

# An elementary topological property of Hilbert space, with applications to Erdős space, and some generalizations

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This paper is centered on an elementary yet striking interplay between the norm topology and the natural product topology on Hilbert space,  $\ell_2$ . The natural product topology is associated with the classical definition of Hilbert space as the space of square-summable sequences with the norm  $\|\cdot\|_2$  which we will simply write as  $\|\cdot\|$ . This gives an embedding (in the algebraic, not the topological sense) of  $\ell_2$  into the product space  $\mathbb{R}^\omega$ . [For notational convenience, we make the set  $\omega$  of non-negative integers the domain for all our sequences.] The natural product topology is strictly coarser than the weak topology, but is well known to be equivalent to it on norm-bounded sets.

Since  $\ell_2$  is paracompact, every locally finite collection of subsets expands to a locally finite collection of open sets [E]. The other topologies are also paracompact, since they are coarser than the norm topology and hence Lindelöf. There are locally finite collections in the norm and weak topologies that are not locally finite in the coarser topologies; however, within each topology one can expand any locally finite collection to a locally finite collection of open sets. Our first theorem extends this fact by expanding any collection  $\mathcal{S}$  of sets that is locally finite in the norm topology to a family of sets that is open in the product topology and hence in the weak topology, yet is locally finite in the norm topology. This result is apparently new even where the weak topology is concerned. Yet it is a corollary of an even more general result [Lemma 1] which states that any countable family of subsets of  $\ell_2$  can be expanded to a family of product-open sets that is norm-locally finite at every point where the original family was norm-locally finite.

The proofs easily extend to any separable Banach space with a Schauder basis, provided that the product topology is defined with respect to the Schauder basis in the natural way. In Section 2 we also extend the first theorem to the Hilbert spaces  $\ell_2(\Gamma)$  and some other classes of non-separable Banach spaces, via a more complicated proof.

The sets in these expansions of can be made to exclude any given point not in the union of the original family, simply because the product topology is Hausdorff. Subsequent applications use the ability to exclude the zero point, which we designate  $\vec{0}$ . These applications involve Erdős space, the subgroup  $\mathbb{E}$  of  $\ell_2$  consisting of those points whose coordinates are all rational. Already back in 1940 [H o], Erdős showed that  $\mathbb{E}$  does not have a base of clopen sets in the norm topology, even though any two points can be separated by a clopen set—that is, if  $x \neq y$  then there is a

clopen subset  $C$  of  $\mathbb{E}$  with the norm topology such that  $x \in C$  and  $y \notin C$ . Beyond this, nothing seems to have been done until now to improve our understanding of the clopen subsets of  $\mathbb{E}$ . The results in Section 3 represent a big jump in our understanding, but our knowledge of the clopen subsets of  $\mathbb{E}$  is still fragmentary. Section 4 uses that knowledge to produce new TVS topologies on Hilbert space that are between the norm topology and the product topology. With few exceptions, the topologies are not locally convex.

### 1. The basic theorem and some generalizations.

We will employ the following notation. Given  $x = \langle x_n : n \in \omega \rangle \in \ell_2$ , we define  $\text{spt}(x)$ , the support of  $x$ , to be its set of nonzero coordinates. Given a subset  $A$  of  $\omega$ , we define  $x \upharpoonright A$  to be the point in  $\ell_2$  which agrees with  $x$  on  $A$  and is 0 elsewhere. We use the von Neumann convention that every natural number equals the set of its predecessors in  $\omega$ , so that  $x \upharpoonright n$  means  $x \upharpoonright [0, n) \cap \omega$ .

**Definition 1.** Let  $X$  be a topological space, let  $x \in X$  and let  $A$  be an indexing set. A family of sets  $\{S_a : a \in A\}$  is *locally finite at  $x$*  if  $x$  has a neighborhood meeting  $S_a$  for only finitely many  $a \in A$ . A family is *locally finite* if it is locally finite at every point of  $X$ . An *expansion* of  $\{S_a : a \in A\}$  is a family  $\{R_a : a \in A\}$  such that  $S_a \subset R_a$  for all  $a \in A$ .

One feature of Definition 1 is that a set cannot be repeated infinitely many times in the indexing of a locally finite family. Some papers require that distinct  $S_a$  expand to distinct  $R_a$  in the definition of an expansion. In the interests of simplicity, we do not require this here; however, if  $\{R_a : a \in A\}$  is locally finite, then each  $R_a$  extends at most finitely many members of  $\{S_a : a \in A\}$ .

**Lemma 1.** *Let  $\{S_n : n \in \omega\}$  be a family of subsets of  $\ell_2$ . There is a choice of product-open sets  $U_n \supset S_n$  such that if  $x$  is a point of  $\ell_2$  at which  $\{S_n : n \in \omega\}$  is locally finite in the norm, then  $\{U_n : n \in \omega\}$  is also norm-locally-finite at  $x$ .*

*Proof.* Let  $V_n = \{x \in \ell_2 : \exists s \in S_n (\|s - x\| < 1/2^n)\}$ . It is easy to see that  $V_n$  is norm-open and that  $\{V_n : n \in \omega\}$  is norm-locally finite at every point at which  $\{S_n : n \in \omega\}$  is locally finite in the norm. Let  $P_n = \{p \in V_n : \text{spt}(p) \text{ is finite}\}$ . For each  $p \in P_n$  let  $k_n(p) = \max\{n, \max(\text{spt}(p)) + 1\}$ . Let  $U_n$  be the set of all points in  $\ell_2$  which agree with some point  $p$  of  $P_n$  in the first  $k_n(p)$  coordinates. Clearly,  $V_n \subset U_n$ .

