ω_1-compactness in Type I manifolds

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Abstract

In this paper we establish that any well-pruned ω_1-tree, T, admits an ω_1-compact Type I manifold if T does not contain an uncountable antichain. If T does contain an uncountable antichain, it has been shown that whether or not T admits an ω_1-compact manifold is undecidable in ZFC.

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1 Introduction

In [G] the question of whether or not an arbitrary tree admits a Type I manifold was examined. First, an arbitrary tree admits a Type I manifold if and only if it is an ω_1-tree and well-pruned. It was found that under the axiom $\diamondsuit$, any well-pruned ω_1-tree admits an ω_1-compact manifold, whereas under the axiom $(*)$, if a well-pruned tree T admits an ω_1-compact manifold, then T does not contain an uncountable antichain. It was also shown, in ZFC, that any well-pruned ω_1-tree which did not contain an uncountable antichain or a Souslin subtree admits an ω_1-compact manifold. We now show that it does not matter whether T contains a Souslin subtree.

A *tree*, $\langle T, \leq \rangle$, which we will denote simply as T, is a partially ordered set such that for each $t \in T$ the set of predecessors of t, $\{s \in T : s < t\}$, is well-ordered. We will assume throughout that T has a single least element called the *root* of T.

Definition 1.1. Let T be a tree.

(a) If $t \in T$, $\hat{t} = \{s \in T : s < t\}$.

(b) For each ordinal α, the *α-th level* of T, or $T(\alpha)$, is the set $\{t \in T(\alpha) : \hat{t} \text{ has order type } \alpha\}$.

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(c) T_α is the subtree $\bigcup_{\beta<\alpha} T(\beta)$.

(d) If $t \in T(\alpha)$ for some ordinal α, $t^+ = \{s \in T(\alpha+1) : s > t\}$.

(e) The *height* of T is the least ordinal α such that $T(\alpha) = \varnothing$.

An ω_1-*tree* T has height ω_1 and each level $T(\alpha)$ is countable.

A space X is of *Type I* if it is the union of an ω_1-sequence $\Sigma = \{U_\alpha : \alpha < \omega_1\}$ of open subspaces such that $\overline{U_\alpha} \subset U_\beta$ whenever $\alpha < \beta$, and such that $\overline{U_\alpha}$ is Lindelöf for all α. If in addition $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$ for any limit ordinal α, then Σ is a *canonical sequence* for X.

We will assume from now on that M is a nonmetrisable manifold. Given any canonical sequence Σ for a manifold M, it is possible to relate Σ to an ω_1-tree. Nyikos [N] defines such a tree, $\Upsilon(\Sigma)$, as follows:

Definition 1.2. Let M be a Type I manifold, and let $\Sigma = \langle U_\alpha : \alpha < \omega_1\rangle$ be canonical for M. The *tree of non-metrisable-component boundaries associated with* Σ, denoted $\Upsilon(\Sigma)$, is the collection of all sets of the form ∂C such that C is a nonmetrisable component of $M \smallsetminus \overline{U_\alpha}$ for some α, with the following order: if $\tau, \sigma \in \Upsilon(\Sigma)$, then $\tau < \sigma$ iff σ is a subset of a component whose boundary is τ.

We usually denote this tree by just Υ if Σ is clear, and refer to it as an Υ-*tree*. We say that a tree *admits a manifold with property* P, if there exists a Type I manifold with property P, and with an Υ-tree isomorphic to T.

2 ω_1-compactness

Definition 2.1. A space is ω_1-*compact* if it does not contain an uncountable closed discrete subset.

Theorem 2.2. *If T is a well-pruned ω_1-tree which does not contain an uncountable antichain, then there exists an ω_1-compact Type I manifold whose Υ-tree is T.*

Proof. We will use induction to construct a manifold M by defining an open Lindelöf subset U_α for each $\alpha \in \omega_1$, so that $\bigcup_{\alpha\in\omega_1} U_\alpha$ is the underlying set for M, and $\Sigma = \langle U_\alpha : \alpha \in \omega_1\rangle$ is a canonical sequence.

Let $U_0 = (-1,1) \times (-1,0)$ with the usual topology.

