SOME SCREENABLE, σ-RELATIVELY DISCRETE DOWKER SPACES

Peter Nyikos

Abstract.

Dowker spaces (that is, normal spaces that are not countably metacompact) are constructed which are also screenable and the countable union of discrete subspaces. The constructions and proofs are easy modifications of the ones for Balogh's screenable Dowker space [B1]. Balogh's example is presented alongside the modifications in a new exposition with several new results common to these spaces, including the fact that each has a discrete collection of $2^{\aleph_0} = \mathfrak{c}$ clopen sets. It is also shown that some of the new spaces do not have α-normal product with $\omega + 1$. A conjecture of Balogh's concerning a submetrizable space is discussed along with several open problems.

Introduction

Before his death of a heart attack at age 48, Zoltán ("Zoli") Balogh produced a remarkable assortment of Dowker spaces in ZFC, including four published examples, [B0] [B1] [B2] [B3]. All but the first were constructed using a technique that Balogh pioneered and mastered and used to produce a number of other spaces in ZFC, including a paracompact Q-set space and a space that solved two of the three Morita Conjectures. See [BG] for an exposition on all of these spaces and many other noteworthy accomplishments of Balogh.

This paper describes modifications in his screenable Dowker space [B1], that make them the union of countably many discrete (but far from closed) subspaces. In this way they combines one of the properties of his Dowker space in [B0] with the properties of his screenable Dowker space, which will be designated $\langle X, \tau_B \rangle$ while our two main ones are designated $\langle X, \tau_\sigma \rangle$ and $\langle X, \tau_\sigma^* \rangle$.

Incidentally, the paragraph in [BD] where Balogh's natural Dowker space [B3] is introduced may create the false impression that it is a re-working of Balogh's first

---

1991 *Mathematics Subject Classification.* (Updated to 2000.) Primary: 54D20, 54G20, 54G99; Secondary: 03E75, 54D15, 54E99, 54G12.

Key words and phrases. Dowker, σ-relatively discrete, screenable, normal, ultranormal, control pair, complete neighborhood, elementary submodel, M, N-reflection, ξ-homogeneous, coherent δ-sequence, submetrizable.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Dowker space [B0]. Actually, the natural Dowker space is a bare-bones Dowker space, not designed to have any other properties besides Dowkerness and "smallness" in the sense of being of cardinality $\mathfrak{c}$. The space in [B0], on the other hand, was designed to have not only these properties but also to be hereditarily normal and the countable union of discrete subspaces.

Remarkably enough, Balogh's proof that $\langle X, \tau_B \rangle$ is a screenable Dowker space goes through for $\langle X, \tau_\sigma \rangle$ and $\langle X, \tau_\sigma^* \rangle$ with only a few changes. However, it seemed worthwhile to revise the proof and add more information about the three spaces, in a unified treatment of all three. Thanks to this unified treatment, it is possible to define $2^\mathfrak{c}$ "hybrids" X_Y $(Y \subset X)$ of $\langle X, \tau_B \rangle$ and $\langle X, \tau_\sigma \rangle$, as described briefly in Section 1.

It is hoped that all this will stimulate research that uses this fascinating but very demanding Balogh technique – so demanding, in fact, that until very recently, no one but Balogh has used it successfully! Now, besides this paper, there is the very recent paper by four co-authors, featuring a ZFC example of a Dowker space with normal (hence also Dowker) square [OSWY].

Balogh's technique can be conveniently divided into three parts, which will be referred to as "stages" in what follows. The first stage is very straightforward and simple. An underlying set X is defined; and each cofinite subset of X and each member W_n of an ascending sequence of sets W_n covering X is declared to be open. This sequence is destined to be an open cover without a point-finite refinement. [Recall that a space is **countably metacompact** if every countable open cover has a point-finite refinement, and that a **Dowker** space is a normal space that is not countably metacompact.]

The second stage consists of building a subbase for the topology in $2^{|X|}$ steps, to build in certain "desirable" properties, including normality. On the surface, this is also straightforward. In the case of normality, what is obtained in Zoli's examples (as well as in our space $\langle X, \tau_\sigma \rangle$) is ultranormality: the property that every binary open cover can be refined to a partition. This is done by listing all proper binary covers in a sequenceof length $2^{|X|}$, each repeated $2^{|X|}$ times, and the first time (if any) that both sets in the cover are open, one adds a binary partition of X to the subbase, each member of which is contained in one of the members of the cover. A similar technique is used to obtain screenability of the various $\langle X, \tau_Y \rangle$. But in the case of $\langle X, \tau_\sigma \rangle$ we really need only countably many families to ensure screenability, and $\langle X, \tau_\sigma^* \rangle$ (Example 5 in Section 2) takes advantage of this.

What makes this and other Balogh constructions intricate is the need to make the choices of these subbasic sets in a sufficiently "generic" way so that certain "unwanted" features not built into the space do not appear by accident, as it were. [In the case of the Dowker examples, the "undesirable" characteristic is, necessarily, countable metacompactness.] In each of Balogh's examples, this kind of "genericity" is produced by defining control tuples (pairs in the case of his screenable Dowker space, triples in most other places) which are listed in a sequence of length $|X|$. [In the case of the Dowker examples, $|X| = \mathfrak{c}$.]

The third stage takes up by far the greatest part of Balogh's papers where he uses the technique. It consists of a very clever proof of the negative feature (in the Dowker papers, lack of countable metacompactness). It is a three-part process, involving two new ideas (complete neighborhoods and reflections of sequences, explained in Sections 3 and 4), and the heavy use of countable elementary submodels and control-tuples.

We will follow the numbering of Balogh's paper [B1] for the definitions, lemmas, propositions, theorems, etc. that go through without change (modulo two minor corrections) for the two spaces. These were given a 2-digit numbering. Examples of certain covers and their refinements, whose purpose is to give readers a better feel for the spaces, are numbered in single digits, and they are not needed for the main results. Problems, presented in Sections 6 and 7, are also numbered in single digits. Other new definitions, etc. will be given a 3-digit numbering. In many cases they merely single out concepts Balogh introduced in the running text of [B1], but there are also some new definitions, notably at the end of Section 3 and in Section 6.

Section 1. The first and second stages

The underlying set for X is $\mathfrak{c} \times \omega$. We adopt the following notation: $L_n = \mathfrak{c} \times \{n\}$ and $W_n = \mathfrak{c} \times n$ $(= \bigcup_{i=0}^{n-1} L_i)$. In particular, $W_0 = \emptyset$. In the base for the topology τ_σ, we include $B_x = \{x\} \cup W_{n-1}$ for each $x \in W_n$. This makes each W_n open in X and each L_n discrete in its relative topology.

In Balogh's space $\langle X, \tau_B \rangle$ being modified here, B_x was defined as $X \setminus \{x\}$ and declared to be open, thereby ensuring that X is a T_1 space. The starting point for his base was:

$$\mathfrak{B}_0 = \{B_x : x \in X\} \cup \{W_n : n \in \omega\}.$$

For our construction of τ_σ, we might as well keep this notation, with the understanding that B_x is being defined very differently.

The rest of Balogh's article now goes through for τ_σ, except for the following changes:

(A) a change in the meaning of $V_{t,K}(x)$ and $V_{t,K,\xi}(x)$ [see Notation 3.0.1], and

(B) in the proof of 5.1, having $y[n]$ stand for $\{\gamma_n\} \times k_n$ instead of $\{\gamma_n\} \times (k_n + 1)$, with a resulting modification in the last few sentences of Case 1 and Case 2 in [B1]; see the last sentence in the proof of Claim 5 in Section 5.

Even with change (B), our proof of 5.1 works just as well for Balogh's $\langle X, \tau_B \rangle$ as it does for our modification $\langle X, \tau_\sigma \rangle$. It will also work for the $2^\mathfrak{c}$ "hybrids" $\langle X, \tau_Y$ $(Y \subset X)$ mentioned in the introduction. In X_Y, the initial basic neighborhood B_x follows the τ_σ description if $x \in Y$ and otherwise follows the τ_B description. With this notation, $\tau_B = \tau_\emptyset$ and $\tau_\sigma = \tau_X$, but we will adhere to the B, σ subscipts below because of their mnemonic features.

There are also two corrections to be made in Balogh's paper [B1]:

(1) In the definition of $t(\beta, n)$ in [B1], there should be at least one $\xi \geq \beta$ included.

(2) In the proof of Lemma 3.2 in [B1], "smallest" should read "greatest".

We now embark on the unified description of $\langle X, \tau_B \rangle$ and $\langle X, \tau_\sigma \rangle$. The space $\langle X, \tau_\sigma^* \rangle$, which cuts down drastically on the number of Type 2 families (described below), will be Example 5 in Section 2.

Notation 1.0. Let $\langle \mathcal{S}_\xi \rangle_{\xi < 2^\mathfrak{c}}$ be a listing of two kinds of families of subsets of X, with each family listed $2^\mathfrak{c}$ times. The families are called, respectively, **Type 1 pairs** and **Type 2 sequences**. Type 1 pairs are binary covers of X; and each Type 2 sequence is indexed by $\mathfrak{c}$, its union is W_n for some $n > 0$, and it is said to be **of height n**.

The base $\mathfrak{B}$ for X is built in a $2^\mathfrak{c}$-step induction commencing with $\mathfrak{B}_0$. If ν is a limit ordinal, then $\mathfrak{B}_\nu = \bigcup \{\mathfrak{B}_\xi : \xi < \nu\}$, and the final subbase [actually a base, see Example 2 below] is $\mathfrak{B} = \mathfrak{B}_{2^\mathfrak{c}}$.

When $\mathfrak{B}_\xi$ is defined, $\mathcal{S}_\xi$ is analyzed to determine whether to make $\mathfrak{B}_{\xi+1}$ equal $\mathfrak{B}_\xi$, or whether to add new open sets to $\mathfrak{B}_\xi$ to produce $\mathfrak{B}_{\xi+1}$. We make $\mathfrak{B}_{\xi+1}$ equal $\mathfrak{B}_\xi$ whenever some member of $\mathcal{S}_\xi$ fails to be open in the topology generated by $\mathfrak{B}_\xi$, and also when all members of $\mathcal{S}_\xi$ were already open in some earlier $\mathcal{S}_\eta$. Otherwise, we will define a clopen partition of X [*resp.* an open partition of W_n] refining $\mathcal{S}_\xi$ and will add it to the existing subbase $\mathfrak{B}_\xi$ to produce $\mathfrak{B}_{\xi+1}$. The notation for these partitions is $\langle B^0_\xi, B^1_\xi \rangle$ [*resp.* $\langle E^\rho_\xi : \rho < \mathfrak{c} \rangle$]. They are precise refinements; this means $B^i_\xi \subset S^i_\xi$ [*resp.* $E^\rho_\xi \subset S^\rho_\xi$] for all i, ρ. As shown in Section 2, partitions that come from Type 1 families ensure normality of X while those that come from Type 2 families ensure **screenability,** *viz.,* the property that every open cover has an open refinement that is the countable union of disjoint collections.