$U_n$  is open in the product topology. If  $x \in U_n$ , let  $p \in P_n$  satisfy  $p = x \upharpoonright k_n(p)$ . Let  $m = k_n(p)$ . There exists  $\varepsilon > 0$  such that the  $\varepsilon$ -ball centered on  $p$  is a subset of  $V_n$ . Then  $y \in P_n$  whenever  $\text{spt}(y) \subset m$  and  $|p(i) - y(i)| < \frac{\varepsilon}{m}$  for all  $i \in m$ .

Therefore, the basic product-open set  $\{z \in \ell_2 : |p(i) - z(i)| < \frac{\varepsilon}{m} \text{ for all } i \in m\}$  is a subset of  $U_n$  containing  $x$ .

Finally, suppose that  $\{U_n : n \in \omega\}$  is not locally finite at  $x$ . Let  $z_j \rightarrow x$  in the norm, where  $z_j \in U_{n(j)}$  for some  $n(j) \geq j$ . Let  $m(j) \geq n(j)$  satisfy  $z_j \upharpoonright m(j) \in V_{n(j)}$ . Then

$$\|x \upharpoonright m(j) - z_j \upharpoonright m(j)\| = \|(x - z_j) \upharpoonright m(j)\| \leq \|x - z_j\| \rightarrow 0$$

and  $x \upharpoonright m(j) \rightarrow x$  and so  $z_j \upharpoonright m(j) \rightarrow x$  all in the norm. But  $z_j \upharpoonright m(j) \in V_{n(j)}$ , hence  $\{V_n : n \in \omega\}$  is not norm-locally finite at  $x$ .  $\square$

The following is immediate from Lemma 1.

**Corollary 1.** *Let  $\mathcal{V}$  be a denumerable family of norm-open subsets of  $\ell_2$ . There is an expansion  $\mathcal{U} = \{U_V : V \in \mathcal{V}\}$  of  $\mathcal{V}$  to a family of product-open sets, such that if  $x$  is in the closure of  $\bigcup \mathcal{U}$  without being in the closure of any  $U_V$ , then  $x$  is already in the closure of  $\bigcup \mathcal{V}$ .  $\square$*

**Theorem 1.** *Each locally finite collection of subsets of  $\langle \ell_2, \|\cdot\| \rangle$  expands to a family of product-open sets that is locally finite in  $\langle \ell_2, \|\cdot\| \rangle$ .*

*Proof.* Since  $\ell_2$  is separable, every locally finite collection of subsets is countable. Now apply Lemma 1.  $\square$

With one tiny change in the proof, we can extend Lemma 1 to all separable Banach spaces with Schauder bases, and then both Corollary 1 and Theorem 1 then extend immediately. To define the product topology in this setting, one uses the natural isomorphism between a Banach space  $E$  with Schauder basis  $\{e_n : n \in \omega\}$  and the set  $X$  of all sequences  $s$  from  $\omega$  to  $\mathbb{R}$  for which there is  $y_s \in E$  such that  $s(n)$  is the coordinate of  $y_s$  with respect to  $e_n$ . The norm is simply transferred from  $E$  to  $X$  and the product topology is even coarser than the weak topology on  $X$ : the coordinate map taking each  $s$  to  $s(n)$  is obviously both linear and continuous.

Of course,  $X$  and the isomorphism depend very much on the basis chosen. But whatever the basis, we can extend the proof of Lemma 1 by multiplying  $\|x - z_j\|$  by a constant factor  $L$  in the displayed formula. This is because every Schauder basis can be associated with a constant  $L$  such that for every vector  $x$  and every positive integer  $n$  we have  $\|x \upharpoonright n\| \leq L\|x\|$  [S, ] [Of course, we can take  $L = 1$  in the case of  $\ell_p$  spaces.] And so we have:

**Corollary 2.** *Let  $E$  be a Banach space with a denumerable Schauder basis  $B$  and let  $x \in E$ . Each countable family of subsets of  $E$  which is locally finite at  $x$  with*

respect to the norm has an expansion to a family of sets which are open in the relative product topology associated with  $B$  and norm-locally-finite at  $x$ .  $\square$

Without a Schauder basis there is usually no natural definition for a product topology, but we can use the fact that every separable Banach space isometrically and isomorphically embeds into  $C[0, 1]$  with the supremum norm [M] to show that every locally finite collection of sets in a separable Banach space (or even just a separable normed vector space) expands to a family of weakly open sets that is locally finite in the norm. Lemma 1 also extends:

**Theorem 2.** *Let  $X$  be a separable normed vector space and let  $\{S_n : n \in \omega\}$  be a (countable) family of subsets of  $X$ . There is a choice of weakly open sets  $U_n \supset S_n$  such that if  $x$  is a point of  $X$  at which  $\{S_n : n \in \omega\}$  is locally finite in the norm, then  $\{U_n : n \in \omega\}$  is also norm-locally-finite at  $x$ .*

*Proof.* Let  $X$  be identified with a subspace of  $C[0, 1]$ , via the embedding of  $X$  in its completion and [M, ]. Let  $\mathcal{T}$  be the product topology that is defined on  $C[0, 1]$  with respect to some Schauder basis and let  $\mathcal{T} \upharpoonright X$  be the relative topology on  $X$ . Extend Lemma 1 to produce  $\mathcal{T}$ -open sets  $U'_n$  that are locally finite (in the norm) at any point of  $C[0, 1]$  at which the family of  $S_n$ 's is locally finite. Each  $U'_n$  is weakly open in  $C[0, 1]$ , so  $U_n = X \cap U'_n$  is weakly open in  $X$ .  $\square$

## 2. Other product topologies

Of course, the product topology used in the foregoing proof has very little to do with the product topology most naturally associated with  $C[0, 1]$ , which is not even metrizable: the topology of pointwise convergence. It is therefore perhaps surprising that there is a similar proof of a similar theorem with respect to this product topology as well. The proof uses much the same ideas of squeezing the points of the norm-open sets progressively more strongly as  $n$  increases.

