of* $\mathbb{R}^n$, *we have that* $\log(\rho(\cdot,0)) \in \mathrm{BMO}(\Theta)$ *where* ρ *is the induced quasi-distance and* $\|\log(\rho(\cdot,0))\|_{\mathrm{BMO}(\Theta)} \le c(p(\Theta))$.

Proof For any $\theta \in \Theta$, let $x_\theta \in \theta$ such that $\rho(x_\theta,0) := \min_{x\in\theta} \rho(x,0)$.
Case I: $|\theta| \le \rho(x_\theta,0)$. Observe that $\log(\rho(x_\theta,0)) := \min_{x\in\theta} \log(\rho(x,0))$. Since for any $x \in \theta$, $\rho(x,0) \le \kappa(\rho(x,x_\theta) + \rho(x_\theta,0))$, where $\kappa \ge 1$ is defined in (2.4), we have that

$$
\begin{aligned}
M(\theta) &:= \frac{1}{|\theta|}\int_\theta (\log(\rho(x,0)) - \log(\rho(x_\theta,0)))dx \\
&\le \frac{1}{|\theta|}\int_\theta (\log\kappa(\rho(x,x_\theta)+\rho(x_\theta,0)) - \log(\rho(x_\theta,0)))dx \\
&\le \log\kappa + \frac{1}{|\theta|}\int_\theta \log\left(\frac{\rho(x,x_\theta)}{\rho(x_\theta,0)} + 1\right)dx \\
&\le \log\kappa + \frac{1}{|\theta|}\int_\theta \log\left(\frac{|\theta|}{\rho(x_\theta,0)} + 1\right)dx \\
&\le \log\kappa + \log 2.
\end{aligned}
$$

Case II: $\rho(x_\theta,0) \le |\theta|$. By the triangle inequality (2.4), $\theta \subset B(0, 2\kappa|\theta|)$ and therefore

$$
\begin{aligned}
&\frac{1}{|\theta|}\int_\theta \log\big(2\kappa|\theta|\big) - \log\big(\rho(x,0)\big)dx \\
&\quad \le c\frac{1}{|B(0,2\kappa|\theta|)|}\int_{B(0,2\kappa|\theta|)} \log\big(2\kappa|\theta|\big) - \log\big(\rho(x,0)\big)dx.
\end{aligned}
$$

Applying Theorem 2.7 we have

$$
\begin{aligned}
&\frac{1}{|B(0,2\kappa|\theta|)|}\int_{B(0,2\kappa|\theta|)} (\log(2\kappa|\theta|) - \log\rho(x,0))dx \\
&\quad = \log(2\kappa|\theta|) + \frac{1}{|B(0,2\kappa|\theta|)|}\sum_{j=1}^\infty \int_{B(0,2\kappa|\theta|2^{-j+1})\setminus B(0,2\kappa|\theta|2^{-j})} \log\rho(x,0)^{-1}dx
\end{aligned}
$$

$$
\begin{aligned}
&\le \log(2\kappa|\theta|) + \frac{1}{|B(0, 2\kappa|\theta|)|} \\
&\quad \times \sum_{j=1}^{\infty} |B(0, 2\kappa|\theta|2^{-j+1}) \backslash B(0, 2\kappa|\theta|2^{-j})| \log((2\kappa|\theta|)^{-1}2^{j}) \\
&\le \log(2\kappa|\theta|) - \log(2\kappa|\theta|) + c' \frac{1}{2\kappa|\theta|} \sum_{j=1}^{\infty} 2\kappa|\theta|2^{-j+1} j \\
&\le c' \sum_{j=1}^{\infty} 2^{-j+1} j = c''.
\end{aligned}
$$

□

5.3 A Classification Result

First, we recall some basic definitions from convex analysis.

Definition 5.5 Let $K \subset \mathbb{R}^n$ be a bounded domain with piecewise C^1 boundary. Let $L \subset \mathbb{R}^n$ be an hyperplane through the origin, with normal N. For each $x \in L$ let the perpendicular line through $x \in L$ be $G_x = \{x + yN : y \in \mathbb{R}\}$, and let $l_x := \mathit{length}(K \cap G_x)$. The **Steiner Symmetrization** of K, with respect to L is

$$S_L(K) = \{x + yN : x \in L, K \cap G_x \neq \emptyset, -(1/2)l_x \le y \le (1/2)l_x\}.$$

It is not hard to see that whenever K is convex so is $S_L(K)$ and that the Steiner Symmetrization preserves volume, i.e. $|S_L(K)| = |K|$, (see [2]).