A very naïve way of defining the partitions is to adopt a "first come, first served" policy. This would defeat our purpose by making X a discrete space: if $\mathcal{S}_\xi = \langle X \setminus \{x\}, X \rangle$, then putting each point $\langle \beta, n \rangle$ into the first B^i_ξ that contains it would give $B^i_\xi = \{x\}$, making x an isolated point.

Balogh's technique makes subtle use of control pairs to shuffle the points $\langle \beta, n \rangle$ around by moving some member of $\mathcal{S}_\xi$ to the front of the line, so to speak, in a way that depends on $\beta \in \mathfrak{c}$.

For each $A \subset X$ and every $\mathcal{S}_\xi$, let $\mathcal{S}_\xi \restriction A$ be the sequence $\langle S^\gamma_\xi \cap A \rangle_{\gamma \in \Gamma}$ where $\Gamma = \{0, 1\}$ if $\mathcal{S}_\xi$ is a Type 1 pair and $\Gamma = \mathfrak{c}$ if $\mathcal{S}_\xi$ is a Type 2 sequence.

Definition 1.1. *A pair $\langle A, d \rangle$ is called a* **control pair** *if the following conditions are met:*

(C-1) A is a countably infinite subset of X;

(C-2) d is a countable function with
$dom(d) \subset \{\mathcal{S} \restriction A : \mathcal{S}$ is a Type 1 or Type 2 sequence$\}$, and $range(d) \subset \mathfrak{c}$.

If $x = \langle \beta, k \rangle \in X$, we write $x(\xi) = \gamma$ (or $\langle \beta, k \rangle(\xi) = \gamma$) to mean $x \in B^\gamma_\xi$ [or $x \in E^\gamma_\xi$, as the case may be]. We write $d_\beta(\_,\_)$ for $d_\beta(\langle \_,\_ \rangle)$.

Let $\langle A_\beta, d_\beta \rangle$ list all control pairs, mentioning each one $\mathfrak{c}$ times.

The definition of $\mathfrak{B}$ will be complete once we tell how $\langle B^0_\xi, B^1_\xi \rangle$ is defined for ξ falling into what Balogh called Case 1 and how $\langle E^\rho_\xi : \rho < \mathfrak{c} \rangle$ is defined for ξ in Balogh's Case 2. We use the term **ξ-open** to denote "open in the topology generated by $\mathfrak{B}_\xi$." It is immaterial which of $\{\tau_Y\}$ is referred to in the following description, as long as we are consistent for all $\xi \in 2^{\mathfrak{c}}$.

Case 1. Assume that $\mathcal{S}_\xi = \langle S^0_\xi, S^1_\xi \rangle$ is a Type 1 pair, S^0_ξ and S^1_ξ are both ξ-open, and there is no $\eta < \xi$ such that $\mathcal{S}_\eta = \mathcal{S}_\xi$ and S^0_ξ and S^1_ξ are both η-open.

Subcase 1a. Suppose that $\mathcal{S}_\xi \restriction A_\beta \in dom(d_\beta)$ and $d_\beta(\mathcal{S}_\xi \restriction A_\beta) \in \{0, 1\}$.

In this subcase, we move S^i_ξ to the front of the line, so to speak, where $i = d_\beta(\mathcal{S}_\xi \restriction A_\beta)$. To be precise: if $x = \langle \beta, k \rangle$ then $x(\xi) = i$ (and thus $\langle \beta, k \rangle \in B^i_\xi$) if $x \in S^i_\xi$, otherwise we let $x(\xi) = 1 - i$.

Subcase 1b. If Subcase 1a does not hold, it is "first come, first served": $\langle \beta, k \rangle \in B^0_\xi$ unless $\langle \beta, k \rangle \notin S^0_\xi$, in which case $\langle \beta, k \rangle \in B^1_\xi$.

Case 2. Assume that $\mathcal{S}_\xi = \langle S^\rho_\xi : \rho < \mathfrak{c} \rangle$ is a Type 2 sequence, of some height n, S^ρ_ξ is ξ-open for every $\rho < \mathfrak{c}$, and there is no $\eta < \xi$ such that $\mathcal{S}_\eta = \mathcal{S}_\xi$ and S^ρ_η is η-open for all $\rho < \mathfrak{c}$.

In this case, we only define $x(\xi)$ for $x \in W_n$; in other words, only when $x = \langle \beta, k \rangle, k < n, \beta < \mathfrak{c}$. We put x in $E^{x(\xi)}_\xi$, in the following way.

Subcase 2a. Suppose $\mathcal{S}_\xi \restriction A_\beta \in dom(d_\beta)$. Then, with $d_\alpha(\mathcal{S}_\xi \restriction A_\beta) = \rho$, we move S^ρ_ξ to the front of the line by letting $\langle \beta, k \rangle(\xi) = \rho$ (whence $\langle \beta, k \rangle \in E^\rho_\xi$) if $\langle \beta, k \rangle \in S^\rho_\xi$, otherwise we take the minimum σ such that $\langle \beta, k \rangle \in S^\sigma_\xi$ and let $\langle \beta, k \rangle(\xi) = \sigma$, whence $\langle \beta, k \rangle \in E^\sigma_\xi$.

Subcase 2b. If Subcase 2a does not hold, we let $\langle \beta, k \rangle(\xi) = min\{\sigma : \langle \beta, k \rangle \in S^\sigma_\xi\}$ for every $k < n$.

Having finished the construction of $\langle \mathfrak{B}_\xi : \xi < 2^{\mathfrak{c}} \rangle$ and thus of the respective topologies on X, let us set:

$$H_1 = \{\xi < 2^{\mathfrak{c}} : \text{ Case 1 holds for } \xi\},$$

$$H_2 = \{\xi < 2^{\mathfrak{c}} : \text{ Case 2 holds for } \xi\},$$

and let $H = H_1 \cup H_2$.

It follows from the minimality of ξ in Cases 1 and 2 that:

Observation 1.3.
(a) If $\xi \in H_1$ then, for $\langle \beta, k \rangle \in B^i_\xi$ to hold, it is sufficient that $\langle \beta, k \rangle \in S^i_\xi$ and $d_\beta(\mathcal{S}_\xi \restriction A_\beta) = i$.
(b) If $\xi \in H_2$ then, for $\langle \beta, k \rangle \in E^\rho_\xi$ to hold, it is sufficient that $\langle \beta, k \rangle \in S^\rho_\xi$ and $d_\beta(\mathcal{S}_\xi \restriction A_\beta) = \rho$.

Observation 1.3 and the following observation together give an alternative criterion for which member of the refining partition contains $\langle \beta, k \rangle$.

Observation 1.3.1.
(a) If $\xi \in H_1$ and $\mathcal{S}_\xi \restriction A_\beta \notin dom(d_\beta)$ or $d_\beta(\mathcal{S}_\xi \restriction A_\beta) \neq 1$ then $\langle \beta, k \rangle \in B^0_\xi$ iff $\langle \beta, k \rangle \in S^0_\xi$.
(b) If $\xi \in H_2$ and $\mathcal{S}_\xi \restriction A_\beta \notin dom(d_\beta)$ then $\langle \beta, k \rangle \in E^\rho_\xi$ iff $\langle \beta, k \rangle \in S^\rho_\xi$ and $\langle \beta, k \rangle \notin S^\sigma_\xi$ for all $\sigma < \rho$.

In particular, if $\mathcal{S}_\xi$ is Type 2 of height n, and $k \geq n$, then $\langle \beta, k \rangle(\xi)$ is undefined and $\langle \beta, k \rangle \notin \bigcup\{E^\rho_\xi : \rho < \mathfrak{c}\}$.

The following examples are presented here to give the reader a better feel for the respective spaces. Examples 3 and 4 will be used in Section 2 but not in any of the subsequent sections.

Example 1. There is nothing in the definitions to prevent $\langle X, X \rangle$ from being, say, $\mathcal{S}_0$. Then we are in Case 1, so

$$B^1_0 = \bigcup\{\{\beta\} \times \omega \mid d_\beta(A_\beta, A_\beta) \text{ exists and } = 1\}$$

while $B^0_0 = X \setminus B^1_0$.

Example 2. The final subbase $\mathfrak{B}$ is actually a base, as can be seen from the following kind of cover. For $x = \langle \beta, k \rangle$ let U be any open neighborhood of x.

Let $\mathcal{S} = \langle U, X \setminus \{x\} \rangle$. Then $\mathcal{S} = \mathcal{S}_\xi$ for some $\xi < 2^{\mathfrak{c}}$ for which Case 1 holds [see the opening paragraph of Section 2, below], and $x \in B^0_\xi \subset U$. Thus X is 0-dimensional; that is, it has a base of clopen sets.

Example 3. Here is an example of a closed discrete subspace of X of cardinality $\mathfrak{c}$ in τ_σ, consisting of isolated points. Let $\mathcal{S}_\xi = \langle W_1, X\rangle$. Then B^0_ξ is a closed discrete subspace of L_0, and

$$B^0_\xi = \{\langle \alpha, 0\rangle : d_\alpha(A_\alpha \cap L_0, A_\alpha) \text{ either does not exist, or exists and } = 0\}.$$

and

$$B^1_\xi = \{\langle \beta, k\rangle : \text{ either } k \neq 0, ord_\alpha(A_\alpha \cap L_0) \text{ exists and } = 1\}.$$

Of course, $B^1_\xi = X \setminus B^0_\xi$.

If we use $W_n (n > 1)$ in Example 3 in place of W_1, then $B^0_\xi \cap L_{n-1}$ is a closed discrete subspace of L_{n-1}. We will see from the proof of Theorem 2.2.5 that there are subspaces of cardinality $\mathfrak{c}$ that are also τ_B-closed discrete.

Example 4. In $\langle X, \tau_\sigma\rangle$, let $\mathcal{S}_0 = \{\langle\{\langle\beta, 1\rangle\} \cup L_0 | \beta \in \mathfrak{c}\rangle\}$. Then $\langle E^\rho_0 | \rho \in \mathfrak{c}\rangle$ partitions W_2 into $\mathfrak{c}$-many open sets, each of which meets L_1 in a single point. Although they are relatively clopen in W_2, each one has every point of $X \setminus W_2$ in its closure. This is because the neighborhoods of these points are unchanged from the topology generated by $\mathfrak{B}_0$, and so they meet L_0 in a cofinite subset, while each E^ρ_0 has cardinality $\mathfrak{c}$.

The disjoint open sets E^ρ_0 in this last example can be shrunk individually to X-clopen nbhds of the respective $\langle\beta, 1\rangle$ by the process in Example 2. For instance, we can let $\mathcal{S}_{\beta+1} = \langle E^\beta_0, X\rangle$ for all $\beta < \mathfrak{c}$.