**Theorem 3.** *Let  $C_p[0, 1]$  denote  $C[0, 1]$  with the topology of pointwise convergence. Let  $\{S_n : n \in \omega\}$  be a (countable) family of subsets of  $C_p[0, 1]$ . There is a choice of  $C_p$ -open sets  $U_n \supset S_n$  such that if  $\{S_n : n \in \omega\}$  is locally finite in the norm at  $g \in C[0, 1]$ , then  $\{U_n : n \in \omega\}$  is also locally finite at  $g$  in the norm.*

*Proof.* As in Lemma 1, let  $V_n = \{x \in \ell_2 : \exists s \in S_n (\|s - x\| < 1/2^n)\}$ . As before,  $V_n$  is norm-open and  $\{V_n : n \in \omega\}$  is norm-locally finite at every point at which  $\{S_n : n \in \omega\}$  is locally finite in the norm.

Let  $\mathbb{Q} \cap [0, 1] = \{q_n : n \in \omega\}$ . Each  $f \in C[0, 1]$  is uniformly continuous, so that if  $f \in V_n$ , there exists  $m = m(f) \geq n$  such that  $|f(r) - f(q_i)| < 1/2^n$  for all

$r$  between  $q_i$  and the adjacent member(s) of  $\{q_0, \dots, q_m\}$ . Choose  $m$  so that also  $g \in V_n$  whenever  $\|f - g\| < 1/2^m$ . Let

$$U_n = \{h \in C[0, 1] : \exists f \in V_n \text{ such that } |f(q_i) - h(q_i)| < \frac{1}{2^{m(f)}} \text{ for all } i \leq m(f)\}.$$

Now let  $g$  have points from infinitely many  $U_n$  in every norm neighborhood. For each  $j \in \omega$  let  $k(j) \geq j$  be so large that  $|g(r) - g(q_i)| < 1/2^j$  for all  $r$  between  $q_i$  and the adjacent member(s) of  $\{q_0, \dots, q_{k(j)}\}$ . Choose  $n(j) \geq k(j)$  so that there is  $h_j \in U_{n(j)}$  satisfying  $g(q_i) - h_j(q_i) < 1/2^j$  for all  $i \leq n(j)$ . Let  $f_j \in V_{n(j)}$  satisfy

$$|f_j(q_i) - h_j(q_i)| < \frac{1}{2^{m(f_j)}} \text{ for all } i \leq n(j).$$

Since  $j \leq m(f_j)$  we have  $|g(q_i) - f_j(q_i)| < 1/(2^{j-1})$  for all  $i \leq n(j)$ . Now if  $r \in [0, 1]$  let  $r \in [q_i, q_j]$ ; then

$$|g(r) - f_j(r)| \leq |g(r) - g(q_i)| + |g(q_i) - f_j(q_i)| + |f_j(q_i) - f_j(i)| \leq \frac{1}{2^{j-2}}.$$

Hence  $\|g - f_j\| \leq 1/(2^{j-2})$  and  $f_j \rightarrow g$  as  $j \rightarrow \infty$ .  $\square$

The proof of Theorem 2 established something a little stronger than its statement: it showed that there is a single metrizable linear topology on  $X$  that is coarser than the weak topology, from which the sets  $U_n$  can always be taken. While the product topology in Theorem 3 is not metrizable, we really only used the basic product neighborhoods associated with rational numbers. Indeed, there is a natural algebraic embedding of  $C[0, 1]$  into the metrizable space  $\mathbb{R}^{\mathbb{Q} \cap [0, 1]}$  due to the fact that every continuous real-valued function on  $[0, 1]$  is determined by its values on  $\mathbb{Q}$ . Hence our proof of Theorem 3 gives an alternative way of proving the strengthened version of Theorem 2.

In some spaces, there is a natural product topology to which Theorem 1 does not extend. For example,  $C[0, 1]$  has a natural product topology in which every nonempty open set is dense in the  $L_p$  norm [ $1 \leq p < \infty$ ]. Thus it is impossible to even have an infinite collection of product-open sets that is locally finite in the  $L_p$  norm. Of course,  $\langle L_p[0, 1], \|\cdot\|_p \rangle$  does have a Schauder basis and it is possible to define a product topology with respect to that.

### 3. Extensions to some non-separable spaces

There is a natural generalization of Theorem 1 to  $\ell_p(\Gamma)$  for every set  $\Gamma$  and every  $p$  such that  $1 \leq p < \infty$ .

**Theorem 4.** *Let  $\{V_a : a \in A\}$  be a family of subsets of  $\ell_p(\Gamma)$  that is locally finite with respect to the norm. There is an expansion to a family  $\{U_a : a \in A\}$  of product-open sets that is locally finite with respect to the norm.*

*Proof.* Since  $\ell_p(\Gamma)$  is metrizable, it is paracompact, and so every locally finite collection of subsets expands to a locally finite collection of open sets. So we may assume without loss of generality that each  $V_a$  is open. For each  $a \in A$  let  $P_a = \{p \in V_a : \text{spt}(p) \text{ is finite}\}$ . Let  $U_a$  be the set of all points which agree with some point of  $P_a$  on its support. Since each point  $x$  of  $\ell_p(\Gamma)$  can be approximated arbitrarily closely by  $x \upharpoonright F$  for some finite subset  $F$  of  $A$ ,  $V_a$  is a subset of  $U_a$ , and the proof that  $U_a$  is open is like the proof that  $U_n$  is open in Lemma 1, with  $\text{spt}(p)$  replacing  $k_n(p)$ .