For any hyperplane of the form $H := \{(y_1, \dots, y_{n-1}, h) : y_i \in \mathbb{R}\}$, with h fixed, we denote $H^+ := \{(y_1, \dots, y_{n-1}, y_n) : y_n \ge h\}$, and $H^- := \{(y_1, \dots, y_{n-1}, y_n) : y_n \le h\}$. For a set K, ∂K denotes the boundary of K.

Lemma 5.6 *Let θ be an ellipsoid in $\mathbb{R}^n$. For $1 \le i \le n-1$, let L_i be the hyperplane $L_i := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\}$. Then the following hold*

(a) *The convex body $K := S_{L_1} \circ S_{L_2} \circ \cdots \circ S_{L_{n-1}}(\theta)$ is symmetric with respect the x_i-axis for every $1 \le i \le n$.*

(b) *For every two hyperplanes of the form $H_i = \{(y_1, \dots, y_{n-1}, h_i)\}$, $i = 1, 2$, we have that*

$$|H_1^- \cap H_2^+ \cap \theta| = |H_1^- \cap H_2^+ \cap K|.$$

(c) *For every two hyperplanes of the form $H_i = \{(y_1, \dots, y_{n-1}, h_i) : y_i \in \mathbb{R}\}$, $i = 1, 2$, where $h_1 > h_2$, we have that*

$$|H_1^- \cap H_2^+ \cap \theta| \le n!((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n))|\theta|,$$

where $\tilde{x}_n = \min_{(y_1,\dots,y_n)\in\theta} y_n$, and $\tilde{z}_n = \max_{(y_1,\dots,y_n)\in\theta} y_n$.

Proof Assertions (a) and (b) follow from the construction of K. We now prove (c). First we show that

$$|K_2| \le n!|K|, \tag{5.1}$$

where K_2 is the minimal (with respect to volume) box that contains K. For convenience we can assume that K is centered at the origin. Let $a_1, \dots, a_n > 0$ be positive numbers such that the points $(a_1, 0, \dots, 0), (0, a_2, 0, \dots, 0), \dots, (0, \dots, 0, a_n)$ belong to ∂K. Let K_1 denote the convex hull of $\pm(a_1, 0, \dots, 0), \dots, \pm(0, \dots, 0, a_n)$ i.e. $K_1 := \operatorname{conv}\{\pm(a_1, 0, \dots, 0), \dots, \pm(0, \dots, 0, a_n)\}$ and let $K_2 := \operatorname{conv}\{(\pm a_1, \pm a_2, \dots, \pm a_n)\}$. Obviously K_2 is the minimal box that contain K, and

$$K_1 \subset K \subset K_2.$$

A simple integral calculation shows that $|K_1| = (\prod_{i=1}^n a_i)2^n/n!$, and $|K_2| = (\prod_{i=1}^n a_i)2^n$, which implies (5.1). Thus, from (5.1) and (b) we have

$$\begin{aligned}
|H_1^- \cap H_2^+ \cap \theta| &= |H_1^- \cap H_2^+ \cap (S_{L_1} \circ \cdots \circ S_{L_{n-1}}(\theta))| \\
&\le |H_1^- \cap H_2^+ \cap K_2| = ((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n))|K_2| \\
&\le n!((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n))|S_{L_1} \circ S_{L_2} \circ \cdots \circ S_{L_{n-1}}(\theta)| \\
&= n!((h_1 - h_2)/(\tilde{z}_n - \tilde{x}_n))|\theta|.
\end{aligned}$$

□

Lemma 5.7 *Let Θ be a cover of $\mathbb{R}^n$ such that $B^* \in \Theta_0$. For $1 \le i \le n$ define*

$$g_i(x_1, \dots, x_n) := \begin{cases} \log|x_i| & (x_1, \dots, x_n) \in B^*, \\ 0 & (x_1, \dots, x_n) \notin B^*. \end{cases} \tag{5.2}$$

Then $g_i \in \mathrm{BMO}(\Theta)$, with $c_1 \le \|g_i\|_{\mathrm{BMO}(\Theta)} \le c_2(p(\Theta))$.