We will use the following notation:

• For each $t \in T$ let I_t be a copy of $(-1,1) \times [0,1)$ with the usual topology, and denote the point in I_t corresponding to $\langle x, y\rangle$ by $I_t\langle x, y\rangle$. (For each α the underlying set for U_α is $\cup\{I_s : s < t, t \in T(\alpha)\} \cup U_0$, and $(-1,1) \times \{0\}$ will be the boundary component in $\Upsilon(\Sigma)$ corresponding to t.)

• For each $\alpha \in \omega_1$ and $t \in T(\alpha)$, if α is a successor ordinal define $i_t : I_t \to (-1,1) \times [\alpha, \alpha+1)$ by $i_t(I_t\langle x, y\rangle) = \langle x, \alpha + y\rangle$. If α is a limit ordinal define $j_t : I_t \to (-1,1) \times [1,2)$ by $j_t(I_t\langle x, y\rangle) = \langle x, 1 + y\rangle$.

• For each $t \in T$ let $B_t = (\bigcup\{I_s : s \leq t\} \cup U_0)$. In each case B_t will be associated with an embedding $h_t : B_t \to (-1,1) \times (-1, \alpha+1)$, where $t \in T(\alpha)$, and will therefore have a coordinate system determined by this embedding. We will denote $h_t^{-1}(\langle x, y\rangle)$ by $\langle x, y\rangle_t$.

- For each $t \in T$ let $\{t^n : n < |t^+|\}$ denote the successors of t.
- For each successor ordinal α, pick a homeomorphism

$$f_\alpha : (-1, 1) \times (-1, \alpha + 1) \to (-1, 1) \times (-1, \alpha),$$

satisfying the following properties:

(i) $f_\alpha((-1, 1) \times (\alpha, \alpha + 1)) = A$, where

$$A = \left\{ \langle x, \alpha - 1 + y\rangle : (-1 < x < 0) \wedge \left(\frac{1}{2}(x^2 + 1) < y < \frac{1}{2}\left(\sqrt[4]{|x|} + 1 \right) \right) \right\};$$

(ii) $f_\alpha((-1, 1) \times (\{\alpha\}) = \partial A$, where

$$\partial A = \left\{ \langle x, \alpha - 1 + y\rangle : (-1 < x < 0) \wedge \left(\frac{1}{2}(x^2 + 1) = y \right) \vee \left(y = \frac{1}{2}\left(\sqrt[4]{|x|} + 1 \right) \right) \right\};$$

(iii) $f_\alpha((-1, 1) \times (\alpha - 1, \alpha) = (-1, 1) \times (\alpha - 1, \alpha) \setminus (A \cup \partial A)$;

(iv) $\forall x \leq 0,\ f_\alpha(\{x\} \times (\alpha - 1, \alpha)) = \{x\} \times (\alpha - 1, \alpha - 1 + \frac{1}{2}(x^2 + 1))$;

(v) $\forall x > 0,\ f_\alpha(\{x\} \times (\alpha - 1, \alpha)) \subset (-x, x) \times (\alpha - 1, \alpha)$; and

(vi) $f_\alpha \restriction (-1, 1) \times (0, \alpha - 1]$ is the identity.

Denote the root of T by r and define $h_r : B_r \to (-1, 1) \times (-1, 1)$ to be the identity on U_0 and let $h_r(I_r\langle x, y\rangle) = \langle x, y\rangle$ for each $I_r\langle x, y\rangle$.

Suppose α is a successor ordinal, U_α has been defined and for each $t \in T(\alpha - 1)$, h_t has been defined. Define $h_{t^0} : B_{t^0} \to (0, 1) \times (0, \alpha + 1)$ by letting $h_{t^0} \restriction B_t = h_t$ and $h_{t^0} \restriction I_{t^0} = i_{t^0}$. If t has more than one successor, we will now define for each $n > 0$ a function

$$\varphi_{t^n} : \left(B_t \cup \bigcup_{m \leq n} I_{t^m} \right) \to (-1, 1) \times (-1, \alpha + 1)$$

and h_{t^n} is then $\varphi_{t^n} \restriction B_{t^n}$.