Section 2. Proofs of some elementary properties

Normality and screenability are built into the space, in a way that makes the proofs short. The building-in takes advantage of the fact that the cofinality of $2^{\mathfrak{c}}$ is greater than $|X| = \mathfrak{c}$, so that every collection of $\mathfrak{c}$ or fewer open sets is already an open collection at some initial stage $\xi < 2^{\mathfrak{c}}$.

Proposition 2.1. *X is normal.*

Proof. Let F_0 and F_1 be disjoint closed sets; then $\langle X \setminus F_1,\ X \setminus F_0\rangle$ is an open cover of X and hence is $\mathcal{S}_\xi$ for cofinally many ξ. The first such ξ is in H_1, so $F_i \subset B^i_\xi$ for this ξ; the disjoint open sets B^i_ξ thus witness normality for the pair F_0, F_1. $\square$

Proposition 2.2. *X is screenable; that is, every open cover can be refined to an open cover that is the union of countably many disjoint collections.*

Proof. If $\mathcal{U}$ is an open cover of X, and $n \in \omega$, let $\mathcal{U} \restriction W_n = \{U \cap W_n : U \in \mathcal{U}\}$. We can refine each $\mathcal{U} \restriction W_n$ to an open (in W_n, hence in X) partition $\mathcal{V}_n$ of W_n by a similar argument as for normality, using the appropriate $\mathcal{S}_\xi \in H_2$. Then $\bigcup_{n=1}^\infty \mathcal{V}_n$ is the desired open refinement of $\mathcal{U}$. $\square$

A very different proof of screenability is available for the following simplification of $\langle X, \tau_\sigma \rangle$.

Example 5. The underlying set is $\mathfrak{c} \times \omega$ as always, and the Type 1 families and control pairs are as always, but instead of $2^{\mathfrak{c}}$ Type 2 families, there are only countably many. These are the families

$$\mathcal{S}_n = \langle S_n^\beta : \beta < \mathfrak{c} \rangle \quad \text{where} \quad S_n^\beta = W_n \cup \{(\beta, n)\}$$

where the individual S_n^β are included in the base for the topology at the outset, just as in τ_σ. This is enough to make each row $L_n = \mathfrak{c} \times \{n\}$ into a discrete space in its relative topology, and each $\mathcal{E}_n = \langle E_n^\beta : \beta \in \mathfrak{c} \rangle$ is an expansion of L_n to a disjoint collection of open sets whose union is W_{n+1}. The resulting topology τ_σ^* is easily seen to be screenable: given an open cover $\mathcal{U}$, just intersect each E_n^β with a member of $\mathcal{U}$ containing $\langle \beta, n \rangle$ to get a σ-disjoint refinement of $\mathcal{U}$. The Type 1 families give normality as before.

Normality and screenability together imply the following concept.

Definition 2.2.1. *Given a subset D of a set X, an* **expansion** *of D is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = \{d\}$ for all $d \in D$. A space X is* [**strongly**] **collectionwise Hausdorff** *(abbreviated* [**strongly**] **cwH***) if every closed discrete subspace has an expansion to a disjoint [resp. discrete] collection of open sets.*

A well-known, simple fact is that every normal, cwH space is strongly cwH: if $\{U_d : d \in D\}$ witnesses cwH for D, let V be an open set containing D whose closure is in $\bigcup\{U_d : d \in D\}$; then $\{U_d \cap V : d \in D\}$ is a discrete open expansion of D.

Corollary 2.2.2. *X is strongly cwH.*

Proof. This follows from the foregoing observation and the general fact that every normal, screenable space is collectionwise normal with respect to discrete collections of non-Dowker closed sets. [T, Lemma 1.6] □

Proposition 2.2.3. *The topologies τ_Y and τ_σ^* do not depend on the order in which the $\mathcal{S}_\xi$ are listed.*

Proof. The following proof works for any $Y \subset X$ as well as for τ_σ^*. Let $\mathfrak{S} = \langle \mathcal{S}_\xi : \xi < 2^{\mathfrak{c}} \rangle$ be one listing, $\mathfrak{T} = \langle \mathcal{T}_\xi : \xi < 2^{\mathfrak{c}} \rangle$ another. $\mathcal{S}_0$ consists of 0-open sets and so when it first comes up in $\mathfrak{T}$ as $\mathcal{T}_{\xi_0}$, it gets refined to a partition of X or of some W_n into open sets. The partition does not depend on ξ_0 but only on $\mathcal{S}_0 = \mathcal{T}_{\xi_0}$.

Proceed by induction, defining H with respect to $\mathfrak{S}$ and, for each $\alpha \in H$, defining ξ_α to be the first ordinal η for which $\mathcal{T}_\eta = \mathcal{S}_\alpha$ and for which every member of $\mathcal{T}_\eta$ is open. We are guaranteed such an η by the induction hypothesis that we have been successful up to α: even if the members of $\mathcal{T}_\xi$ are never open for any $\xi < sup\{\xi_\beta : \beta < \alpha\}$, there will occur another η above this supremum for which $\mathcal{T}_\eta = \mathcal{S}_\alpha$ and by then all members of $\mathcal{T}_\eta$ will be open. So, although the ξ_α need not be listed in ascending order, defining τ_B using $\mathfrak{T}$ gives a finer topology than the one that uses $\mathfrak{S}$. Reversing the roles now shows that the two topologies are the same. $\square$

Corollary 2.2.4. *Whenever $Y \subset Y'$, τ'_Y is finer than τ_Y. In particular, τ_σ is the finest and τ_B is the coarsest of the τ_Y.* $\square$

We will see in the next section that τ_σ is strictly finer than τ_B. There we will develop machinery that enables us to show easily that X is not discrete (!) in any of the topologies. At the present stage, the easy results go in the opposite direction:

Theorem 2.2.5. *Every subspace of X of cardinality $< \mathfrak{c}$ is closed discrete in τ_B, hence in τ_Y for all $Y \subset X$.*

Proof. Clearly, it is enough to show "closed," but with a little extra effort we can verify a much stronger fact. Note that the following subspace of X contains $\mathfrak{c}$ columns, *viz.,* sets of the form $\{\gamma\} \times \mathfrak{c}$.

For each $\beta \in \mathfrak{c}$, let

$$X_\beta = \{\langle \gamma, n\rangle : d_\gamma = d_\beta, A_\gamma = A_\beta, n \in \omega\}.$$

Claim. *Every union of fewer than $\mathfrak{c}$ sets of the form X_β is a closed discrete subspace of X.*

Proof of Claim. Let $\kappa < \mathfrak{c}$ and let $Y = \bigcup_{\eta<\kappa} X_{\beta_\eta}$. Let $x = \langle \delta, n\rangle \in X$. We will show that x is not in the closure of $Y \setminus \{x\}$.

Pick $\theta \notin \bigcup\{ran(d_{\beta_\eta}) : \eta < \kappa\} \cup ran(d_\delta)$ so that $\theta \geq 2$ and let $\mathcal{S}$ be defined by:

$$S^\theta = W_{n+1}, \text{ and } S^\alpha = W_{n+1} \setminus \{x, \langle \alpha, 0\rangle\} \text{ if } \alpha \neq \theta.$$

Then each S^ρ is already open in the topology whose subbase is $\mathfrak{B}_0$, so the first time $\mathcal{S}$ is listed, say as $\mathcal{S}_\xi$, we have $\xi \in H_2$. Now if $y \in Y, y \neq x$, then the first S^ρ_ξ in which y is found has $\rho < 2$, so the only way y could get into E^θ_ξ is for $y = \langle \nu, m\rangle$ for some ν such that $d_\nu(\mathcal{S}_\xi \restriction A_\nu) = \theta$. But θ was chosen in such a way as to exclude this possibility.

On the other hand, θ is the least ρ such that $\langle \delta, n\rangle = x \in S^\rho_\xi$ and since $\theta \notin ran(d_\delta)$, it follows that E^θ_ξ is a clopen neighborhood of x missing $Y \setminus \{x\}$. $\square$

Corollary 2.2.6. *The cellularity of X is $\mathfrak{c}$; in fact, X has a discrete collection of $\mathfrak{c}$-many open sets.*

Proof. Let D be a closed discrete subspace of X of cardinality $\mathfrak{c}$: for the various τ_Y, let $D = X_\beta$ as in the proof of 2.2.5; for τ_σ^*, use Example 3. Using the fact that X is strongly cwH, expand D to a discrete collection of open sets. □

Definition 2.2.7. A space is **left-separated** if it can be well-ordered in a way that makes every initial segment closed. Such a well-ordering is called **a left separation**. A **neighborhood assignment** or **neighbornet** on a space X is a family of sets indexed by the points of X, each one a neighborhood of the indexing point. A **D-space** [resp. **dually discrete space**] is a space X such that for every neighbornet $\mathcal{V} = \{V_x : x \in X\}$ there is a closed discrete subset [*resp.* a discrete subset] D of X such that $\{V_x : x \in D\}$ covers X.

Corollary 2.2.8. *$\langle X, \tau_Y\rangle$ is left-separated for all $Y \subset X$ and hence is a D-space.*

Proof. By Theorem 2.2.5, every well-ordering of X of the form $\{x_\alpha : \alpha < \mathfrak{c}\}$ is a left separation. The rest is immediate from the following well-known fact:

Theorem. *Every left-separated space is a D-space.*

The proof of this theorem is straightforward. Let $\{x_\alpha : \alpha < \kappa\}$ be a left-separation of X. If $V_x : x \in X$ is a neighbornet, decide on D by induction: put x_0 in D and put x_α in D iff it is not in $\bigcup_{\beta<\alpha}\{V(x_\beta)$. Clearly $\{V_x : x \in D\}$ covers X. To show D is closed discrete, take any $\xi < \mathfrak{c}$, and let α be the least ordinal such that $x_\xi \in V(x_\alpha)$; then $V(x_\alpha) \setminus \{x_\eta : \eta < \xi\}$ cannot contain any point of D besides x_α, and if $\alpha < \xi$ it does not even contain that. □

I do not know whether $\langle X, \tau_\sigma^*\rangle$ is a D-space. If it is not, it would answer a number of questions that have been posed about D-spaces, most obviously:

Problem 1. [A, Problem 1.22] [E, Problem 6] *Is every screenable Tychonoff space a D-space?*

Mary Ellen Rudin's screenable Dowker space using $\diamondsuit^{++}$ [Ru1] is left-separated, so Example 5 stands alone here as a possible counterexample.

Section 3. The third stage, part 1: complete neighborhoods

In order to show the failure of countable metacompactness, it seems necessary to go much deeper into the topology of X than we have done so far. Balogh did this through the ingenious concept of complete neighborhoods, which afford a glimpse into the $\mathcal{S}_\xi$'s that are far along in the inductive process and depend on a rich assortment of earlier $\mathcal{S}_\eta$ to make their members open.