Proof Without loss of generality, we assume that $n > 1$ (the univariate case is known [15]) and $i = n$ and for the rest of the proof we denote $g := g_n$. From the definition of the BMO space

$$\|g\|_{\mathrm{BMO}(\Theta)} \ge \frac{1}{|B^*|} \int_{B^*} \left|g(x) - c_{B^*}\right| dx =: c_1,$$

where $c_{B^*} = \frac{1}{|B^*|} \int_{B^*} g(y)dy$.

In the other direction, if $\theta \cap B^* = \emptyset$, then $g(x) = 0$ on θ and we're done. Else, $\theta \cap B^* \ne \emptyset$. Assume $\theta = \theta(x, t)$. If $t \le 0$, then

$$\frac{1}{|\theta|} \int_\theta \left|g(x) - c_\theta\right| dx \le \frac{1}{|\theta|} \int_{B^*} \left|g(x)\right| dx \le c.$$

We now deal with the case $\theta = \theta(x, t)$ with $t \ge 0$. Let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \theta$ such that $\tilde{x}_n = \min_{(y_1, \dots, y_n) \in \theta} |y_n|$. There are two cases:

Case I: $\max_{(y_1,\dots,y_n)\in\theta}|y_n-\tilde{x}_n|\le|\tilde{x}_n|$. Here we have

$$\begin{aligned}\frac{1}{|\theta|}\int_\theta(\log|y_n|-\log|\tilde{x}_n|)dy &\le \frac{1}{|\theta|}\int_\theta(\log(|y_n-\tilde{x}_n|+|\tilde{x}_n|)-\log|\tilde{x}_n|)dy\\ &\le \frac{1}{|\theta|}\int_\theta \log\left(\frac{|y_n-\tilde{x}_n|}{|\tilde{x}_n|}+1\right)dy\le\log 2.\end{aligned}$$

Case II: $\max_{(y_1,\dots,y_n)\in\theta}|y_n-\tilde{x}_n|>|\tilde{x}_n|$. This condition implies that $x+3M_{x,t}(B^*)$ intersects the hyperplane $\{y=(y_1,\dots,y_{n-1},0)\}$, where $M_{x,t}$ is the matrix associated with $\theta(x,t)$. Let $(z_1,\dots,z_{n-1},0)$ be some point in the intersection. From Lemma 2.2 there exists $\tilde{c}:=\tilde{c}(\mathrm{p}(\Theta))>0$ such that $\theta\subseteq B:=B((z_1,\dots,z_{n-1},0),\tilde{c}|\theta|)$. Let us explain this last fact. By Lemma 2.2 there exist $c>0$ that depends only on the parameters of the cover such that $x+3M_{x,t}(B^*)\subseteq\theta(x,t-3c)$. Since $|\theta(x,t-3c)|\le a_1^{-1}a_2 2^{3c}|\theta(x,t)|$, we can choose $\tilde{c}:=a_1^{-1}a_2 2^{3c}$ to obtain $\theta\subseteq B((z_1,\dots,z_{n-1},0),\tilde{c}|\theta|)$ as claimed.

Let $\tilde{z}:=(\tilde{z}_1,\dots,\tilde{z}_n)\in B$ such that $|\tilde{z}_n|:=\max_{(y_1,\dots,y_n)\in B}|y_n|$. With this definition,

$$\frac{1}{|\theta|}\int_\theta\big(\log|\tilde{z}_n|-\log|y_n|\big)dy\le c\frac{1}{|B|}\int_B\big(\log|\tilde{z}_n|-\log|y_n|\big)dy.$$

Denoting

$$H_j:=B\cap\big\{(y_1,\dots,y_n)\in\mathbb{R}^n:|y_n|\le 2^{-j}|\tilde{z}_n|\big\},\quad j\ge 0,$$

we may apply Lemma 5.6 to conclude

$$\begin{aligned}\frac{1}{|B|}\int_B(\log|\tilde{z}_n|-\log(|y_n|))dy &= \log|\tilde{z}_n|+\frac{1}{|B|}\sum_{j=1}^\infty\int_{H_{j-1}\backslash H_j}\log|y_n|^{-1}dy\\ &\le \log|\tilde{z}_n|+\frac{1}{|B|}\sum_{j=1}^\infty|H_{j-1}\backslash H_j|\log(|\tilde{z}_n|^{-1}2^j)\\ &\le \log|\tilde{z}_n|-\log|\tilde{z}_n|+n!\frac{1}{|B|}\sum_{j=1}^\infty 2^{-j}|B|j\\ &\le n!\sum_{j=1}^\infty 2^{-j}j=c(n).\end{aligned}$$