Let $\varphi_{t^0} = h_{t^0}$. If φ_{t^n} has been defined, let $\varphi_{t^{n+1}} \restriction \left(B_t \cup \bigcup_{m \leq n} I_{t^m} \right) = f_\alpha \circ \varphi_{t^n}$ and $\varphi_{t^{n+1}} \restriction I_{t^{n+1}} = i_{t^{n+1}}$.

Topologise $U_{\alpha+1}$ so that each φ_{t^n} is a homeomorphism. Note that this is well defined and consistent with the topology on B_t. Let $U_{\alpha+1} = \bigcup_{n < |t^+|} B_{t^n}$ have the direct limit topology.

If α is a limit ordinal, let $U_\alpha = \cup_{\beta < \alpha} U_\beta$ with the direct limit topology. It remains to define $U_{\alpha+1}$.

Consider the equivalence classes: $[t] = \{s \in T(\alpha) : \hat{s} = \hat{t}\}$. For each equivalence class pick a member $t = t^0$ and denote the other members t^1, t^2 etc. Pick a homeomorphism

$$g_\alpha : (-1, 1) \times (-1, \alpha + 1) \to (-1, 1) \times (-1, 2)$$

which preserves the first coordinate of each point, preserves horizontal lines, and such that $g_\alpha((-1,1)\times(-1,\alpha]) = (-1,1)\times(-1,1]$, and define $e_{[t]} : \bigcup_{s<t} B_s \to (-1,\alpha)$ by

$$e_{[t]}(I_s\langle x,y\rangle) = h_s(I_s\langle x,y\rangle).$$

For each equivalence class $[t]$, define $h_t : B_t \to (0,1)\times(0,\alpha+1)$ by letting $h_t \restriction \bigcup_{s<t} B_s = e_{[t]}$ and $h_t \restriction I_t = i_t$.

If $|[t]| > 1$ we define for each $n > 0$ a function

$$\psi_{t^n} : (\cup_{s<t} B_s) \cup (\cup_{m\leq n} I_{t^m}) \to (-1,1)\times(-1,2)$$

and h_{t^n} will be $(g_\alpha^{-1} \circ \psi_{t^n}) \restriction B_{t^n}$. Let $\psi_t = g_\alpha \circ h_t$ and if ψ_{t^n} has been defined let

$$\psi_{t^{n+1}} \restriction \cup_{s<t} B_s \cup (\cup_{m\leq n} I_{t^m}) = f_1 \circ \psi_{t^n}$$

and $\psi_{t^{n+1}}|I_{t^{n+1}} = j_{t^{n+1}}$. Topologise $U_{\alpha+1}$ so that each ψ_{t^n} is a homeomorphism.

For each $\alpha \in \omega_1$, U_α is Lindelöf since $T(\alpha)$ is countable and for each $t \in T(\alpha)$, $B_t \setminus I_t$ is homeomorphic to $(-1,1)\times(-1,\alpha)$. U_α is connected since $U_0 \subset B_t \setminus I_t$.

Clearly M is a manifold with $\Upsilon(\Sigma) = T$.

We now show that M is ω_1-compact. For any $p \in M$, if $s = \min\{t \in T; p \in B_s\}$, let $p = \langle p(x), p(y)\rangle_s$. Then for every $t > s$, if $p = \langle x', y'\rangle_t$, $|x'| \leq |p(x)|$ by properties (iv) and (v) of f_α. Suppose X is an uncountable discrete set, then there exist $a, b \in (-1,1)$ such that $Y = \{p \in X : a < p(x) < b\}$ is uncountable.

In the order inherited from T, Y is a tree in which every antichain is countable because it is an antichain of T. Since Y is uncountable, it is an ω_1-tree. Hence there exists $\alpha < \omega_1$ and $t \in T(\alpha)$ such that $B_t \cap Y$ is infinite, and since $h_t^{-1}((a,b)\times(1,\alpha+1))$ has countably compact closure, $B_t \cap Y$ has a limit in M. Thus M does not contain an uncountable closed discrete set. $\square$

If T contains a Souslin subtree but is not Souslin, then T has at most countably many uncountable paths which cease to branch at some level α say. The manifold constucted in the proof above ensures that for each path P, $P \setminus T_{\alpha+1}$ is ω_1-compact, as well as the Souslin subtree.