To see that  $\{U_a : a \in A\}$  is locally finite, suppose on the contrary that every neighborhood of  $x$  meets  $U_a$  for infinitely many  $a$ . Pick distinct  $a_n \in A$  and  $z_n \in U_{a_n}$  such that  $z_n \rightarrow x$ . Pick  $z_n^* \in F_{a_n}$  agreeing with  $z_n$  on  $\text{spt}(z_n^*)$ . Let  $S_n = \text{spt}(z_n^*)$  and let  $\{\gamma_n : n \in \omega\} = \text{spt}(x) \cup \bigcup_{n \in \omega} S_n$ . Our goal is to define a Cauchy subsequence of  $\langle z_n^* : n \in \omega \rangle$ , contradicting local finiteness of  $\{V_a : a \in A\}$ .

For each positive integer  $m$ , define  $A_m \subset \{\gamma_i : i < m\}$  by induction so that  $S_n \cap \{\gamma_i : i < m\} = A_m$  for infinitely many  $n$ , and so that  $A_n \cap \{\gamma_i : i < m\} = A_m$  whenever  $n \geq m$ . Fix  $\delta > 0$ . Pick  $m$  so that  $\|x - x \upharpoonright \{\gamma_i : i < m\}\| < \delta$  and  $\|x - z_n\| < \delta$  for all  $n \geq m$ .

*Claim.* If  $M, N \geq m$  and  $S_M \cap \{\gamma_i : i < m\} = S_N \cap \{\gamma_i : i < m\} = A_m$ , then  $\|z_M^* - z_N^*\| < 6\delta$ .

Assuming the claim, define  $k_m \geq m$  for all  $m \geq 1$  so that  $S_{k_m} \cap \{\gamma_i : i < m\} = A_m$ . Then  $\langle z_{k_m}^* : m \geq 1 \rangle$  is the desired Cauchy subsequence.

*Proof of Claim.* We have

$$(*) \quad \|x \upharpoonright A_m - z_N \upharpoonright A_m\| < \delta \quad \text{and}$$

$$(**) \quad \|x \upharpoonright (S_N \setminus A_m) - z_N \upharpoonright (S_N \setminus A_m)\| < \delta$$

because  $N \geq m$  implies  $\|x - z_N\| < \delta$ .

Moreover,  $\|x \upharpoonright \{\gamma_i : i \geq m\}\| < \delta$  and so

$$\|x \upharpoonright (S_N \setminus A_m)\| = \|x \upharpoonright (S_N \cap \{\gamma_i : i \geq m\})\| < \delta$$

also. It follows from **(\*\*)** that  $\|z_N \upharpoonright (S_N \setminus A_m)\| < 2\delta$ .

We have the same facts with  $M$  in place of  $N$ . So we have

$$\|z_N \upharpoonright A_m - z_M \upharpoonright A_m\| < 2\delta$$

and

$$\|z_N \upharpoonright (S_N \setminus A_m) - z_M \upharpoonright (S_M \setminus A_m)\| < 4\delta.$$

But  $z_N^* = (z_N \upharpoonright A_m) + (z_N \upharpoonright [S_N \setminus A_{k+1}])$ , and similarly for  $z_M^*$ . So these last two displayed formulas give  $\|z_M^* - z_N^*\| < 6\delta$ , as desired.  $\square$

The foregoing proof also works verbatim for  $c_0(\Gamma)$  with the supremum norm. There is a natural generalization of an unconditional basis to arbitrary index sets  $\Gamma$  which gives a further generalization of Theorem 4. The generalization is to a linearly independent set  $\{e_\gamma : \gamma \in \Gamma\}$  such that for each vector  $x$  there is a family of vectors  $\{x_\gamma = r_\gamma e_\gamma : \gamma \in \Gamma\}$  is *summable to  $x$* .

This is easily seen to be equivalent to having  $r_\gamma = 0$  for all but countably many  $\gamma$  and having  $\sum_n x_{\gamma_n} = x$  no matter how one lists the nonzero coordinates as  $\gamma_0, \gamma_1, \dots$ . Calling such a  $\Gamma$  an unconditional basis even if it is uncountable, and defining the product topology with respect to it in the natural way, we arrive at:

**Theorem 5.** *Let  $X$  be a Banach space with an unconditional basis. Every locally finite collection of subsets of  $X$  can be expanded to a locally finite collection of product-open sets.*

**Problem 2.** *Can Theorem 5 be extended to all Banach spaces in which the norm locally depends on finitely many coordinates?*

#### 4. The clopen subsets of Erdős space.

A corollary of Erdős's results mentioned in the introduction is that the topology  $\tau$  on  $\mathbb{E}$  whose base is the set of all clopen subsets of  $\langle \mathbb{E}, \|\cdot\| \rangle$  is a strictly coarser Tychonoff topology. We will now prove that the two topologies have the same countable closed subsets, and derive some interesting consequences.

**Lemma 2.** *Let  $D$  be a countable norm-closed subset of  $\mathbb{E}$ , not containing  $\vec{0}$ . There is a product-open subset of  $\ell^2$  containing  $D$  whose trace on  $\mathbb{E}$  is a norm-clopen set missing  $\vec{0}$ .*

*Proof.* Let  $D = \{d_n : n \in \omega\}$ . In Lemma 1, let  $S_n = \{d_n\}$  and follow the proof, choosing  $V_n$  so that its closure misses  $\vec{0}$ . For each  $n$  let  $W_n \subset U_n$  be a basic product-open set containing  $d_n$  whose trace on  $\mathbb{E}$  is clopen. This can simply be arranged by having the coordinates in which  $W_n$  is restricted use intervals with

irrational endpoints for doing the restricting. Now by Lemma 1, the only points of  $\ell_2$  which have each neighborhood meeting infinitely many  $W_n$  are in the norm-closure of  $D$ , hence are either in  $D$  or else have at least one irrational coordinate. So  $\bigcup\{W_n : n \in \omega\}$  traces a clopen set on  $\mathbb{E}$  containing  $D$  and missing  $\vec{0}$ .  $\square$