Notation 3.0.1. Let $x \in X$. Recall that $\xi \in H_1$ codes a unique clopen neighborhood of x, either B^0_ξ or B^1_ξ, and each $\xi \in H_2$ codes a unique open neighborhood of x. This neighborhood is E^ρ_ξ for a unique $\rho = x(\xi)$. If $\mathcal{S}_\xi$ is of height m, and $x \in L_n$ where $n \geq m$, then $\rho = \infty$ and $E^\rho_\xi = X$. In any case, we denote the coded neighborhood by $T_\xi(x)$.

Notation 3.0.2. For each $x = \langle \beta, k\rangle \in X, t \in [H]^{<\omega}$, and $K \in [W_{k+1}]^{<\omega}$ such that $x \notin K$, let

$$V_{t,K}(x) = \bigcap_{\xi \in t} T_\xi(x) \cap (U_k \setminus K) = V_{t \cap \xi, K}(x).$$

where $U_k = W_{k+1}$ if the topology is τ_B and and $U_k = \{x\} \cup W_k$ if $x \in Y$ and the topology is τ_Y (which is always the case with $\tau_X = \tau_\sigma$).

For each $\xi < 2^{\mathfrak{c}}$, let

$$V_{t,K,\xi}(x) = \bigcap_{\eta \in t \cap \xi} T_\eta(x) \cap (U_k \setminus K).$$

It is easy to see that the sets $V_{t,K}(x)$ form a base for the neighborhoods of x in the respective topologies. We can restrict ourselves to those t whose H_2 elements do not code any Type 2 $\mathcal{S}_\xi$ of height $\leq n$ when $x \in L_n$.

Definition 3.1. *A neighborhood $V_{t,K}$ of $x \in X$ is said to be* **complete** *if for every $\xi \in t$,*

$$V_{t,K,\xi}(x) \subset S^{x(\xi)}_\xi.$$

Otherwise $V_{t,K}$ is said to be **incomplete**.

In other words, $V_{t,K}(x)$ is complete iff $V_{t,K,\xi}(x)$ is contained in the unique S^γ_ξ for which $x \in B^\gamma_\xi$ [*resp.* $x \in E^\gamma_\xi$]. Completeness takes no account of other S^δ_ξ which may or may not contain x.

The beauty of complete neighborhoods is that we can safely ignore all η not in t in identifying which points these neighborhoods contain. These η may indeed determine whether $S^{x(\xi)}_\xi$ is open; but they are not needed to make $S^{x(\xi)}_\xi$ a neighborhood of x. The earlier ordinals in t are enough to do that when the neighborhood is complete. Hence it is significant that every neighborhood of x contains a complete neighborhood:

Lemma 3.2. *If $V_{t,K}(x)$ is a neighborhood of $x = \langle \beta, k\rangle$, then there are $t^* \supset t$ and $K^* \supset K$ such that $V_{t^*,K^*}(x)$ is a complete neighborhood of x.*

Proof. For every incomplete neighborhood $V_{t',K'}(x)$ with $t' \supset t$ and $K' \supset K$, let $\xi_{t',K'}$ be the greatest $\xi \in t'$ such that $V_{t',K',\xi}(x) \notin S^{x(\xi)}_\xi$. Our lemma now follows from the following claim by the fact that the ordinal $2^{\mathfrak{c}}$ is well-founded.

Claim. If $t' \supset t, K' \supset K$ *and* $V_{t',K'}(x)$ *is an incomplete neighborhood of* x*, then there are* $t'' \supset t', K'' \supset K'$ *such that* $V_{t'',K''}(x)$ *is either a complete neighborhood of* x*, or an incomplete neighborhood with* $\xi_{t'',K''} < \xi_{t',K'}$*.*

To prove the claim, let $\eta = \xi_{t',K'}$. Since $S_\eta^{x(\eta)}$ is η-open, there are $\bar{t} \in [H \cap \eta] < \omega$ and $\overline{K} \in [X]^{<\omega}$ such that $V_{\bar{t},\overline{K}(x)}$ is a neighborhood of x with $V_{\bar{t},\overline{K}(x)} \subset S_\eta^{x(\eta)}$. Then $t'' = t' \cup \bar{t}$ and $K'' = K \cup \overline{K}$ are as required. $\square$

Remark. The above is taken verbatim from [B1], except that Balogh had "smallest" in place of "greatest," rendering the definition of t'' and K'' incorrect except in the simplest case where $\xi_{t',K'}$ is the *only* ordinal ξ for which $V_{t',K',\xi}(x)$ is incomplete.

Our next three results illustrate the power of complete neighborhoods.

Lemma 3.2.1. *If* $x \in L_n$*, then every* τ_B*-neighborhood [*resp. τ_Y*-neighborhood or* τ_s^**-neighborhood] of* x *meets* L_k *in a set of cardinality* $\mathfrak{c}$ *for all* $k \le n$ *[*resp. $k < n$*].*

Proof. Let $V_{t,K}(x)$ be a basic nbhd of x, with $t = \{\xi_1, \dots \xi_n\}$ listed in ascending order. For each pair ξ_i, ξ_j find γ such that $S_{\xi_i}^\gamma \ne S_{\xi_j}^\gamma$ and let $x_{ij} \subset S_{\xi_i}^\gamma \Delta S_{\xi_j}^\gamma$ [as usual, Δ denotes symmetric difference]. Let A be a denumerable set containing all the x_{ij} and let α be any one of the $\mathfrak{c}$ ordinals such that $\langle \alpha, k\rangle \notin K$ for $k \le n$ and such that $A_\alpha = A$ and $d_\alpha(\mathcal{S}_{\xi_i} \restriction A)$ exists and equals $x(\xi_i)$ for $i = 1, \dots n$. Completeness of $V_{t,K}$ then implies that $\langle \alpha, k\rangle \in V_{t,K}$ for all such α and for all $k \le n$ [*resp.* $k < n$]. This follows from an easy induction beginning with

$$V_{t,K,\xi_1}(x) = V_{\emptyset,K}(x) = U_n \setminus K$$

and use of Observation 1.3 at each step. $\square$

Corollary 3.2.2. τ_σ *is strictly finer than* τ_B*.*

Proof. By Corollary 2.2.2, τ_σ is finer than τ_B. On the other hand, Lemma 3.2.1 shows that no point of L_0 is isolated in τ_B, while every point of L_0 is isolated in τ_σ. $\square$

Theorem 3.2.3. *No point of* X *is a* G_δ *in* τ_B*, and no point of* $X \setminus L_0$ *is a* G_δ *in* τ_σ^* *or any of the* τ_Y*.*

Proof. Let $\{V_{t_n,K_n}(x) : n \in \omega\}$ be a family of basic nbhds of $x = \langle \beta, k\rangle$. Let A be a denumerable set with the following property: for each pair $\{\xi, \eta\} \in T = \bigcup_{n=0}^\infty t_n(x)$ there exists γ such that A meets $S_\xi^\gamma \Delta S_\eta^\gamma$.

Now let $A = A_\alpha$ for some α such that $K_n \cap (\{\alpha\} \times \omega) = \emptyset$ for all n and such that $d_\alpha(\mathcal{S}_\xi \restriction A_\alpha) = x(\xi)$ for all $\xi \in T$. There is no problem doing this even if ξ appears in a number of different $t_n(x)$, because we get $x(\xi)$ each time with any given ξ. Argue by induction for each $t_n(x)$ separately, as in the proof of 3.2.1, to show that $\{\alpha\} \times k \subset \bigcap_{n=0}^{\infty} V_{t_n,K_n}(x)$. $\square$

4. The third stage, Part 2: reflecting sequences $\langle \beta_n, k_n \rangle$

When I first went through Balogh's preprint for [B1], I hoped that some such maneuvering as in the proofs of 3.2.1 and 3.2.3 would suffice to show the failure of countable metacompactness. The proof of this failure in [B1] uses the following characterization of countable metacompactness, whose necessity follows very easily from the usual characterization of every countable open cover having a point finite refinement: any $\subset$-descending family $\langle F_n \rangle_{n=0}^{\infty}$ of closed sets with empty intersection can be followed down by open sets: $\exists$ open $G_n \supset F_n$ such that $\bigcap_{n=0}^{\infty} G_n = \emptyset$.

Balogh's procedure (which will be used here as well) was to let $F_n = X \setminus W_n$ and to find $cof(\mathfrak{c})$-many whole columns in $\bigcap_{n=0}^{\infty} G_n$ whenever the sets $G_n \supset F_n$ are open for all n. I told him at the 1996 Prague Topological Symposium that this seemed "like using a cannonball to shoot a fly" at first, but that I could not simplify his proof by settling for just one point in the intersection. That still holds, for all τ_Y and for τ_σ^*.

The proof begins by letting ξ_k denote the unique element of H_1 such that $\mathcal{S}_{\xi_k} = \langle W_k, G_k \rangle$. Then for each $x = \langle \beta, k \rangle \in X$, we choose a complete basic nbhd $V(\beta, k) = V(x) = V_{t(x),K(x)}(x)$ such that

(4-0) $\{\xi_j : j \le k\} \subset t(x) = t(\beta, k)$.

This ensures that $V(x) \subset B^1_{\xi_k} \subset G_k$. Let $t_i(x) = t(x) \cap H_i$ for $i = 1, 2$. Let $V_\xi(x) = V_{t(x),K(x),\xi}(x)$ for every $\xi \in 2^{\mathfrak{c}}$.

For every $C \in [X]^\omega$, let $\langle \zeta_j(C) \rangle_{j \in \omega}$ be a list, with repetitions permitted, of $t_1(C) = \bigcup_{x \in C} t_1(x)$. Since replacing $V(x)$ by a smaller nbhd of x preserves $V(x) \subset B^1_{\xi_k} \subset G_k$, we can ensure that conditions (4-1), (4-2), and (4-3$^+$) below hold for every $\beta \in \mathfrak{c}$. We follow Balogh's numbering [in which (4-3) is simply the requirement that $V(x)$ be complete] as closely as practical:

(4-1) if $\beta > sup(\pi(A_\beta))$ then $\{\zeta_j(A_\beta) : j < k\} \subset t_1(\beta, k)$ for every $k \in \omega$.

(4-2) $j < k < \omega$ implies $t_1(\beta, j) \subset t_1(\beta, k)$.

(4-3$^+$) $V(x) = V(\beta, k)$ is complete, and there exists $\theta \ge \beta$ in $t_1(x) = t_1(\beta, k)$.

Next, let M and N be countable elementary submodels of

$H(2^{2^{\mathfrak{c}}}) = \{S : S$ is a set whose transitive closure has cardinality $\le 2^{2^{\mathfrak{c}}}$

with $M \in N$, such that

$$\mathfrak{c},\ \langle \mathcal{S}_\xi : \xi \le 2^{\mathfrak{c}} \rangle, H_1,\ t : X \to [H]^{<\omega},\ K : X \to [X]^{<\omega},\ \langle x(\xi) : \xi \in H_1, x \in X \rangle$$

are all elements of M.