$\diamondsuit$ implies that any well-puned ω_1-tree, T, admits an ω_1-compact manifold. $\diamondsuit$ is needed for the case when T contains an uncountable antichain. We will consider one case in which $\clubsuit$ is adequate to obtain an ω_1-compact manifold from a tree with an uncountable antichain.

Let $\clubsuit_2$ denote the following axiom: there is a family $\{\lambda_\alpha : \alpha < \omega_1\}$ of countable limit ordinals, and a family $\{A_\alpha : \alpha < \omega_1\}$ such that:

1. both $dom(A_\alpha)$ and $ran(A_\alpha)$ are cofinal subsets of λ_α, of order type ω; and,
2. if X is a subset of $\omega_1 \times \omega_1$ that meets each subset of the form $[\alpha,\omega_1) \times [\alpha,\omega_1)$, then there exists α such that $A_\alpha \subset X$.

We will call such a family $\{A_\alpha : \alpha < \omega_1\}$, a $\clubsuit_2$-sequence.

Theorem 2.3. $\clubsuit$ *implies* $\clubsuit_2$.

We use the following formulation of $\clubsuit$: there exists a family

$$\{L_\gamma : \gamma < \omega_1,\ \gamma \text{ is a limit ordinal}\}$$

such that each L_γ is a cofinal subset of γ of order type ω, and such that if X is an uncountable subset of ω_1, then there exists γ such that $L_\gamma \subset X$.

We will make use of the lemma below, [D, Theorem 4.3].

Lemma 2.4. *If*

$$\{L_\gamma : \gamma < \omega_1,\ \gamma \textit{ is a limit ordinal}\}$$

witnesses $\clubsuit$, and X is an uncountable subset of ω_1, then $S = \{\gamma : L_\gamma \subset X\}$ is stationary.

Proof of Theorem 2.3. Let $\varphi : \omega_1 \to \omega_1 \times \omega_1$ be a bijection, and let

$$C = \{\gamma \in \omega_1 : \varphi([0,\gamma)) = [0,\gamma) \times [0,\gamma)\}.$$

A standard leapfrog argument shows C is a club. Let C' be the derived set of C and for each $\gamma \in C'$ let $B_\gamma = \varphi(L_\gamma)$. Since L_γ is cofinal in γ of order type ω, at most finitely many members of B_γ can be in $[0,\alpha) \times [0,\alpha)$ for any $\alpha < \gamma$; hence at least one projection of B_γ is unbounded in $[0,\gamma)$. Let $\{\lambda_\alpha : \alpha < \omega_1\}$ list in order all members of C' such that both projections are unbounded ω-sequences in λ_α. Let $A_\alpha = B_{\lambda_\alpha}$ for all α.
Claim. $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ witnesses $\clubsuit_2$.

Proof of Claim. Suppose X meets every set of the form $[\alpha,\omega_1) \times [\alpha,\omega_1)$. Let $Y = \{(\beta_\alpha,\delta_\alpha) : \alpha \in \omega_1\}$ be a subset of X such that

$$\alpha \leq \min\{\beta_\alpha,\delta_\alpha\} \leq \max\{\beta_\alpha,\delta_\alpha\} < \min\{\beta_\varepsilon,\delta_\varepsilon\}$$

whenever $\alpha < \varepsilon$.

Since $\varphi^{-1}(Y)$ is uncountable, there is a stationary set S such that $L_\sigma \subset \varphi^{-1}(Y)$ for all $\sigma \in S$. Let $\gamma \in C' \cap S$. Then $B_\gamma \subset Y$, and at least one projection of B_γ is unbounded in γ. This forces both $dom(B_\gamma)$ and $ran(B_\gamma)$ to be ω-sequences with supremum γ. So $B_\gamma \in \mathcal{A}$ and $B_\gamma \subset X$, as desired. $\square$

The following example demonstrates that the full strength of $\diamondsuit$ in Theorem [G, Theorem 5.4], need not be necessary.

Example 2.5. Assume $\clubsuit$. There exists an ω_1-compact Type I manifold M, such that if Σ is any canonical sequence for M, $\Upsilon(\Sigma)$ contains an uncountable antichain.