**Theorem 6.** *A countable subset of  $\mathbb{E}$  is  $\tau$ -closed iff it is norm-closed.*

*Proof.* By translation-invariance, Lemma 2 implies every norm-closed countable subset of  $\mathbb{E}$  is an intersection of norm-clopen sets, and is therefore  $\tau$ -closed.  $\square$

If one follows the proof of Lemma 1, one may wind up shrinking the sets  $W_n$  more than necessary. For example, if  $D$  is closed discrete in  $\ell_2$ , then one can let each  $V_n$  be of diameter  $r$  times the distance from  $d_n$  to its nearest neighbor(s) where  $r$  is any positive number  $< 1/2$ . Then any basic open set of the following form can be chosen for  $W_n$ . Let  $A$  be a finite subset of  $\omega$  such that there are irrational numbers  $p_n(i)$  and  $q_n(i)$  satisfying  $p_n(i) < d_n(i) < q_n(i)$  for each  $i \in A$ , and such that the following set is a subset of  $V_n$ :

$$Y(p_n, q_n) = \{y \in L : \text{spt}(y) \subset A \text{ and } p_n(i) < y(i) < q_n(i) \text{ for all } i \in A.\}$$

Let  $W_n = \{z \in \ell_2 : z \upharpoonright A \in Y(p_n, q_n)\}$ .

Alan Dow observed the following corollary of Theorem 4.

**Theorem 5.** *Every  $\tau$ -convergent sequence in  $\mathbb{E}$  is norm-convergent. Thus  $\langle \mathbb{E}, \tau \rangle$  is not a sequential space.*

*Proof.* Suppose there were a  $\tau$ -convergent sequence  $\sigma$  that is not norm-convergent. Since  $\tau$  is coarser than the norm topology, this would imply that the range of some one-to-one subsequence of  $\sigma$  is closed discrete in the norm topology. But then by Theorem 4, this range is also  $\tau$ -closed-discrete, contradicting  $\tau$ -convergence to  $\vec{0}$ .  $\square$

Theorem 4 leads even more directly to the negation of a generalization of sequentiality, introduced by Moore and Mrówka in [MM].

**Definition 2.** A topology is *determined by countable closed sets* [resp. *countably tight*] if a set  $A$  is closed if (and only if)  $\text{cl}(B)$  is a subset of  $A$  whenever  $B \subset A$  and  $\text{cl}(B)$  is countable [resp. and  $B$  is countable].

In what came to be called “the Moore-Mrówka problem,” they asked whether every compact Hausdorff countably tight space is determined by countable closed sets. This has been shown to be independent of the usual (ZFC) axioms of set theory

[BDFN] [B]. They also remarked that the problem was open for arbitrary Hausdorff spaces. A ZFC counterexample was provided by I. Juhász and Weiss [JW] [N]. Just from the name, one might infer that  $\langle \mathbb{E}, \tau \rangle$  is another counterexample, inasmuch as it has the same countable closed sets as the norm topology does. This is indeed the case: one can let  $A$  be any set which is norm-closed but not  $\tau$ -closed, and any subset of  $A$  with countable  $\tau$ -closure has the same norm-closure which is thus a subset of  $A$ .

It is interesting to compare and contrast the counterexample in [JW] with this one. The one in [JW] is constructed by transfinite induction and is neither hereditarily separable nor hereditarily Lindelöf.  $\langle \mathbb{E}, \tau \rangle$  has both properties and is defined in an elementary way using only ZF; only the countable axiom of choice is needed to verify that  $\tau$  is not determined by countable closed sets. On the other hand, the Juhász-Weiss space is pseudo-radial [JW] while  $\langle \mathbb{E}, \tau \rangle$  is not.

**Definition 3.** A space is *pseudo-radial* if its topology is determined by well-ordered nets. In other words, if a set  $A$  is not closed, there is a point  $x$  outside  $A$  and a well-ordered net in  $A$  converging to  $x$ .

**Theorem 6.** *The space  $\langle \mathbb{E}, \tau \rangle$  is hereditarily separable (hence countably tight) and hereditarily Lindelöf, but not pseudo-radial.*

*Proof.* Erdős space is separable metrizable and hence both hereditarily separable and hereditarily Lindelöf. Since  $\tau$  is a coarser topology, it has both of the latter properties.

To show that  $\langle \mathbb{E}, \tau \rangle$  is not pseudo-radial, we use the fact that every point is a  $G_\delta$ . So, if  $\xi$  is a net of uncountable cofinality that is not eventually constant and  $p$  is any point of  $\mathbb{E}$ , then  $p$  has a neighborhood missing a cofinal subnet of  $\xi$ . Thus the only convergent well-ordered subnets that are not eventually constant have cofinal convergent subsequences. But Theorem 5 shows that these are not enough to determine the topology. For instance, the complement of the open unit ball in  $\mathbb{E}$  is norm-closed and hence sequentially closed (but is also dense!) in  $\langle \mathbb{E}, \tau \rangle$ .  $\square$

The following generalization of pseudo-radiality has been studied in connection with a famous unsolved problem of general topology, the  $M_1$ - $M_3$  problem [MSK]:

**Definition 4.** A space  $X$  is said to be WAP if for every non-closed subset  $A$  there is a point  $x \in \text{cl}(A) \setminus A$  and a subset  $B$  of  $A$  such that  $x \in \text{cl}(B)$  and  $x$  is the only point of  $\text{cl}(B)$  that is not also in  $A$ .