Let $A = N \cap X (= (N \cap \mathfrak{c}) \times \omega))$, let $R = t_1(A) \cap M$. Note that by (4-1),

(4-4$^+$) $\beta > sup(N \cap \mathfrak{c})$ and $A_\beta = A$ imply $R \subset \bigcup_{k\in\omega} t_1(\beta, k)$.

In the case of τ_s^*, we even have $t(A) \cap M = R$ since $H_2 = \omega$ here.

Lemma 4.0.1. $\{\xi_k : k \in \omega\} \subset R(\subset M)\}$.

Proof. Let $\alpha \in M \cap \mathfrak{c}$. Then $t_1(\alpha, k) \in M$ for all k, and so by (4-0), $\{\xi_k : k \in \omega\} \subset t_1(A) \cap M = R$. $\square$

Definition 4.1. Let $\beta_n > sup(N \cap \mathfrak{c})$ $(n \in \omega)$ be a sequence of ordinals $< \mathfrak{c}$. We say that a sequence $\langle x_n\rangle_{n\in\omega}$, with $x_n = \langle \alpha_n, k_n\rangle$ for all n is **an increasing M, N-reflection** of $\langle\beta_n\rangle_{n\in\omega}$ if $\{x_n : n \in \omega\} \subset N$ and the following conditions hold:

(4-5) For every $n \in \omega$, $t_1(\beta_n, k_n) \cap M = t_1(x_n) \cap M$, and $x_n(\xi) = \langle\beta_n, k_n\rangle(\xi)$ for all $\xi \in t_1(\beta_n, k_n) \cap M$;

(4-6) The sets $t_1(x_n) \setminus M$ are pairwise disjoint.

(4-7) $k_0 < \cdots < k_n < \cdots$, and, for every $n \in \omega$ and $\xi \in \bigcup_{j<n} t_2(x_j)$, $k_n >$ height of S_ξ.

Elementarity of N, together with (4-3$^+$) and (4-6), implies

(4-6$^+$) The sets $t_1(x_n) \setminus M$ are pairwise disjoint and nonempty.

Also, (4-1), (4-5) and the first part of (4-7) imply:

(4-8) if $A_{\beta_n} = A$ for every $n \in \omega$, then for every $\theta \in R$, we have $\theta \in t_1(x_n)$ for all but finitely many $n \in \omega$.

Indeed, $\theta = \zeta_j(A)$ for some $j \in \omega$, and if $n > j$, then it follows from (4-1) that $\theta \in t_1(\beta_n, k_n)$. So, from $\theta \in R \subset M$, it follows that $\theta \in t_1(\beta_n, k_n) \cap M = t_1(x_n) \cap M$.

Remark. The use of the word "reflection" is a bit misleading because (4-6) makes the sets $t_1(x_n) \setminus M$ pairwise disjoint while the sets $t_1(\beta_n, k_n) \setminus M$ could even be an ascending sequence. This happens in the case where $\beta_n = \beta_0$ and $A_{\beta_n} = A$ for all n, as in the successor case of the proof of Lemma 5.1, by (4-1).

Lemma 4.2. *For every sequence of ordinals β_n such that $sup(N \cap \mathfrak{c}) < \beta_n < \mathfrak{c}$ for all $n \in \omega$, there is an increasing M, N-reflection $\langle x_n\rangle_{n\in\omega}$ of $\langle\beta_n\rangle$.*

Proof. Let $k_0 = 0$. Since $t_1(\beta_0, 0) \cap M$ is finite, it is an element of M. Elementarity of N implies that there exists $x_0 \in N$ satisfying (4-5) and (4-6$^+$), the latter because of a $\theta \geq \beta_0$ as in (4-3$^+$), while (4-7) and (4-8) are vacuously satisfied.

If k_{n-1} and x_{n-1} have been defined, let $k_n > k_{n-1}$ be such that $k_n >$ height of S_ξ for all $\xi \in \bigcup_{j<n} t_2(x_j)$. Define a finite function r_n by $dom(r_n) = t_1(\beta_n, k_n) \cap M$ and for every $\xi \in dom(r_n)$, $r_n(\xi) = \langle \beta_n, k_n \rangle(\xi)$. Note that $r_n \in M$.

Consider the property

$$\varphi(\alpha, n) \qquad t_1(\alpha, k_n) \supset dom(r_n) \text{ and } \langle \alpha, k_n \rangle(\xi) = r_n(\xi) \text{ for every } \xi \in dom(r_n)$$

Since $\varphi(\alpha, n)$ can be described by a formula with parameters in M, and since $\varphi(\beta_n, n)$ holds, it follows that $\Phi \in M$ where $\Phi = \{\alpha \in \mathfrak{c} : \varphi(\alpha, n)\}$ and that Φ is uncountable. Again by elementarity of M, there is a $D \in M$ such that D be a subset of Φ which is maximal with respect to the property that the collection $\{t_1(\alpha, k_n) \setminus dom(r_n) : \alpha \in D\}$ is pairwise disjoint. Then D is uncountable; were it not so, we would have $D \subset M$, but then $D \cup \{\beta\}$ would contradict maximality of D. So there exists $\alpha_n \in D$ such that

$$(*) \qquad t_1(\alpha_n, k_n) \setminus dom(r_n) \text{ is disjoint from } M \cup \bigcup_{j<n} t_1(x_j)$$

Since $M \in N$ we can find such an α_n in N. Let $x_n = \langle \alpha_n, k_n \rangle$. As in the case $n = 0$, $t_1(\beta_n, k_n) \setminus dom(r_n) \neq \emptyset$. Now $x_n = \langle \alpha_n, k_n \rangle$ satisfies (4-5) because of $\varphi(\alpha_n, k_n)$ and $\varphi(\beta_n, k_n)$ and the fact that $dom(r_n) = t_1(\beta_n, k_n) \cap M$. It satisfies (4-6$^+$) because of $(*)$ and the preceding paragraph, while (4-7) was built into the definition of k_n. $\square$

Aside. Balogh used (4-6) instead of (4-6$^+$) in the preceding proof [B1]. This led to a minor hole in his proof: under his hypotheses, $t_1(\beta_n, k_n)$ could be a subset of M for small n; and for those n, we would have $\{t_1(\alpha, k_n) \setminus dom(r_n) : \alpha \in D\} = \{\emptyset\}$, and $D \cup \{\beta\} = D$, which no longer contradicts maximality of D.

The following fact will play a key role in the next section.

Proposition 4.3. *If $\xi, \eta \in N \cap H$ and $S_\xi \restriction A = S_\eta \restriction A$, then $\xi = \eta$.*

Proof. Since $\langle \mathcal{S}_\xi : \xi < 2^{\mathfrak{c}} \rangle \in N$, and $\xi, \eta \in N$, it follows that $\mathcal{S}_\xi \restriction A = \mathcal{S}_\eta \restriction A$ implies $N \models \mathcal{S}_\xi = \mathcal{S}_\eta$, which in turn implies $\mathcal{S}_\xi = \mathcal{S}_\eta$. Then $\xi = \eta$ by Proposition 1.2. $\square$

5. The third stage, Part 3: homogeneity, coherent δ-sequences, and the proof that X is not countably metacompact.

We are almost ready to embark on the intricate proof that open sets G_k described at the beginning of Section 4 have nonempty intersection in all the τ_Y. Here is a concept that will play a key role in it.

Definition 5.0.1. Let $\beta, \gamma \in \mathfrak{c}$ and $\xi \in H_1$. We write $\beta \approx_\xi \gamma$ iff $\langle \beta, k\rangle(\xi) = \langle \gamma, k\rangle(\xi)$ for every $k \in \omega$. We say γ **is ξ-homogeneous** to mean that $\gamma \times \omega \subset B^i_\xi$ for either $i = 0$ or $i = 1$. Otherwise we say γ **is ξ-splitting** and say that n **is above the ξ-split of** γ to mean that $\{\gamma\} \times n$ meets both B^0_ξ and B^1_ξ. We say γ is **R-homogeneous** if it is ξ-homogeneous for every $\xi \in R = t_1(A) \cap M$.

This concept leads to the main result via the following lemma:

Lemma 5.1. *There is a $\gamma > sup(N \cap \mathfrak{c})$ which is R-homogeneous.*

This lemma applies to τ_B and τ_σ and every τ_Y in between. Once it is proven, we have:

Corollary 5.2$^+$. *$\langle X, \tau_Y\rangle$ is not countably metacompact, for any $Y \subset X$.*

Proof. With $G_k (k > 0)$ the descending sequence of open sets fixed at the beginning of Section 4, we need only show that their intersection is empty. With ξ_k as defined there, it is enough to show that $\bigcap_{k\in\omega} B^1_{\xi_k} \neq \emptyset$, since $B^1_{\xi_k} \subset G_k$.

Let γ be as in 5.1. Then, by 4.0.1, γ is ξ_k-homogeneous for all k. Since $B^0_{\xi_k} \subset W_k$, the whole column $\{\gamma\} \times \omega$ is in $B^1_{\xi_k}$ for all k. $\square$

A natural question arises in the wake of the foregoing proof: why does 5.1 speak of R-homogeneity when it is enough to have ξ_k-homogeneity for infinitely many k? After all, we can make $\langle G_k : k > 0\rangle$ be a $\subset$-descending sequence by taking finite intersections.

One answer is that we cannot get at the sets $B^i_{\xi_k}$ directly: the open sets G_k are too "generic" for that. All we have to work with are the sets $V(x)$, and their definition potentially involves all of H. With careful maneuvering, including copious use of elementarity of N and especially of M, we can get by with just R, but that seems to be the best we can do. Part of our strategy will be to use M, N-reflections $\langle x_k : k \in \omega\rangle$ to cover the column $\{\gamma\} \times \omega$ piecemeal with sets of the form $V(x_k)$. Of course, each $V(x_k)$ only covers a finite subset of the column.

What makes the proof of 5.1 so lengthy is that we cannot seem to get at γ directly, either. Instead, we will be approximating it inductively with γ_ν's which are $R \cap \Theta_\nu$-homogeneous for increasingly large ordinals Θ_ν, until we arrive at the desired $\gamma = \gamma_\upsilon$. Our process takes only countably many steps, taking advantage of countability of R.

To facilitate the process, we introduce the following concepts.

Notation 5.1.1. For every $\gamma \in \mathfrak{c}$ that is R-homogeneous, let $\Theta(\gamma) = \infty$ with the convention that ∞ is greater than any ordinal. Otherwise, let $\Theta(\gamma)$ denote the least $\theta \in R$ such that γ is not θ-homogeneous.

Definition 5.1.2. Let δ be a countable ordinal. A sequence $\{\gamma_\nu : \nu < \delta\}$ is **a coherent δ-sequence** if the following conditions are satisfied for all $\nu < \delta$:

(5-1) if $\mu < \nu$, then $sup(N \cap \mathfrak{c}) < \gamma_\mu < \gamma_\nu$;

(5-2) $\mu < \nu \implies \Theta(\gamma_\mu) < \Theta(\gamma_\nu)$

(5-3) $A_{\gamma_\nu} = A$ and

(5-4) if $\mu < \nu$, and $\xi \in R \cap \Theta(\gamma_\mu)$ then $\gamma_\mu \approx_\xi \gamma_\nu$.

