A COUNTABLE PRODUCT THEOREM FOR ANTI-PONDEROUS SPACES

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Abstract. An anti-ponderous space is a space in which every infinite countably compact subspace has a one-to-one convergent sequence. It is shown that a countable family of anti-ponderous spaces has an anti-ponderous product, with only a modest separation axiom assumed. This still leaves a wide range of uncertainty if the Continuum Hypothesis is not assumed. Questions about this and about sequentially compact spaces are discussed.

This paper continues the theme begun in [1] and [2] of the effects of cardinal functions on the convergent sequences in countably compact spaces without necessarily assuming that the spaces in question are Hausdorff or better. Often, as in the case of the main new result of this paper, weaker axioms are adequate for various theorems, and stronger axioms do not seem to lead to stronger results. Our new theorem is a natural variation on the classical result that the product of countably many sequentially compact spaces is sequentially compact. It involves a natural weakening of sequential compactness, given in Definition 4.

Definition 1. A space is *countably compact* if every countable open cover has a finite subcover. Equivalently, every infinite sequence has a cluster point.

Definition 2. A space is *sequentially compact* if every infinite sequence has a convergent subsequence.

The following property was introduced in [2], and some cardinal invariants associated with it were discussed there.

Definition 3. A **ponderous** space is an infinite, countably compact space in which every convergent sequence is eventually constant.

Definition 4. An **anti-ponderous** space is one that has no ponderous subspaces.

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Definition 5. A space has *Property S* if every convergent sequence has a unique cluster point.

Clearly, every ponderous space is T_1, as is every space with Property S. As is well known, every KC space (that is, a space in which every compact subset is closed) has Property S, and every Hausdorff space is KC. From the T_1 property of ponderous spaces it easily follows that a countably compact space is anti-ponderous if, and only if, every infinite, countably compact subspace contains a one-to-one convergent sequence. This is because, in a T_1 space, a convergent sequence cannot repeat more than one point infinitely many times.

Well known examples of ponderous spaces include $\beta\mathbb{N}$ and its countably compact subspaces, including the Novak-Teresaka example of a countably compact Tychonoff space whose square is not countably compact (nor even pseudocompact). The famous Efimov Conjecture, still not completely disproven, is equivalent to the conjecture that every ponderous compact Hausdorff space contains a copy of $\beta\mathbb{N}$.

The following is a strengthening, in some models of set theory, of the classical theorem mentioned above. The proof of that theorem goes through with only minor changes.

Theorem 0. [3, Theorem 6.9(a)] *The product of fewer than $\mathfrak{t}$-many sequentially compact spaces is sequentially compact.*

The cardinal $\mathfrak{t}$ is, as in [3] and [4], the least cardinality of a complete tower on ω. Closely related is $\mathfrak{h}$, the least height of a splitting tree on ω [5]. In [2] it was shown that $\mathfrak{h}$ is the least cardinality (also the least net weight) of a countably compact space that is not sequentially compact, and a KC example was given of a space that witnesses this.

On the other hand, new ideas are needed for any strengthening of the following result along these lines:

Theorem 1. *Let $\{X_n : n \in \omega\}$ be a countable family of spaces with Property S. If none of the X_n contains a ponderous subspace, then neither does their product.*

This is as far as we are able to go at present:

Problem 1. Is there a model of set theory in which the product of $\aleph_1$ anti-ponderous spaces is anti-ponderous?

This is in marked contrast to Theorem 0 and the definitive result for KC spaces involving $\mathfrak{h}$. For example, Martin's Axiom implies $\mathfrak{t} = \mathfrak{h} = \mathfrak{c}$, and Martin's axiom is compatible with $\mathfrak{c}$ being "arbitrarily large". We are somewhat better off in the opposite direction.

Problem 2. Is it consistent that there is a product of $< \mathfrak{s}$ anti-ponderous spaces that contains a ponderous subspace?

Here $\mathfrak{s}$ is the splitting number, which satisfies $\mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}$ [3], [5]. Ponderous compact subsets of $2^{\mathfrak{s}}$ have been constructed in models of $\aleph_1 = \mathfrak{s} < \mathfrak{c}$, [6], [7]. The latter was shown to exist in any model obtained by adding random reals to a model of CH in the usual way, hence $\mathfrak{c}$ could be arbitrarily large. This also provides an upper bound for strengthenings of Theorem 0, but the window of uncertainty is wide open between the two:

Problem 3. Is it consistent that there is a family of $< \mathfrak{s}$ sequentially compact spaces whose product is not sequentially compact?

Problem 4. Is it consistent that any product of $\mathfrak{t}$ sequentially compact spaces is sequentially compact?

These last two problems were already implicit in [3, Question 6.10] and there does not seem to have been any progress on them since then. All four problems ask only for consistency, because otherwise the CH would imply negative answers for all of them.

The proof of Theorem 1 involves the following lemmas, the first two of which are elementary and well known.

Lemma 1. *Every countable, countably compact space is compact and sequentially compact.* □

Lemma 2. *A continuous image of a [countably] compact space is [countably] compact.* □

Lemma 3. *In any space X with Property S, the range of a convergent sequence with infinite range, together with its (unique) limit point, is a closed copy of $\omega + 1$.*

Proof. Since X is T_1, the sequence σ minus its limit point p is closed discrete in its relative topology. Therefore, any one-to-one map from ω to $ran(\sigma) \setminus \{p\}$ is a homeomorphism, and its obvious extension to $\omega + 1$ is also a homeomorphism since every neighborhood of p contains all but finitely many points in the range of σ, but is not in the closure of any finite subset of $ran(\sigma) \setminus \{p\}$. □

By the way, Property S should not be confused with the weaker property [to which Lemma 3 does not extend] that no sequence can converge to more than one point.