□

Theorem 5.8 *Let Θ_1 and Θ_2 be two covers and let ρ_1 and ρ_2 be the corresponding induced quasi-distances. Then following are equivalent*:

(i) *The quasi-distances ρ_1 and ρ_2 are equivalent.*
(ii) *$H^1(\Theta_1)\sim H^1(\Theta_2)$, i.e., there exist constants $0<A<B<\infty$ such that for all $f\in\mathcal{S}'$, $A\|f\|_{H^1(\Theta_1)}\le\|f\|_{H^1(\Theta_2)}\le B\|f\|_{H^1(\Theta_1)}$.*
(iii) *$H^p(\Theta_1)\sim H^p(\Theta_2)$ for all $0<p\le 1$.*

Remark Notice that in fact Theorem 5.8 really characterizes only the case $p = 1$. Further generalization of the proof is needed to show that the quasi-distances are equivalent iff the Hardy spaces are equivalent for some $0 < p \le 1$.

Proof It is obvious that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and so it remains to show that (ii) $\Rightarrow$ (i). First, observe that for $n = 1$ any cover induces a quasi-distance which is equivalent to the Euclidian distance, so the result is obvious. For $n \ge 2$ assume to the contrary that (ii) holds but (i) does not hold. Then w.l.g. there exists a sequence of pairs of points $u_m, v_m \in \mathbb{R}^n$, $m \ge 1$, such that

$$\frac{\rho_1(u_m, v_m)}{\rho_2(u_m, v_m)} \underset{m\to\infty}{\to} 0. \tag{5.3}$$

Assuming (5.3) holds we will construct a sequence of compactly supported piecewise constant functions $\{f_m\}$ such that $\|f_m\|_{H^1(\Theta_1)}/\|f_m\|_{H^1(\Theta_2)} \to 0$ as $m \to \infty$ thereby contradicting our assumption that $H^1(\Theta_1) \sim H^1(\Theta_2)$.

Let $\varepsilon > 0$, and let $m \ge 1$ such that $\rho_1(u_m, v_m)/\rho_2(u_m, v_m) \le \varepsilon$. Let $\theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$, such that $\rho_1(u_m, v_m) = |\theta_1|$ and $\rho_2(u_m, v_m) = |\theta_2|$. We now construct three ellipsoids centered at $z_m := (u_m + v_m)/2$ as follows:

(i) $\tilde{\theta}_1 := \theta(z_m, t_1) \in \Theta_1$, such that $|\tilde{\theta}_1| \sim |\theta_1|$, and $u_m, v_m \in \tilde{\theta}_1$,
(ii) $\tilde{\theta}_2 := \theta(z_m, t_2) \in \Theta_2$, such that $|\tilde{\theta}_2| \sim |\theta_2|$, with $u_m, v_m \in (\tilde{\theta}_2)^c$,
(iii) $\hat{\theta}_2 := \theta(z_m, t_2 + c) \in \Theta_2$, with minimal c (depending on the parameters of the cover Θ_2) such that $2M_{z_m,t_2+c}(B^*) \subset M_{z_m,t_2}(B^*)$.

Select the affine transformation, A_m, incorporating a rotational element, that satisfies:

(i) $A_m(B^*) = \hat{\theta}_2$,
(ii) $A_m^{-1}(\tilde{\theta}_1)$ is symmetric with respect to the $x_n = 0$ hyperplane.

We define new covers $\Theta_1' := A_m^{-1}\Theta_1$, $\Theta_2' := A_m^{-1}\Theta_2$ with equivalent parameters to Θ_1, Θ_2, respectively and new points $\tilde{u}_m := A_m^{-1}(u_m)$, $\tilde{v}_m := A_m^{-1}(v_m)$. We now have the following geometric objects 'at the origin' with the following properties:

(i) $B^* = A_m^{-1}(\hat{\theta}_2) \in \Theta_2'$,
(ii) $\tilde{\theta}_1' := A_m^{-1}(\tilde{\theta}_1) \in \Theta_1'$, with $\tilde{u}_m, \tilde{v}_m \in \tilde{\theta}_1'$ and $|\tilde{\theta}_1'| < c\epsilon$,
(iii) $\theta_2' := A_m^{-1}(\tilde{\theta}_2) \in \Theta_2'$ with $2B^* \subset \theta_2'$, $\tilde{u}_m, \tilde{v}_m \in (\theta_2')^c \subset (2B^*)^c$ and $|\theta_2'| \sim 1$.