Note that (5-1) implies $\Theta(\gamma_\nu) < \gamma_0$ for all ν unless $\Theta(\gamma_\nu) = \infty$, and that (5-4) implies that if $\{\eta, \xi\} \subset H_1$, and $\{\eta, \xi\} \subset R \cap \Theta(\gamma_\mu$, then $\{\gamma_\mu, \gamma_\nu\} \times \omega \subset B^i_\xi$ and $\{\gamma_\mu, \gamma_\nu\} \times \omega \subset B^j_\eta$ for some $i, j \in \{0,1\}$, but we need not have $i = j$.

Sublemma 5.1.3. *Let δ be a countable ordinal. If $\sigma = \{\gamma_\nu : \nu < \delta\}$ is a coherent δ-sequence, then either δ has a greatest element υ and $\Theta(\gamma_\upsilon) = \infty$, or σ can be extended to a coherent $\delta + 1$-sequence.*

Once this is proven, Lemma 5.1 follows quickly: let γ_0 satisfy (5-1) and (5-3). We are done if it so happens that $\Theta(\gamma_0) = \infty$; otherwise, we build coherent sequences by induction, each extending the earlier ones, and taking unions at limit ordinals; and, since R is countable, we eventually arrive at υ as described, and γ_υ is R-homogeneous.

Proof of 5.1.3. We adopt the notation Θ_ν for $\Theta(\gamma_\nu)$. If $\delta = 0$ then $\sigma = \emptyset$, and we can let γ_0 be any ordinal $\eta > sup(N \cap \mathfrak{c})$ for which $A_\eta = A$. This ensures that $\sigma' = \langle \gamma_0 \rangle$ satisfies (5-3) and the relevant part of (5-1), *viz.,* $sup(N \cap \mathfrak{c}) < \gamma_\nu = \gamma_0$, while the rest of 5.1.2 is vacuously satisfied.

The rest of the proof of the Sublemma breaks up into the case where δ is a successor and where it is a limit ordinal.

Case 1. $\delta = \mu + 1$ for some μ.

We will define γ_δ so that (with $\delta = \nu$) we have (5-1) through (5-4) for this ν and μ. Then this, together with coherence of σ, takes care of all other pairs.

Apply Lemma 4.2 to the case where $\beta_n = \gamma_\mu$ for all n to obtain x_n for all n. Let $t^*(x_n) = t(x_n) \cap \Theta_\mu$. Fix $i \in \{0,1\}$ such that $x_n(\Theta_\mu) = i$ for infinitely many $n \in \omega$. Note that if $\Theta_\mu = \xi_k$ for some k then we have no choice but to let $i = 1$ — recall that $\mathcal{S}_{\xi_k} = \langle W_k, G_k \rangle$ — but otherwise we might have a choice or even be forced to let $i = 0$.

Define the function d (to go into the control pair $\langle A, d \rangle$) as follows.

1. $dom(d) = \{\mathcal{S}_\xi \restriction A : \xi \in \bigcup_{n\in\omega} t^*(x_n) \cup \{\Theta_\mu\}\}$
2. $d(\mathcal{S}_\xi \restriction A) = i$ if $\xi = \Theta_\mu$, otherwise let $d(\mathcal{S}_\xi \restriction A) = x_n(\xi)$ for the unique n satisfying $\xi \in t^*(x_n) \setminus \bigcup_{j<n} t^*(x_j)$.

Now d is well-defined because if $\eta \neq \xi$, then $\mathcal{S}_\eta \restriction A \neq \mathcal{S}_\xi \restriction A$ by Proposition 4.3. There is also no "vertical conflict": the following claim shows that we could have let $d(\mathcal{S}_\xi \restriction A) = x_m(\xi)$ for any m for which $\xi \in t(x_m)$ when $\xi \neq \Theta_\mu$.

Claim 1. If $\xi \in t^*(A)$, then $x_m(\xi) = x_n(\xi)$ for all m, n for which $\xi \in t(x_m)$ and *$\xi \in t(x_n)$.*

$\vdash$ *Proof.* By (4-7), $t(x_n) \cap t(x_m) \cap H_2 = \emptyset$ if $n \neq m$, so we may assume $\xi \in H_1$. Also, if $\xi \in N \setminus M$, then (4-6) implies ξ cannot be in $t(x_\ell)$ for more than one ℓ. Thus we may assume $\xi \in R = t_1(A) \cap M$. Now γ_μ is ξ-homogeneous because $\xi < \Theta_\mu$. Hence $\langle \gamma_\mu, k_m \rangle(\xi) = \langle \gamma_\mu, k_n \rangle(\xi)$. Next, recall that $\beta_\ell = \gamma_\mu$ for all ℓ. Therefore, (4-5) implies $x_m(\xi) = \langle \gamma_\mu, k_n \rangle(\xi)$ and also $\langle \gamma_\mu, k_n \rangle(\xi) = x_n(\xi)$. $\dashv$

Next we pick $\gamma_\delta > \gamma_\mu$ to satisfy $\langle A_{\gamma_\delta}, d_{\gamma_\delta} \rangle = \langle A, d \rangle$. Clearly (5-1) and (5-3) hold for δ in place of ν.

For our next claim, recall that $T_\xi(x_n)$ stands for either $B_\xi^{x_n(\xi)}$ or $E_\xi^{x_n(\xi)}$ depending on whether $\xi \in H_1$ or $\xi \in H_2$. Recall also that

$$V_{\Theta_\mu}(x_n) = V_{t(x_n), K(x_n), \Theta_\mu}(x_n) = \bigcap_{\xi \in t^*(x_n)} T_\xi(x_n) \cap U_{k_n} \setminus K(x_n).$$

Claim 2. Let $y[n] = \{\gamma_\delta\} \times k_n$. Then $y[n] \subset V_{\Theta_\mu}(x_n)$.

$\vdash$ *Proof.* We have $\gamma_\delta \notin N$ whereas $K(x_n) \in N$, so $y[n] \subset W_n \setminus K(x_n) \subset U_{k_n} \setminus K(x_n) = V_{\emptyset, K(x_n)}(x_n)$.

We will show by induction on $\xi \in t^*(x_n)$ that

$$(I_\xi) \qquad y[n] \subset T_\xi(x_n).$$

and this will finish the proof of Claim 2.

So suppose (I_η) for all $\eta < \xi$. [This is vacously true when $\xi = min(t^*(x_n))$.] Then completeness of $V_{\Theta_\mu}(x_n)$ gives $y[n] \subset S_\xi^{x_n(\xi)}$. So it is enough to show that $d(\mathcal{S}_\xi \restriction A) = x_n(\xi)$. If $\xi \notin \bigcup_{j<n} t^*(x_j)$ this follows from the definition of d; the alternative is that $\xi \in t^*(x_j) \setminus \bigcup_{i<j} t^*(x_i)$ for some $j < n$. Then $d(\mathcal{S}_\xi \restriction A) = x_j(\xi) = x_n(\xi)$ by Claim 1. $\dashv$

The next two claims establish (5-2) for Case 1.

Claim 3. γ_δ is ξ-homogeneous for all $\xi \in R \cap \Theta_\mu$.

$\vdash$ *Proof.* By (4-8), $\xi \in t_1^*(x_n)$ for all sufficiently large n. Then $y[n] \subset T_\xi(x_n)$ for these n by (I_ξ), but $T_\xi(x_n) = B_\xi^{x_n(\xi)}$ because $\xi \in H_1$. So Claim 1 implies there exists i such that $y[n] \subset B^\ell_\xi$ for all sufficiently large n, but the $y[n]$ form a chain whose union is $\{\gamma_\delta\} \times \omega \subset B^\ell_\xi$. $\dashv$

In the foregoing proof, it would have been enough to have $y[n] \subset T_\xi(x_n)$ for infinitely many n. This is like what is used in the proof of our next claim.

Claim 4. γ_δ *is* Θ_μ*-homogeneous.*

$\vdash$ *Proof.* Suppose not. Pick $n \in \omega$ such that $\Theta_\mu \in t_1(x_n)$ and $x_n(\Theta_\mu) = i$ and k_n is above the split of Θ_μ at γ_δ. By Claim 2, $y[n] \subset V_{\Theta_\mu}(x_n)$, and completeness of $V_{t(x_n),K(x_n}(x_n)$ implies $V_{\Theta_\mu} \subset S^{x_n(\Theta_\mu)}_{\Theta_\mu} = S^i_{\Theta_\mu}$, giving us $y[n] \subset S^i_{\Theta_\mu}$. And so, by Observation 1.3(a) $y[n] \subset B^i_{\Theta_\mu}$; but this contradicts the claim that k_n is above the split of Θ_μ at γ_δ. $\dashv$

The proof in Case 1 will be done once we show (5-4) for this specific μ and $\nu = \delta$:

Claim 5. $\gamma_\mu \approx_\xi \gamma_\delta$ *for all* $\xi \in R \cap \Theta_\mu$.

$\vdash$ *Proof.* Since $\xi < \Theta_\mu < \Theta_\delta$, both γ_μ and γ_δ are ξ-homogeneous, we need only find k and k' such that $\langle \gamma_\mu, k\rangle(\xi) = \langle \gamma_\delta, k'\rangle(\xi)$.

Note that (4-5) implies $\xi \in t_1(x_n)$ for sufficiently large n, for which $x_n(\xi) = \langle \gamma_\mu, k_n\rangle(\xi)$. On the other hand, $\xi \in t^*(x_n)$, and so by (I_ξ), $\{\gamma_\delta\} \times k_n (= y[n]) \subset T_\xi(x_n)$, and in particular, $\langle \gamma_\delta, k_n - 1\rangle(\xi) = x_n(\xi)$. $\dashv$

The proof in our remaining case is very similar to that in Case 1.

Case 2. δ is a limit ordinal.

Pick $\delta_n \nearrow \delta$ and let $\beta_n = \gamma_{\delta_n}$ for all $n \in \omega$. Let $\{\langle x_n\rangle$ be an increasing M, N-reflection of $\langle \beta_n : n \in \omega\rangle$, with $x_n = \langle \alpha_n, k_n\rangle$. This time, let $t^*(x_n) = t(x_n) \cap \Theta_{\delta_n}$. Let $J = \bigcup_{n\in\omega} t^*(x_n)$, and let d be defined by:

(1*) $dom(d) = \{\mathcal{S}_\xi \restriction A : \xi \in J\}$.

(2*) For all $\xi \in J$, $d(\mathcal{S}_\xi \restriction A) = x_n(\xi)$ for the least n such that $\xi \in t^*(x_n)$.