**Problem 3.** *Is  $\langle \mathbb{E}, \tau \rangle$  a WAP space?*

But the most basic unsolved problem about this space is the following.

**Problem 4.** Is  $\langle \mathbb{E}, \tau \rangle$  a topological group (equivalently, a topological vector space over  $\mathbb{Q}$ )?

Because translation and scalar multiplication are separately continuous in  $\ell^2$ ,  $\langle \mathbb{E}, + \rangle$  is a semitopological group with continuous inverse in the topology  $\tau$ . So Problem 4 boils down to asking whether addition is jointly  $\tau$ -continuous. This is a problem even where the clopen sets of Theorem 4 are concerned:

**Problem 5.** Let  $\mathbb{E}$  be Erdős space and let  $C$  be a clopen nbhd of  $\vec{0}$  in  $\mathbb{E}$  defined by  $C = \mathbb{E} \setminus \bigcup \{W_n : n \in \omega\}$ , where  $W_n$  is as in the proof of Lemma 2. Is there a clopen nbhd of  $\vec{0}$  whose sum with itself is a subset of  $C$ ?

If  $D$  is finite there is a simple solution: follow the remark preceding Theorem 5 and let  $K$  be the set of all points of  $\mathbb{E}$  that are between  $\frac{1}{2}p_n$  and  $\frac{1}{2}q_n$  for some  $n$ , in every coordinate where  $\frac{1}{2}p_n$  (equivalently,  $\frac{1}{2}q_n$ ) is nonzero. But this idea is not feasible for infinite  $D$ , not even in the following example where the members of  $D$  are all a distance of 1 from their nearest neighbors.

**Example 1.** Let  $D = \{d_n : n \in \omega\}$  be the following closed discrete subspace of  $\ell^2$ :  $d_n$  is the point which is  $1/2^n$  in the first  $2^{2n}$  terms, and 0 in all other terms. The points of  $D$  are on the unit sphere of  $\ell_2$ , any pair of successive points is one unit apart, and other pairs are even further from each other. In defining  $W_n$  as in the remark following the proof of Theorem 4, we can let  $p_n$  be the point all of whose nonzero coordinates are  $1/(2^n\pi)$  below the nonzero coordinates of  $d_n$ . Given  $d_n$ , one might try replacing  $W_n$  with the set  $G_n$  of all points which are of absolute value greater than  $\frac{1}{2}p_n(i)$  in at least one coordinate  $i \in \text{spt}(p_n)$ . This is a set that is open in the product topology and whose trace on  $\mathbb{E}$  is clopen in  $\mathbb{E}$ . Also,  $H_n = \mathbb{E} \setminus G_n$  is symmetrical with respect to the origin, and  $H_n + H_n \subset \mathbb{E} \setminus G_n$ . It is easy to see that the set of  $G_n$ 's does not have clopen union. The following point  $x$  is in the closure of the  $G_n$  without being in any  $G_n$ : let  $x(k) = 1/2^{n+2}$  when  $k = 2^{2n}$  and  $x(j) = 0$  whenever  $j$  is not a number of this form. If we let  $z_n$  be the point of  $G_n$  which satisfies  $z_n(2^{2n}) = 1/2^n$  and agrees with  $x$  elsewhere, it is easy to see that the sequence of  $z_n$ 's converges to  $x$  in norm.

However, the following set  $K$  is clopen and does solve the problem in the affirmative where this particular choice of  $W_n$  is concerned.

**Example 2.** Let  $f : \omega \rightarrow \mathbb{R}$  be defined as follows.  $f(0) = 1/4\pi$ ,  $f(i) = 1/8\pi$  for  $i \in \{1, 2, 3\}$ , and in general  $f(n) = 1/(2^{n+2}\pi)$  for  $i \in [2^{n-1}, 2^n - 1] \cap \omega$ . Let  $K$  be the set of all sequences in  $\mathbb{E}$  such that the absolute value of the  $k$ th coordinate is greater than  $f(k)$  in less than half of the coordinates  $k$  in the interval  $[2^{n-1}, 2^n - 1]$ , no matter what  $n$  is.

Now if  $x$  and  $y$  are in  $K$ , then for each  $n$  there exists  $k \in [2^{n-1}, 2^n - 1]$  such that  $x(k)$  and  $y(k)$  are both less than  $f(k)$ , which in turn is less than half of  $1/2^n - 1/(2^n\pi)$ . Thus if  $W_n$  is as in Example 1, then  $x + y \notin W_n$  for any  $n$ . Therefore  $K + K$  is contained in the set  $C = \mathbb{E} \setminus \bigcup\{W_n : n \in \omega\}$ . Moreover,  $K$  is clopen. The proof of this is very similar to that of the following lemma, which extends the notation  $x \upharpoonright A$  to any function from  $\omega$  to  $\mathbb{R}$  in the obvious way for all finite  $A$ :  $f \upharpoonright A$  is the element of  $\ell^2$  which agrees with  $f \upharpoonright A$  on  $A$  and is zero elsewhere.

**Lemma 3.** *Let  $f$  be any positive real sequence which is not in  $\ell^2$ ; in other words, the sequence of  $\ell^2$  norms  $\|f \upharpoonright [0, n]\|$  increases without bound. For  $x \in \ell_2$  and any interval  $I = [m, n] \subset \omega$ , let*

$$(1) \quad A_I(x) = \{k : k \in I \text{ and } |x(k)| \geq f(k)\}.$$

For positive real numbers  $r \leq 1$  and  $\epsilon \leq 1$ , let  $G(f, r, \epsilon)$  be the set of all sequences  $\sigma$  in  $\ell_2$  such that, for each  $I$  satisfying  $\|f \upharpoonright I\| \geq r$ , we have  $\|f \upharpoonright A_I(\sigma)\| < \epsilon\|f \upharpoonright I\|$ . If  $f(i)$  is irrational for all  $i \in \omega$ , then  $K = \mathbb{E} \cap G(f, r, \epsilon)$  is clopen in  $\mathbb{E}$ .