We write $\tilde{\theta}_1' = \tilde{\theta}_1'(0, \tilde{t}_1') = M_{0,\tilde{t}_1'}(B^*)$, where $\tilde{t}_1' \in \mathbb{R}$. Since $\tilde{\theta}_1' \cap (2B^*)^c \neq \emptyset$, we may define

$$s' := \sup\{s \ge 0 : (2B^*)^c \cap M_{0,\tilde{t}_1'+s}(B^*) \neq \emptyset\}$$

and

$$\theta_1' := M_{0,\tilde{t}_1'+s'}(B^*).$$

The newly constructed ellipsoid θ_1' has the following properties:

(i) $(2B^*)^c \cap \theta_1' \neq \emptyset$,
(ii) From the properties of covers and maximality of s', there exists a constant C_0 depending on the parameters of the ellipsoid cover Θ_1' (which are equivalent to the parameters of Θ_1), such that $\theta_1' \subset C_0 B^*$,
(iii) W.L.G (by rotation), the distance between antipodal points on θ_1' is maximal along the x_1 axis,
(iv) $|\theta_1'| \leq c\varepsilon$.

The properties of θ_1' imply that

$$|B^* \cap \theta_1' \cap \{x \in \mathbb{R}^n : x_1 > 0\}| \sim |(B^*)^c \cap 2B^* \cap \theta_1' \cap \{x \in \mathbb{R}^n : x_1 > 0\}| \sim |\theta_1'|,$$

and therefore the existence of two boxes Ω_1 and Ω_2 that are symmetric to the main axes and of dimensions $a_1 \times \cdots \times a_n$ with the following properties:

(i) $\Omega_1 \subset B^* \cap \theta_1'$,
(ii) $\Omega_2 \subset (B^*)^c \cap 2B^* \cap \theta_1'$,
(iii) There exists $2 \leq i \leq n$ such that $a_i \leq c \sqrt[n-1]{\epsilon}$,
(iv) $|\Omega_1| = |\Omega_2| \sim |\theta_1'|$, which implies that $1/a_i \sim \frac{a_1 \times \cdots \times a_{i-1} \times a_{i+1} \times \cdots \times a_n}{|\theta_1'|}$,
(v) $\Omega_1 \cap \{x \in \mathbb{R}^n : x_i = 0\} \neq \emptyset$ and $\Omega_2 \cap \{x \in \mathbb{R}^n : x_i = 0\} \neq \emptyset$.

We shall now construct an atom a_m in $H^1(\Theta_1')$ (which implies $\|a_m\|_{H^1(\Theta_1')} \leq 1$) for which $\|a_m\|_{H^1(\Theta_2')} \geq c' \log(c''\epsilon^{-1})$. This will mean that for $f_m := a_m(A_m^{-1}\cdot)$ we will have $\|f_m\|_{H^1(\Theta_1)} / \|f_m\|_{H^1(\Theta_2)} \leq c''' \log(c''\epsilon^{-1})^{-1}$.

We define the atom a_m in $H^1(\Theta_1')$ by $a_m := |\theta_1'|^{-1}(1_{\Omega_1}(x) - 1_{\Omega_2}(x))$ (a_m satisfies the conditions of Definition 4.1). By Lemma 5.7, the function g_i defined by (5.2) is in $\mathrm{BMO}(\Theta_2')$, with $\|g_i\|_{\mathrm{BMO}(\Theta_2')} \sim 1$. From the properties of g_i and the boxes Ω_1 and Ω_2 we have

$$\begin{aligned}
\|a_m\|_{H^1(\Theta_2')} &= \sup_{\phi \in \mathrm{BMO}(\Theta_2')} \frac{|\langle a_m, \phi\rangle|}{\|\phi\|_{\mathrm{BMO}(\Theta_2')}} \geq c|\langle a_m, g_i\rangle| \\
&\geq -c\frac{1}{\theta_1'} \int_{\Omega_1} \log|x_i| dx \\
&\geq -c\frac{a_1 \times \cdots \times a_{i-1} \times a_{i+1} \times \cdots \times a_n}{\theta_1'} \int_0^{a_i} \log(x_i) dx_i \\
&\geq -c(1/a_i) \int_0^{a_i} \log(x_i) dx_i \\
&\geq c' \log(c''\varepsilon^{-1}). \qquad \square
\end{aligned}$$

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