Correction to "Complete normality and metrization theory of manifolds"

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The claim in this article [1] that the combination of SSA + PFA^+ is shown in [2, p. 660] to be consistent, modulo large cardinals, is incorrect. Moreover, Paul Larson has shown that the SSA is even incompatible with $\mathrm{MA}(\omega_1)$, and it is not known whether the weaker Axiom S is compatible with the PFA.

Fortunately, the topological results in [1] are all consistent. In fact, the PFA is already enough to imply every statement derived from the combination of SSA + PFA^+ in the article, and it can also be shown that the large cardinal strength of PFA is not needed. The key to these new discoveries is the following ZFC theorem of [3]:

2.3. Theorem. *Let X be a space which is either hereditarily normal (abbreviated T_5) or hereditarily strongly cwH, for which there are a continuous $\pi : X \to \omega_1$ and a stationary subset S of ω_1 such that the fiber $\pi^{\leftarrow}\{\sigma\}$ is countably compact for all $\sigma \in S$. Then X cannot contain an infinite family of disjoint closed countably compact subspaces with uncountable π-images.*

This can be combined with the results of [1] in the following way.

1. In [1, Lemma 2.5] $MA(\omega_1)$ is used to show that if M is a hereditarily cwH nonmetrizable manifold, then M is of Type I. That is, M is the union of a strictly ascending ω_1-sequence of open subspaces $M_\alpha (\alpha \in \omega_1)$ such that M_α has Lindelöf closure contained in all M_β such that $\beta > \alpha$.

2. In [1, Lemma 2.6] it is shown how M_α can be chosen so that $M_\alpha = \bigcup\{M_\xi : \xi < \alpha\}$ whenever α is a limit ordinal, and so that each point of $B_\alpha = \overline{M_\alpha} \setminus M_\alpha$ is contained in a compact, connected, infinite subset K_α of B_α so long as $dim(X) > 1$. [Actually, compactness of K_α is not needed for the new proof.]

3. The following is implicit in the proof of Lemma 2.7 in [1]:

Lemma A. *If CC_{22} holds and M is a locally compact space in which every countable subset has Lindelöf closure, and S is a stationary subset of ω_1 and $\{x_\alpha : \alpha \in S\}$ is a subset of M, then there is a stationary subset E of S such that either:*

(1) *$\{x_\alpha : \alpha \in E\}$ is a closed discrete subspace of M, or*
(2) *every countable subset of $\{x_\alpha : \alpha \in E\}$ has compact closure in M.*

This is used in the proof of [1, Theorem 2.7], along with the axiom (which follows from the PFA, see [4, Corollary 6.6]) that every 1st countable perfect preimage of ω_1 contains a copy of ω_1. These axioms are used there to produce a copy W of ω_1 in any hereditarily cwH nonmetrizable Type I manifold M. For any such copy $W = \{p_\alpha : \alpha \in \omega_1\}$ the following set is a club: $C_W = \{\alpha : p_\alpha \in B_\alpha\}$. Again using CC_{22}, a stationary subset S_1 of C_W is produced along with points $\{q_\alpha : \alpha \in S_1\}$, such that such that $F_1 = c\ell\{q_\alpha : \alpha \in S_1\}$ is disjoint from W and countably compact and hence closed in M, and such that both p_α and q_α are contained in a connected subset K_α of B_α for all $\alpha \in S_1$.

4. Also in the proof of [1, Theorem 2.7], assuming also the normality of M, a continuous real-valued function f from M to $[0,1]$ is constructed which is 0 on W and 1 on F_1. Since K_α is connected and meets both W and F_1 whenever $\alpha \in S_1$, this function f takes on all intermediate values on K_α.

In [1] it was shown that CC_{22} follows from PFA$^+$, but it can be derived just from the PFA, as explained in [3]. Also in [1], PFA$^+$ was mis-stated. Correct statements can be found in [2] and [5].

Now comes the new proof of the main theorem of [1], with altered set-theoretic hypothesis:

Main Theorem. [PFA] *Every T_5, hereditarily cwH manifold of dimension greater than 1 is metrizable.*

From each $K_\alpha(\alpha \in S_1)$ pick a point x_α so that $f(x_\alpha)$ is different from all $f(x_\beta)$, $\beta < \alpha$. Use the fact that M is cwH and the Pressing-Down Lemma to eliminate alternative (1) of Lemma A as in the proof of Theorem 2.7 of [1]. Alternative (2) then gives a stationary subset S of S_1 such that every countable subset of $\{x_\alpha : \alpha \in S\}$ has compact closure in M. In particular, the closure X of $\{x_\alpha : \alpha \in S\}$ in M is countably compact and so is $X \cap B_\alpha$ for all $\alpha \in \omega_1$.

Claim. The map $\pi : X \to \omega_1$ which takes $X \cap B_\alpha$ to α is continuous.

Once the claim is proven, we get a contradiction to Theorem 2.3 above as follows. The image under f of $\{x_\alpha : \alpha \in S\}$ is an uncountable subset of $[0,1]$, hence it has $\mathfrak{c}$-many condensation points. For each condensation point p and each countable ordinal α_0, there is a strictly ascending sequence of ordinals $\langle \alpha_n : n \in \omega \rangle$ and points $x_{\alpha_n} \in K_{\alpha_n}$ for $n > 0$ such that $|p - f(x_{\alpha_n})| < \frac{1}{n}$.

Let $\alpha = sup\{\alpha_n : n \in \omega\}$. Since X is countably compact, there is a point of $X \cap B_\alpha$ which is sent to p by f. Thus the sets $X \cap f^{-1}\{p\}$ are a family of $\mathfrak{c}$-many disjoint closed countably compact sets with uncountable π-range.

$\vdash$ *Proof of Claim.* If C is any closed subset of ω_1, then $Y_C = \bigcup\{B_\gamma : \gamma \in C\}$ is closed in M because $M \setminus Y_C$ falls apart into the open sets $M_\gamma \setminus \overline{M_\delta}$ where δ and γ are successive members of C. We then get a natural map $\pi^* : Y_C \to \omega_1$ taking each B_γ to γ. This map is continuous because the preimage of each closed set is closed. If C is the closure of S in ω_1, then the map π of the Claim is the restriction of π^* to X. $\dashv$

The foregoing proof allows us to slightly weaken the hypotheses on M in the main theorem: it is enough for M to be normal and hereditarily strongly cwH. [Recall that a space is termed *strongly cwH* if every closed discrete subspace D expands to a discrete collection of open sets U_d such that $U_d \cap D = \{d\}$ for all $d \in D$.] This is a weakening of hypotheses because every normal, cwH space is strongly cwH. It is an open problem whether normality can be dropped from this weakening. In [3] it is proven that it can be dropped under PFA + Axiom F, but it is not known whether this combination of axioms is consistent, even modulo large cardinals.

REFERENCES

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