We will define a Type I manifold M with the underlying set:

$$\{\langle\alpha,\beta\rangle \in \mathbb{L}^+ \times \mathbb{L}^+ : \alpha,\beta \neq 0\} \smallsetminus \{\langle\alpha,\beta\rangle : \alpha \in [1,\omega_1), \beta \in \omega_1\}.$$

Let $\mathcal{T}$ be the topology on M as a subspace of $\mathbb{L}^+ \times \mathbb{L}^+$ with the usual topology. For each $\alpha < \omega_1$ let $U_\alpha = (0,\alpha)^2 \cap M$. We will define a topology on each U_α such that $\partial U_\alpha = \{\alpha\} \times [(0,\alpha) \cap M] \cup (0,1) \times \{\alpha\}$, $U_\alpha \approx \mathbb{R}^2$, and $\langle U_\alpha : \alpha \in \omega_1\rangle$ is canonical.

Let $M \setminus \bigcup_{\alpha<\omega_1} \partial U_\alpha$ have the topology inherited from $\mathcal{T}$. Note that $U_0 = \varnothing$, and U_1 has the usual topology, and hence $U_1 \approx \mathbb{R}^2$.

Suppose α is a successor ordinal, the topology on U_α has been defined, and $U_\alpha \approx \mathbb{R}^2$. Then let $U_{\alpha+1}$ be the connected sum of $(0, \alpha - \frac{1}{2}]^2 \cap M$, and $U_{\alpha+1} \setminus (0, \alpha - \frac{1}{2})^2$ with the topology inherited from $\mathcal{T}$.

If α is a limit, U_β has been defined for each $\beta < \alpha$, and $U_\beta \approx \mathbb{R}^2$ for each β, let $U_\alpha = \cup_{\beta<\alpha} U_\beta$ with the direct limit topology. Since each $U_\beta \approx \mathbb{R}^2$, $U_\alpha \approx \mathbb{R}^2$ [B].

To complete the construction it remains to define the topology on $U_{\alpha+1}$ when α is a limit. We need only define a neighbourhood base for each $x \in \partial U_\alpha$.

Let $\mathcal{A} = \{A_\alpha : \alpha \in L\}$ be a $\clubsuit_2$-sequence. Choose a function $\phi_\alpha : U_{\alpha+1} \to U_{\alpha+1}$ such that:

(a) $\phi_\alpha | U_\alpha$ is an embedding;

(b) ϕ_α is the identity on the sets $\{(0,1) \times \{\beta\} : \beta \le \alpha+1\}$ and $(0, \frac{1}{4}) \times (0, \alpha+1]$;

(c) for each $\langle x, y \rangle \in U_{\alpha+1}$, $\phi_\alpha(\langle x, y \rangle) = \langle x', y \rangle$ for some $x' \le x$;

(d) if $\beta \in \mathrm{dom} A_\alpha$, $A_\alpha \in \mathcal{A}$, then $\phi_\alpha(\{1\} \times (\beta, \beta+1))$ and $\phi_\alpha(\{\alpha\} \times (\beta, \beta + 1))$ are circular arcs in $(0,1) \times (\beta, \beta+1)$, such that $\phi_\alpha\left(\left\langle 1, \beta + \frac{1}{2} \right\rangle\right) = \left\langle \frac{1}{2}, \beta + \frac{1}{2} \right\rangle$, $\phi_\alpha(\langle \alpha, \beta + \frac{1}{2} \rangle) = \langle \frac{3}{4}, \beta + \frac{1}{2} \rangle$, $\phi_\alpha(\{\alpha+1\}, (\beta, \beta+1)) = (\{2\} \times (\beta, \beta+1))$, $\lim_{x\to 0} \phi_\alpha(\langle \alpha, \beta + x \rangle) = \langle 1, \beta \rangle$, and $\lim_{x \to 1} \phi_\alpha(\langle \alpha, \beta + x \rangle) = \langle 1, \beta + 1 \rangle$.

Such a function ϕ_α exists since $\mathrm{dom} A_\alpha$ is countable. Topologise $U_{\alpha+1}$ so that ϕ_α is an embedding. Clearly $U_{\alpha+1} \approx \mathbb{R}^2$.

This completes the construction.

M is a manifold, since if $x \in M$ then $x \in U_\alpha$ for some α, and since $U_\alpha \approx \mathbb{R}^2$, x has a Euclidean neighbourhood.