*Proof.* If  $p \notin K$ , then for some  $I$  satisfying  $\|f \upharpoonright I\| \geq r$ , we have  $\|f \upharpoonright A_I(p)\| \geq \epsilon\|f \upharpoonright I\|$ . Then clearly the set of all points  $x$  such that  $|p(i) - x(i)| < |p(i)| - f(i)$  for all  $i \in I$  is a product-open set in  $\ell^2$  that contains  $p$  and misses  $K$ . Hence  $K$  is closed even in the product topology of  $\mathbb{E}$ .

To show  $G = G(f, r, \epsilon)$  is norm-open, let  $y \in G$ . Let  $j \in \omega$  be so large that  $\|y \upharpoonright [j, \infty)\| < r\epsilon$  and  $\|f \upharpoonright [0, j-1]\| > r$ . Let  $k \geq j$  be such that  $\|f \upharpoonright [j, k]\| > r$ .

Let  $\delta = \min\{\epsilon, \nu\}$  where  $\epsilon = \min\{f(i) - |y(i)| : i \in [0, k] \setminus A_{[0, k]}(y)\}$  and  $\nu = r\epsilon - \|y \upharpoonright [j, \infty)\|$ . Then  $B(y, \delta) = \{x \in \mathbb{E} : \|y - x\| < \delta\}$  is an open subset of  $G$  containing  $y$ . Indeed, let  $I = [m, n]$  and suppose first that  $n \leq k$ . In this case,  $A_I(x) \subset A_I(y)$  and so  $\|f \upharpoonright A_I(x)\| < \epsilon\|f \upharpoonright I\|$  for all  $x \in B(y, \delta)$ . So now suppose  $n > k$ . Then by the first case,  $\|f \upharpoonright A_{[m, j-1]}(x)\| < \epsilon\|f \upharpoonright [m, j-1]\|$ , while

$$\begin{aligned} \|x \upharpoonright [j, \infty)\| &\leq \|y \upharpoonright [j, \infty)\| + \|x - y \upharpoonright [j, \infty)\| < r\epsilon, \text{ and} \\ \|f \upharpoonright A_{[j, n]}(x)\| &< r\epsilon < \epsilon\|f \upharpoonright [j, n]\|. \end{aligned}$$

Now  $\|f \upharpoonright A_{[m, n]}(x)\| = \sqrt{\|f \upharpoonright A_{[m, j-1]}(x)\|^2 + \|f \upharpoonright A_{[j, n]}(x)\|^2} < \epsilon\sqrt{\|f \upharpoonright [m, j-1]\|^2 + \|f \upharpoonright [j, n]\|^2} = \epsilon\|f \upharpoonright [m, n]\|$ .  $\square$

#### 4. Strange new topologies on Banach spaces

Besides providing us with an abundance of clopen subsets of  $\mathbb{E}$ , the proof of Lemma 3 also provides us with a variety of topologies on  $\ell_2$  making it a topological vector space over  $\mathbb{R}$ . The most straightforward way is to use all the sets  $G(f, r, \epsilon)$  as follows.

**Theorem 7.** *There is a topology  $\mathcal{T}$  on  $\ell_2$  whose base of neighborhoods of  $\vec{0}$  is all finite intersections of sets of the form  $G(f, r, \epsilon)$ , with  $f$ ,  $r$ , and  $\epsilon$  as in Lemma 3, and which makes it a topological vector space over  $\mathbb{R}$ . Moreover, each set  $G(f, r, \epsilon)$  is open in this topology.*

*Proof.* Each member of the base is clearly symmetrical. If  $G = G(f, r, \epsilon)$  and  $G' = G(f/2, r/2, \epsilon/2)$  then  $G' + G' \subset G$  and this relationship is preserved by finite intersections. So, if  $\mathcal{T}$  is the set of all translates of members of the base, we will know  $\langle \ell_2, \mathcal{T} \rangle$  is a topological group once we show that if  $y \in G(f, r, \epsilon)$  then there is a set of the form  $y + G(f', r', \epsilon')$  contained in  $G(f, r, \epsilon)$  [C, 1.11]. Also,  $\lambda G(f, r, \epsilon) = G(f/\lambda, r/\lambda, \epsilon)$ , because the intervals  $I$  on which  $\|f \upharpoonright I\| > r$  are the same as the ones on which  $\|f/\lambda \upharpoonright I\| > r/\lambda$ , and so we will also know that scalar multiplication is continuous.

So let  $y \in G(f, r, \epsilon)$  and  $j \in \omega$  be so large that  $\|y \upharpoonright [j, \infty)\| < \frac{r\epsilon}{3}$  and  $\|f \upharpoonright [0, j-1]\| > r$ . Let  $k \geq j$  be such that  $\|f \upharpoonright [j, k]\| > r$ .

Let  $n \geq 3$  be large enough so that  $|y(i)| + \frac{1}{n}f(i) < f(i)$  for all  $i \leq k$  satisfying  $y(i) < f(i)$ . Let  $r' \leq r/n$  and  $\epsilon' \leq \epsilon/2$  be small enough so that if  $x \in G(\frac{1}{n}f, r', \epsilon')$  then  $|x(i)| < \frac{1}{n}f(i)$  for all  $i \leq k$ . This can be done *e.g.* by letting  $\epsilon = \epsilon/2$  and making  $r' \leq \frac{1}{n}f(i)$  for all  $i \leq k$  in addition to making  $r' \leq r/n$ .