Then d is well-defined, for the same reason as in Case 1. Similarly:

Claim 1′. If $\xi \in t^*(A)$*, then* $x_m(\xi) = x_n(\xi)$ *for all* m, n *for which* $\xi \in t(x_m)$ *and* $\xi \in t(x_n)$.

$\vdash$ Reasoning as in Claim 1, we may assume $\xi \in t_1(A) \cap M$. Hence by (4-5), $x_\ell(\xi) = \langle \gamma_{\delta_\ell}, k_\ell\rangle(\xi)$ for all ℓ. Wo l o g $m < n$, so $\xi < \delta_m < \delta_n$ and $\gamma_{\delta_m} \approx_\xi \gamma_{\delta_n}$. So $(x_m(\xi) =)\langle \gamma_{\delta_m}, k_m\rangle(\xi) = \langle \gamma_{\delta_n}, k_m\rangle(\xi)$. Since $\xi < \Theta_{\delta_m} < \Theta_{\delta_n}$, γ_{δ_n} is ξ-homogeneous, so $\langle \gamma_{\delta_n}, k_m\rangle(\xi) = \langle \gamma_{\delta_n}, k_n\rangle(\xi)$. $\dashv$

Next, let $\beta > sup_n \gamma_{\delta_n}$ be such that $\langle A_\beta, d_\beta\rangle = \langle A, d\rangle$. We let $\gamma_\delta = \beta$. As before, (5-1) and (5-3) are satisfied for δ in place of ν.

Claim 2′. $y[n] \subset V_{\Theta_{\delta_n}}(x_n)$.

$\vdash$ *Proof.* As in Claim 2, with $V_{\Theta_{\delta_n}}$ in place of V_{Θ_μ}. $\dashv$

The following claim establishes (5-2) in this limit case:

Claim 3′. γ_δ *is* ξ*-homogeneous for all* $\xi \in R \cap sup\{\Theta_{\delta_n} : n \in \omega\}$.

$\vdash$ *Proof.* Suppose this fails for ξ. Pick n such that $\xi < \Theta_{\delta_n}$, and such that $\xi \in t(x_n)$ (whence $\xi \in t_1^*(x_n)$) and k_n is above the split of ξ at γ_δ. But Claim 2^* implies that $y[n] \subset T_\xi(x_n) = B_\xi^{x_n(\xi)}$, contradicting the claim that k_n is above the split of ξ at γ_δ. $\dashv$

One can also give a proof similar to that of Claim 3. It remains only to show (5-4).

Claim 5′. *For all* $n \in \omega$ *and* $\xi \in t_1(A) \cap \Theta_{\delta_n}$, $\gamma_{\delta_n} \approx_\xi \gamma_\delta$.

$\vdash$ *Proof.* As in the proof of Claim 5, with δ_n in place of μ (including its use in subscripts). $\dashv$

And now (5-4) follows for δ in place of ν, because if $\mu < \delta$, there exists δ_n such that $\mu < \delta_n$. And so $\langle \gamma_\mu : \mu < \delta \rangle$ is coherent. $\square$

6. A product theorem and some open problems

With a little extra work, we can prove something even stronger than the main theorem about the $\langle X, \tau_Y \rangle$ for which each L_n is discrete. It is that their product with $\omega + 1$ fails to be α-normal.

Definition 6.1.1. A space X is **α-normal** [*resp.* **β-normal**] if for each pair of disjoint closed subsets A, B of X, there are disjoint open sets [*resp.* open sets with disjoint closures] U, V such that $A \cap U$ and $B \cap V$ are dense in A and B respectively.

The following is Theorem 2.3 of [LNP].

Lemma 6.1.2. *Let* X *be a* T_1 *space. The following are equivalent.*

(1) $X \times (\omega + 1)$ *is* α*-normal.*

(2) X *is* α*-normal, and if* $\{F_n : n \in \omega\}$ *is a family of closed sets and* $F = \bigcap_{k\in\omega} c\ell_X(\bigcup_{n=k}^\infty F_n)$*, and* E *is a closed set disjoint from* F*, then there is a family* $G_n (n \in \omega)$ *of open sets such that* $G_n \cap F_n$ *is dense in* F_n *and*

$$E \cap \bigcap_{k\in\omega} c\ell_X(\bigcup_{n=k}^{\infty} G_n)$$

is nowhere dense in E. $\square$

Recall that a space is **scattered** if every nonempty subspace has an isolated point in its relative topology. Clearly, being scattered is hereditary, and every scattered space has a dense set of isolated points. Both $\langle X, \tau_\sigma \rangle$ and $\langle X, \tau_\sigma^* \rangle$ are scattered. More generally, if $X \setminus Y$ is scattered in $\langle X, \tau_B \rangle$, then $\langle X, \tau_Y \rangle$ is also scattered. In particular, if L_n is discrete for all n, then Lemma 3.2.1 implies that X has Cantor-Bendixson height ω, with F_n as its nth Cantor-Bendixson derivative:

Definition 6.1.3. Given a topological space X, and an ordinal α, the **αth Cantor-Bendixson derivative $X^{(\alpha)}$ of X** is defined by induction, as follows. $X^{(0)} = X$; if $\alpha = \beta + 1$ then $X^{(\alpha)}$ is the derived set of $X^(\beta)$, while if α is a limit ordinal, then $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$. The **Cantor-Bendixson height** of a scattered space X is the least α such that $X^{(\alpha)} = \emptyset$.

An alternative definition of "scattered" is that $X^{(\alpha)} = \emptyset$ for some α. Then one can call the Cantor-Bendixson height the **scattered height** and thus convey the fact that the space is scattered.

Lemma 6.1.4. *Let X be a T_1 space, with scattered height ω. If $\bigcap_{k \in \omega} \overline{V_k} \neq \emptyset$ for every family of open sets V_k such that $X^{(k)} \subset V_k$, then $X \times (\omega + 1)$ is not α-normal.*

Proof. If $X^{(n)}$ is put for F_n in Lemma 6.1.2, then we can let $E = F_j$ for any $j \in \omega$. Suppose that, for each $j \in \omega$, there is a choice of G_n^j so that $G_n^j \cap F_n$ is dense in F_n and

$$F_j \cap \bigcap_{k \in \omega} c\ell_X(\bigcup_{n=k}^{\infty} G_n^j)$$

is nowhere dense in F_j. In other words, it is a subset of F_{j+1}. Now $F_n \setminus F_{n+1}$ is a subset of $G_n^j \cap F_n$ for all j, n. So if

$$V_k = \bigcup_{n=k}^{\infty} G_n^1 \cap \cdots \cap G_n^k$$

then $F_k \subset V_k \subset \bigcup_{n=k}^{\infty} G_n^j$ for all $j \leq k$. Then an easy induction on j shows that $\bigcap_{k \in \omega} \overline{V_k} \subset F_j$ for all j, hence $\bigcap_{k \in \omega} \overline{V_k} = \emptyset$, a contradiction. ☐

Corollary 6.1.5. *Let S be $\langle X, \tau_\sigma^* \rangle$ or any $\langle X, \tau_Y \rangle$ such that L_n is discrete for all n. Then $S \times (\omega + 1)$ is not α-normal.*

Proof. An easy induction using 3.11 shows that $X^{(n)} = F_n (= \bigcup_{k=n}^{\infty} L_k)$ for all n. The rest follows from 5.2^+ and 6.1.4. ☐

Lemma 6.1.4 could have been stated more concisely in the case of normal spaces, as the following folklore theorem makes clear:

Theorem 6.1.6. *A normal space of scattered height ω is countably paracompact if, and only if, $\bigcap_{k\in\omega} V_k \neq \emptyset$ for every family of open sets V_k such that $X^{(k)} \subset V_k$.*

Proof. We use Dowker's theorem that the following are equivalent for a normal T_1 space X: (1) $X \times \omega + 1$ is normal (2) X is countably paracompact (3) every countable increasing open cover of X has a countable closed refinement covering X.

Suppose $X \times \omega + 1$ is α-normal. By 6.1.4 there is a family V_n of open sets such that $X^{(n)} \subset V_n$ and $\bigcap_{n\in\omega} V_n = \emptyset$. Let K_n be the complement of V_n. Then each K_n is a closed set of finite Cantor-Bendixson height and the K_n cover X. Now we use a well-known folklore theorem:

Fact 1. Every normal space of finite scattered height is countably paracompact

Now if $\mathcal{W}$ is a countable open cover of X, we can refine the relatively open cover $\{W \cap K_n : W \in \mathcal{W}\}$ to a relatively closed, hence closed cover of K_n and then the union of all these closed refinements is a countable closed cover of X refining $\mathcal{W}$. □

For the sake of completeness, we show Fact 1. Let h be the height of X. If $h = 1$ then X is discrete and the conclusion is obvious. Suppose it is true for $h = n$. Let $\{W_k : k \in \omega\}$ be an increasing open cover of X. Let $F_k = X^{(n)} \cap W_{k+1} \setminus W_k$. Since $X^{(n)}$ is closed discrete, $\{F_k : k \in \omega\}$ is a discrete collection of closed sets. Now we use another well-known folklore theorem:

Fact 2. Every normal space is "countably collectionwise normal". That is, for every countable discrete family of closed sets F_k ($k \in \omega$) there is a disjoint family of open sets $U_k \supset F_k$.

Using Fact 2, let H_k be an open set such that $F_k \subset H_k$ and $\overline{H_k} \subset U_k \cap W_k$ for each k. Then we have a cover of X by the closed set $K = X \setminus \bigcup_{k\in\omega} H_k$ and the countably many closures of the sets H_k. The scattered height of K is $\leq n$ and we use the induction hypothesis to get a countable cover of K that is closed in K and hence in X and which refines $\{W_k : k \in \omega\}$. Since $\overline{H_k} \subset W_k$ for all k, we have our closed refinement.

Proof of Fact 2. Let U_0 and V_0 be disjoint open sets containing F_0 and $\bigcup_{n=1}^{\infty} F_n$ respectively. Then, given V_k, put F_{k+1} and $\bigcup_{n>k+1} F_n$ into disjoint open subsets U_{k+1} and V_{k+1} of V_k. □

Corollary 6.1.7. *Let X be a normal T_1 space of scattered height $\leq \omega$. Then $X \times \omega + 1$ is normal iff it is α-normal. Moreover, if X is of finite height, then $X \times \omega + 1$ is normal.*

The following problems were raised in [LNP].

Problem 2. *Is there a ZFC example of a Dowker space whose product with some infinite compact metric space is β-normal?*

It is easy to see that this problem is equivalent to whether the product with $\omega+1$ is β-normal.