Proof of Theorem 1. Let $X = \Pi_{n=1}^{\infty} X_n$ and suppose no X_n has a ponderous subspace. Let π_n denote the projection of X to X_n and let ρ_n denote its projection to $\Pi_{i=1}^{n} X_n$. Let Y be an infinite, countably compact subspace of X.

We will show that Y has a one-to-one convergent sequence. Our strategy will be to find countable, compact Hausdorff subspaces $Z_n \subset X_n$ such that $\Pi_{n=1}^{\infty} Z_n =: Z$ contains an infinite subset of Y. This subset Y_ω in turn will be the intersection of a descending chain of infinite closed subsets Y_n of Y such that $\rho_n Y_{n+1} \subset Z_1 \times \dots Z_n$. Thus, Y_ω is a subset of the compact metrizable space $Z = \Pi_n Z_n$. Let σ be a one-to-one sequence whose range is in Y_ω, and let τ be a subsequence that converges in Z. Since Y is countably compact, the unique limit of this sequence is in Y, and so we will be done.

To ensure that $Y_\omega = \bigcap_{n=1}^{\infty} Y_n$ is infinite, we will choose points $\{p_n : n \in \mathbb{N}\}$ by induction in Y, making sure p_n is in Y_m for all m. We will also define $Z_n \subset X_n$, $Y_n \subset Y_{n-1}$ by induction. Let $Y_1 = Y$. If $\pi_1 Y$ is finite, let $Z_1 = \pi_1 Y$. Otherwise, let Z_1 be a copy of ω_1 in $\pi_1 Y$. In either case, let $z_1 \in Z_1$ and choose $p_1 \in Y_1$ such that $\pi_1(p_1) = z_1$.

Let $Y_2 = \rho_1^{\leftarrow} Z_1 = \pi_1^{\leftarrow} Z_1$. Clearly, Y_2 is an infinite closed subspace of Y containing p_1. Let $Z_2 = \pi_2 Y_2$ if $\pi_2^{\rightarrow} Y_2$ is finite. Otherwise, let Z_2 be a copy of $\omega + 1$ in $\pi_2^{\rightarrow} Y_2$ that includes $\pi_2(p_1)$. [In spaces satisfying Property S, adding finitely many points to a copy of ω_1 still produces a copy of ω_1.] In either case, let p_2 be a point in Y_2 other than p_1.

In general, suppose that we have defined infinite closed, hence countably compact subspaces $Y_i \subset Y$ and $Z_i \subset \pi_i Y_i$ such that $Y_j \supset Y_i$ for $j \leq i \leq n$ and such that Z_n is either infinite or all of $\pi_n^{\rightarrow} Y_n$. Also suppose that $p_i \in Y_n$ for $i \leq n$. Let $Y_{n+1} = \pi_n^{\leftarrow} Z_n$. Then either $Y_{n+1} = Y_n$ or Y_{n+1} is the preimage of a copy of $\omega + 1$.

Let $Z_{n+1} = \pi_{n+1}^{\rightarrow} Y_{n+1}$ if this image is finite; otherwise, let Z_{n+1} be a copy of $\omega + 1$ in $\pi_{n+1}^{\rightarrow} Y_{n+1}$ that includes $\pi_2(p_i)$ for $i = 1, \dots, n$. In either case, let $p_{n+1} \in Y_{n+1}$, $p_{n+1} \neq p_i$ for $i = 1, \dots, n$. It is routine to show that the induction hypotheses are satisfied. When the induction is complete, we will have ensured that Y_ω is infinite, since it contains all the p_i. □

In the trivial case where all but finitely many of the Z_n are one-point spaces, their product is a countable, compact space, and the chain will stabilize at a Y_n where the $|Z_m| = 1$ for all $m > n$ and we will be done by Theorem 0: Y_n is infinite and sequentially compact and so Y is not ponderous.

Problem 1 asked whether we can improve on "countable" in Theorem 1. The proof does not seem to lend itself to continuation beyond ω; even going to $\omega + 1$ seems to involve sacrificing an essential part of Y_ω and practically starting over. The following problem is also open:

Problem 5. *Can we weaken the topological conditions on the spaces X_n in Theorem 1?*

The condition on uniqueness of limit points of convergent sequences was used in showing that Y_ω is closed in Y and hence countably compact; that Y_ω is sequentially closed in Z; and in getting Z to be metrizable by making sure that a one-to-one sequence and its limit point form a copy of $\omega + 1$. The third use is not really essential since we could also show Z to be sequentially compact using Lemma 1 and Theorem 0.

On the other hand, the first two uses do seem to be essential. We need to avoid a situation where any bijective sequence in Y_ω that is convergent in Z also has so many cluster points outside Z that it has no convergent subsequences in Y. As long as Y_ω is countably compact this is not a problem, but in the absence of strong enough separation axioms this is not easy to achieve.

The T_1 property was used in getting the chain of Y_n's to have nonempty intersection, due to the fact that singletons of X_n are closed and hence so are their preimages in Y_n.

We close with two problems which may be easier to answer than the previous ones. The second is reminiscent both of Problem 2 and of Efimov's conjecture.

Problem 6. *Is there a countable family of anti-ponderous, countably compact spaces whose product is not countably compact?*

Problem 7. *Is it consistent that the product of fewer than $\mathfrak{c}$ spaces can contain a copy of $\beta\mathbb{N}$ without any of the factors containing one?*

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