The proof that  $y + G(f', r', \epsilon') \subset G(f, r, \epsilon)$  is similar to the proof that  $B(y, \delta) \subset G$  in the proof of Lemma 3. Let  $I = [m, n]$  and suppose first that  $n \leq k$ . In this case,  $A_I(y+x) \subset A_I(y)$  for all  $x \in G(f', r', \epsilon')$ , and so  $\|f \upharpoonright A_I(y+x)\| < \epsilon\|f \upharpoonright I\|$ . So now suppose  $n > k$ . Then by the first case,  $\|f \upharpoonright A_{[m, j-1]}(y+x)\| < \epsilon\|f \upharpoonright [m, j-1]\|$ , while if  $i \geq j$  and  $|(y+x)(i)| > f(i)$  then either (a)  $i \in A_{[j, n]}(x)$  or (b)  $|x(i)| \leq f'(i)$  and  $|y(i)| > \frac{2}{3}f(i)$ . Now

$$\|f' \upharpoonright A_{[j, n]}(x)\| < \frac{\epsilon}{2}\|f' \upharpoonright [j, n]\| \text{ because } \frac{f}{n} \geq \frac{r}{n} \geq r'$$

on  $[j, n]$ ; hence also  $\|f \upharpoonright A_{[j, n]}(x)\| < \epsilon/2\|f \upharpoonright [j, n]\|$ . In case (b), let  $A = \{i \in [j, n] : |y(i)| > \frac{2}{3}f(i)\}$ . Then  $\|f \upharpoonright A\| < \frac{1}{2}r\epsilon$  and so

$$\|f \upharpoonright A_{[j, n]}(y+x)\| < \frac{r\epsilon}{2} + \frac{\epsilon}{2}\|f \upharpoonright [j, n]\| \leq \epsilon\|f \upharpoonright [j, n]\|.$$

The rest of the proof of openness is exactly as in Lemma 3.  $\square$

The relationship between  $G(f, r, \epsilon)$  and  $G(f', r', \epsilon')$  is usually much more complicated than when  $f$  and  $f'$  are scalar multiples of each other. In many cases, the

intersection of two sets of the form  $G(f, r, \epsilon)$  does not even contain a set of the same form. Just compare  $G(\bar{\delta}, \delta, 1/2)$  with some  $G(f', r', \epsilon')$  such that  $f'$  converges to 0 pointwise. The former clopen set is simply the set of points in  $\ell_2$  of sup norm  $< \delta$ , while the latter allows points free play on infinitely many coordinates while restricting them more severely on infinitely many other coordinates. Their intersection exhibits a sort of behavior that cannot be obtained by any set of the form  $G(f, r, \epsilon)$ .

Before going on to further examples illustrating this point, let us apply Theorem 7 to  $\mathbb{E}$  and give further information about  $\mathcal{T}$ .

**Corollary 3.** *Let  $\mathcal{T}_{\mathbb{E}}$  be the topology on  $\mathbb{E}$  whose base is all translates of finite intersections of sets of the form  $K(f, r, \epsilon) = \mathbb{E} \cap G(f, r, \epsilon)$  with  $f(i)$  irrational for all  $i \in \omega$ . Then  $\langle \mathbb{E}, \mathcal{T}_{\mathbb{E}} \rangle$  is a topological vector space over  $\mathbb{Q}$ , with a base of clopen sets.  $\square$*

It would be convenient if  $\tau$  were equal to  $\mathcal{T}_{\mathbb{E}}$ , because then we would have positive answers to both Problem 4 and Problem 5. However, there seems little reason to think that  $\tau = \mathcal{T}_{\mathbb{E}}$  might be true. However,  $\mathcal{T}_{\mathbb{E}}$  does have some properties in common with  $\tau$ , as the following theorem shows.

**Theorem 8.** *Every  $\mathcal{T}$ -convergent sequence on  $\ell_2$  is norm-convergent.*

*Proof.* Suppose  $\sigma$  is a sequence of points of  $\ell_2$  that  $\mathcal{T}$ -converges to  $\vec{0}$ . Then  $\sigma$  is pointwise convergent, so that for each  $n \in \omega$  and each  $\epsilon > 0$ , all but finitely many  $\|\sigma(i) \upharpoonright n\|$  are  $< \epsilon$ . If  $\sigma$  were not norm-convergent, there would exist  $\delta > 0$  such that  $\|\sigma(i)\| \geq \delta$  for all  $i$ .

The fact that every  $\mathcal{T}$ -convergent sequence is uniformly convergent played a crucial role in the foregoing proof. If  $\mathcal{T}_0$  is defined as the topology which uses only those  $G(f, r, \epsilon)$  where  $f$  converges to 0, then the sequence  $\langle e_{n^2} : n \in \omega \rangle$  converges to  $\vec{0}$ , where  $e_n$  stands for the unit vector of  $\ell_2$  which is the characteristic function of  $\{n\}$ .

We can also generate uncountably many incomparable topologies on  $\ell_2$ , all of them metrizable, by using scalar multiples of the same sequence  $f$ . The most straightforward way of doing this is to let  $\tau_f$  be the topology which has as a base at  $\vec{0}$  the set of all

$$\mathcal{B} = \{G(f/n, r, \epsilon) : n \in \omega \setminus \{0\} \text{ and } \{r, \epsilon\} \subset (0, 1]\}.$$

[Since we are only using one function  $f$ , there is no need to take finite intersections to get a base at  $\vec{0}$ .] This can be cut down to a countable base in various ways. One

is to take  $\mathcal{B}_0 = \{G(f/n, 1/n, \epsilon/n) : n \in \omega \setminus \{0\}\}$ . Because the second parameter is keeping exactly in step with the first, the set of relevant intervals is always the same, simplifying the proof that every member of  $\mathcal{B}$  contains a member of  $\mathcal{B}_0$ .