Clearly M is Type I and $\Sigma = \langle U_\alpha : \alpha \in \omega_1 \rangle$ is a canonical sequence for M.

$\Upsilon(\Sigma)$ contains an uncountable antichain. $\Upsilon(\Sigma)$ consists of a path P, which may be thought of as the main trunk of $\Upsilon(\Sigma)$, with a path emanating from P at each successor level. These paths do not themselves branch. By choosing a member of $\Upsilon(\Sigma)$ from each of the paths off P, we may obtain an uncountable antichain.

We now establish that M is ω_1-compact. Suppose Z is a closed uncountable discrete subset of M. Then $Z \not\subset U_\alpha$ for any α. We consider 3 cases.

1. $Z \cap (0,1) \times \mathbb{L}^+$ is uncountable.

   For some $a, b \in (0,1)$, the set $Z' = \{\langle x, y \rangle \in Z : a < x < b\}$ is uncountable. If $a \in (0, \frac{1}{4})$, then Z' must have a limit point on $[a, \frac{1}{4}] \times \{\alpha\}$ for some limit ordinal α. If not, Z' must have a limit point on $[\frac{1}{4}, b] \times \{\alpha\}$ for some limit ordinal α.

2. $X \cap ((0, \omega_1) \times (\beta, \beta+1))$ is uncountable for some $\beta \in \omega_1$.

   For some $a, b \in (0,1)$, the set $Z' = \{\langle x, y \rangle \in X : \beta + a < y < \beta + b\}$ is uncountable. This set has a limit point on $\{\alpha\} \times [\beta + a, \beta + b]$ for some limit ordinal α.

3. $Z \cap (0,1) \times \mathbb{L}^+$ is countable, and for each $\beta \in \omega_1$, $X \cap ((0,\omega_1) \times (\beta, \beta+1))$ is countable. Hence $B = \{\beta : Z \cap ([1,\omega_1) \times (\beta, \beta+1)) \neq \varnothing\}$ is uncountable.

   For each $\beta \in B$, pick one point $\langle x_\beta, y_\beta \rangle \in Z \cap ((0,\omega_1) \times (\beta, \beta+1))$. Let $Z' = \{\langle x_\beta, y_\beta \rangle : \beta \in B\}$. Then for some $a, b \in (0,1)$, $Y = \{\langle x_\beta, y_\beta \rangle \in Z' : \beta + a < y_\beta < \beta + b\}$ is uncountable. Let $B' = \{\beta : \langle x_\beta, y_\beta \rangle \in Y\}$. Note that for each $\alpha \in \omega_1$, if $\beta \in \mathrm{dom} A_\alpha$ and $x \in (0,1)$, then the first coordinate of $\phi_\alpha(\langle \alpha, \beta + x \rangle)$ is a fixed value, call it $x(\phi_\alpha)$, since each point in $\phi_\alpha(\{\alpha\} \times (\beta, \beta+1))$ lies on a circular arc through $\langle \frac{3}{4}, \beta + \frac{1}{2} \rangle$, $\langle 1, \beta \rangle$ and $\langle 1, \beta+1 \rangle$. Now let $\lambda^\beta = \min\{\alpha : \langle x_\beta, y_\beta \rangle \in U_\alpha\}$ for each $\beta \in B'$, and observe that $\sup\{\lambda^\beta : \beta \in B'\} = \omega_1$. Let $A = \{\langle \beta, \lambda^\beta \rangle : \beta \in B'\}$. Choose α such that $A_\alpha \in \mathcal{A}$ and $A_\alpha \subset A$. Since for each $\beta \in \mathrm{dom}(A_\alpha)$, $\langle x_\beta, y_\beta \rangle \in ((0,\omega_1) \times (\beta, \beta+1)) \cap U_\alpha$, $1 < x_\beta < \alpha_\beta < \alpha$ and $\beta + a < y_\beta < \beta + b$, if the first coordinate of $\phi_\alpha(\langle x_\beta, y_\beta \rangle)$ is x, then $\frac{1}{2} < x < \max\{a(\phi_\alpha), b(\phi_\alpha)\}$ and hence Z has a limit point on $[\frac{1}{2}, m] \times \{\alpha\}$, where $m = \max\{a(\phi_\alpha), b(\phi_\alpha)\}$.

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