Problem 3. *Is there Dowker space whose product with some compact metric space is α-normal, but not β-normal?*

Dowker spaces of scattered height ω have been constructed under several set-theoretic hypotheses that are independent of ZFC. Perhaps the simplest is Peter de Caux's space constructed using ♣ [dC]. It even fails to satisfy the following conditions:

Definition 6.2.1. A topological space X, τ is **α-countably metacompact** [*resp.* **α-countably paracompact**] if for every decreasing sequence $\langle F_n \rangle_{n=0}^{\infty}$ of closed subsets of X with empty intersection, there exists a sequence $\langle G_n \rangle_{n=0}^{\infty}$ of open sets such that $G_n \cap F_n$ is dense in F_n for all n and

$$\bigcap_{n=0}^{\infty} G_n = \emptyset \quad [\mathit{resp.} \bigcap_{n=0}^{\infty} \overline{G_n} = \emptyset].$$

Obviously, these are weaker conditions than being countably metacompact and countably paracompact, respectively. There is an easy connection between 6.1.1 and 6.2.1 where scattered spaces are concerned.

Theorem 6.2.2. *If X is a scattered T_1-space, and X is not α-countably paracompact, then $X \times (\omega + 1)$ is not α-normal.*

Proof. Let $\langle F_n \rangle_{n \in \omega}$ witness failure of α-countable paracompactness. By bumping the subscripts up if necessary, we let $F_0 = X$. Let $E = X$ in the statement of Lemma 6.1.2. For each n let G_n be an open set such that $G_n \cap F_n$ is dense in F_n. By shrinking G_0 if necessary, we may assume G_0 is the set of isolated points of X. By the failure of α-countable paracompactness, $\bigcap_{n \in \omega} \overline{G_n}$ has nonempty interior, and hence so does the larger set $\bigcap_{n \in \omega} c\ell_X(\bigcup_{k=n}^{\infty} G_k)$. □

Clearly, $\bigcap_{n \in \omega} \overline{G_n} = \bigcap_{n \in \omega} G_n$ in the foregoing proof, so it also shows:

Theorem 6.2.3. *A scattered T_1 space is α-countably paracompact iff it is α-countably metacompact.*

Peter de Caux's space [dC] has the property that every open set containing any set of the form $X^{(n)} \setminus X^{(n+1)}$ contains all but countably many isolated points, of which there are $\aleph_1$-many. So it is not α-countably metacompact.

Problem 4. *Is there a ZFC example of a Dowker space which is not α-countably metacompact? not α-countably paracompact?*

This is not an easy problem even where our scattered examples are concerned. Even the technique that produced "Zoli's cannonball" may fall short of showing that one of our spaces solves it. Where in 5.2^+ it was enough to have points x_n on infinitely many levels such that $\langle \beta, k\rangle \in V(x_n)$ for some β and k, [see the choice of $V(x)$ at the beginning of Section 4], we cannot skip any levels if we want to prove the failure of α-countable paracompactness.

This applies to any space X of scattered height ω. Let $F_n = X^{(n)}$ again and let $Z_n = F_n \setminus F_{n+1}$. Then Z_n is the dense set of isolated points in the relative topology of F_n, so we must have $Z_n \subset G_n$. On the other hand, we may assume $G_n \cap F_n = Z_n$ since there is nothing to be gained by making G_n larger. So we can take G_n to be a union of open neighborhoods $V(x)(x \in Z_n)$ such that $V(x) \setminus \{x\} \subset X \setminus F_n$. In particular, $G_0 = Z_0$, and if $z \in \bigcap_{n=0}^{\infty} \overline{G_n}$ then $z \in Z_0$ and for each n there exists $x_n \in Z_n$ such that $z \in V(z_n)$. In particular, $\langle X, \tau_\sigma^*\rangle$ and those $\langle X, \tau_Y\rangle$ such that each L_n is discrete fall under this rubric and $L_n = Z_n$.

In the proof of 5.1.3, the nearest thing we have to the z_n is the x_n, but we have no control over the size of the k_n. The topology τ_σ^* was formulated partly in the hope of making the k_n more manageable by cutting down the Type 2 families to the barest minimum. Even so, some skipping of levels is inevitable without drastic changes in the proof of 5.1.3.

7. Some more open problems

In [B1], Balogh remarked that $\langle X, \tau_B\rangle$ could be used to get a ZFC example of a normal screenable space that is not collectionwise normal, using a technique of Mary Ellen Rudin [R2]:

Theorem 6.1. *If there is a normal, screenable space that is not paracompact, there is a normal, screenable space that is not collectionwise normal.*

Stsrting with a space X as in the hypothesis, Rudin's technique preserves normality and screenability, and produces a space consisting of a discrete family of $\aleph_1$ copies of X immersed in a sea of isolated points. Thus if X is σ-relatively discrete, so is the resulting space.

Tantalizingly, Balogh made no mention of the question of whether $\langle X, \tau_B\rangle$ is itself collectionwise normal, and this question is open as far as I know. The same applies to all the $\langle X, \tau_Y\rangle$ and to $\langle X, \tau_\sigma^*$. And so, the following problem is still unsolved.

Problem 5. *Is there is a ZFC example of a collectionwise normal, screenable, nonparacompact space?*

Mary Ellen Rudin's screenable Dowker space using $\diamondsuit^{++}$ [R1] is a consistent example of such a space.

The following was asked in [Ny, Classic Problem VII D]:

Problem 6. *Is there a ZFC example of a submetrizable Dowker space?*

Recall that a submetrizable space is a space with a finer metrizable topology. A consistent example for Problem 6 using CH appears in [JKR]. As remarked in [Ny], Balogh left a set of handwritten notes [B4] in which he outlined what he believed to be a ZFC example. Unfortunately, his attempted proof does not work.

Balogh used the square of the Cantor set C as the underlying set and refined the topology. He had a huge "reservoir" as L_0, consisting of co-countably many horizontal lines (in the usual sense of $C^2 \subset \mathbb{R}^2$). The remaining countably many lines became the L_n. He let W_n be the union of L_0 with lines 1 through $n-1$, and declared these W_n to be open.

Right away the trouble comes in: there is a basic product neighborhood of the nth line that cuts out the $n-1$ lines below it. The basic neighborhood still hits the reservoir L_0, but Balogh envisioned using the W_n as the open cover without a point finite refinement, and that is defeated by Fact 2 in 6.1.6, as follows.

Each L_n, $n > 0$, is a closed set, because its complement consists of the complement of W_{n+1} together with the lines above it, and each line above it has a nbhd missing L_n as mentioned above. This makes $\{L_n : n \in \omega\}$ a discrete collection of closed sets. So we can put the L_n, $n > 0$, into disjoint open sets U_n as in the proof of Fact 2, and let $G_n = U_n \cap W_{n+1}$ for all n. These G_n, together with L_0, are a point-finite open refinement of $\{W_n : n \in \omega\}$. In fact, no point is in more than two members of the refinement.

Adapting Balogh's attempted construction to our σ-relatively discrete Dowker spaces yields paracompact spaces, and the following problem remains open:

Problem 7. *Is there a screenable, submetrizable Dowker space?*

Rudin's $\diamondsuit^{++}$ example in [R1], and the $\langle X, \tau_Y \rangle$ and $\langle X, \tau_\sigma^* \rangle$are essentially the only known screenable Dowker spaces, and none is submetrizable; for all except the first, this follows from Theorem 3.2.3. Thus we lack even consistent examples for Problem 7. On the other hand, we also do not know of any axioms that would negate the existence of screenable, submetrizable Dowker spaces.

We also lack consistency results for the following problem, which seeks to combine the salient features of the spaces in [B0] and [B1].

Problem 8. *Is there a screenable, hereditarily normal Dowker space? one that is σ-relatively discrete?*

So far, I have been unable to determine whether any of the $\langle X, \tau_Y \rangle$ is hereditarily normal. There are also natural candidates obtainable by including pairs of subsets $\langle S^0_\xi, S^1_\xi \rangle$ whose union is not X in the Type 1 pairs, and then defining $\langle \alpha, n \rangle(\xi)$ only if $\langle\alpha, n\rangle \in S^0_\xi \cup S^1_\xi$. This gives hereditary normality via the elementary theorem that a space is normal if every open subspace is normal: given an open set U in X, and disjoint relatively closed subsets F_0, F_1 of U, let $S^i = U \setminus F_i$; then $S^0 \cup S^1 = U$ and if ξ is the first ordinal for which both S^i are open, the natural choice for B^i_ξ gives a pair of disjoint open subsets of X containing F_{1-i} respectively.

However, the question of whether X with this modification of any of the τ_Y or of τ^*_σ is a Dowker space not an easy one. If we include these new pairs in H_1, then (4-1) needs to be modified since (as we saw in the proof of 2.2.5) there are open sets which miss $\mathfrak{c}$-many columns, and every column is missed by some open set. On the other hand, non-covering pairs do need to be taken into account if we want Claims 1, 2, $1'$ and $2'$ in the proof of 5.1.1 to continue to hold. Also, the proof of Claim 4 relied on there being infinitely many n for which $x_n(\Theta_\mu)$ is defined. But if Θ_μ indexes a non-covering pair, we cannot very well define $x_n(\Theta_\mu)$ if x_n is not covered by the pair. It remains to be seen whether difficulties like these — and the foregoing is only a small sample — can be successfully overcome.

References

[B0] Z. Balogh, "A small Dowker space in ZFC," Proc. Amer. Math. Soc. **124** (1996) 2555–2560.

[B1] Z. Balogh, "A normal screenable nonparacompact space in ZFC," Proc. Amer. Math. Soc. **126** (1998) no. 6, 1835–1844.

[B2] Z. Balogh, "Dowker spaces and paracompactness questions," Topology Appl. **114** (2001) 49–60.

[B3] Z. Balogh, "A natural Dowker space," Topology Proc. **27** (2003), no. 1, 1–7.

[B4] Z. Balogh, "A submetrizable Dowker space," handwritten notes.

[BG] Z. Balogh and D. Burke, "Zoli," Topology Proc. **27** (2003), no. 1, i-xxiii.

[JKR] I. Juhász, K. Kunen, and M. E. Rudin, "Two more hereditarily separable non-Lindelöf spaces," Canad. J. Math. **28** 998–1005.

[LNP] Lewis D. Ludwig, Peter Nyikos, and John E. Porter, "Dowker spaces revisited," Tsukuba J. Math 34 (2010) No. 1, 1 – 11

[Ny] P. Nyikos, "Classic Problems — 25 Years Later (Part 2)," Topology Proc. **27** (2003) no. 1, 365–378. Available online at http://www.math.sc.edu/~nyikos/publ.html

[OSWY] H. Ohta, P.J. Szeptycki, W. Weiss, and T. Yorioka, "A Dowker space with hereditarily normal square in ZFC," preprint.

[R1] M. E. Rudin, "A normal, screenable, non-paracompact space," Topology Appl. **15** (1983) 313–322.

[R2] M. E. Rudin, "Collectionwise normality in screenable spaces," Proceedings AMS, **87** (1983) 347–350.

[T] F. D. Tall, "Normality versus collectionwise normality," in: *Handbook of Set-theoretic Topology,* K. Kunen and J. Vaughan ed., Elsevier (1984) 685–732.