FRAYED OCTANTS: TEST SPACES IN THE STRUCTURE THEORY OF LOCALLY COMPACT SPACES

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The examples in this paper were originally inspired by the following theorem, which will appear in a forthcoming paper[Ny2].

Theorem A. [PFA] *Let X be a locally compact, T_5, hereditarily cwH, ω_1-compact space of Lindelöf degree $\aleph_1$. There is a discrete collection $\mathcal{W}$ of copies of ω_1 in X such that every closed copy of ω_1 in X meets some member of $\mathcal{W}$ in a closed unbounded subset.*

This paper presents an example (Example 3.4) to show that "ω_1-compact" cannot be eliminated. This is a special case of a more general construction which we introduce here. It always gives a locally compact, Hausdorff, locally countable (hence first countable) space which is the union of an ascending chain of clopen countable subsets. Hence the spaces, which will be generically symbolized $\varDelta$ are of cardinality $\aleph_1$ and pseudonormal–that is, given two disjoint closed subsets F_1 and F_2, one of which is countable, there are disjoint open sets $U_1 \supset F_1$ and $U_2 \supset F_2$.

There are ω_1-compact versions of our construction under some set-theoretic hypotheses, and these will be automatically T_5 (see Corollary 2.7), but no version of it can be ω_1-compact under the axioms for Theorem A (Corollary 3.2). Every version does satisfy all the other hypotheses of Theorem A except, in some cases, "T_5" but none satisfies the conclusion (Lemma 2.8). And Example 3.4 is T_5 under MA(ω_1), hence under the PFA, hence under the axioms for Theorem A.

Consequently, under the special axioms that give us Theorem A, the structure theory of locally compact T_5 hereditarily cwH spaces is substantially more complicated than that of the locally connected spaces in this class: under the set-theoretic hypotheses of Theorem A, the components of these spaces are ω_1-compact [Ny4]. Since a locally connected space is the topological direct sum of its components, there are many things we can conclude about the locally connected case that are either false or unsolved problems about the general case. Among the latter is the question of whether ω_1-compactness can be eliminated from the following consequence of Theorem A, in the same paper [Ny2]:

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Theorem B. [PFA] *Every locally compact, T_5, hereditarily cwH, ω_1-compact space satisfying $L(X) = \aleph_1$ is countably paracompact and collectionwise normal.*

Each version of our construction is hereditarily collectionwise normal and hereditarily countably paracompact if it is normal at all, as will be seen in Section 2. The following theorem from another forthcoming paper [ENy] shows that it is consistent with CH for no version to be ω_1-compact; it would be interesting to know whether CH alone is enough to produce a normal example.

Theorem C. *[Axiom P] Let X be a locally compact Hausdorff space of cardinality $\aleph_1$; then at least one of the following is true:*

(1) *X is the union of countably many ω-bounded, hence countably compact subspaces.*

(2) *X contains an uncountable closed discrete subspace.*

(3) *X has a countable subset with non-Lindelöf closure.*

We will see that (1) fails for our spaces, and it is easy to see from the earlier remarks that (3) fails too; so, under the axiom P of [AT], which is compatible with CH, the spaces cannot be ω_1-compact.

For pseudonormal spaces, (1) can be strengthened to:

(1^+) X is the union of countably many closed ω-bounded, hence countably compact subspaces; in particular, X is a Σ-space.

Σ-spaces are a well-behaved class of "generalized metric" spaces, and while Δ never falls into this class, it is always an $\Sigma^\#$-space:

Definition. A space X is a Σ-*space* [resp. *a* $\Sigma^\#$-*space*] if it has a cover $\mathcal{C}$ by countably compact closed subsets and a σ-locally finite [*resp.* s-closure-preserving] collection $\mathcal{F}$ of closed subsets such that, for every open $U \subset X$ and every $C \subset U$, $C \in \mathcal{C}$, there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$.

In fact, Δ has a closure-preserving cover consisting of closed countably compact subsets.

1. The frayed octants

Despite the scope for variation in our generic construction, the spaces it produces have enough features in common for me to refer to them all by the term "frayed octant". The term is motivated by an informal picture of the horizontal lines in $\omega_1 \times \omega_1$ being the "warp" and the vertical lines being the "woof" in an infinite piece of fabric. In our generic construction, the woof loses its cohesiveness, yet its adhesion to the main diagonal is only somewhat weakened.

1.1. General Example. As the underlying set for our generic construction, we use the set $\varDelta$ of points on or below the diagonal of $\omega_1 \times \omega_1$:

$$\varDelta = \{\langle \xi, \eta \rangle : \langle \xi, \eta \rangle \in \omega_1 \times \omega_1,\ \eta \leq \xi\}.$$

The topology will depend on two parameters. One is the choice of a ladder system $\mathcal{L} = \{L_\gamma : \gamma \in \Lambda\}$, where each ladder L_γ is a set of ordinals of order type ω whose supremum is γ, and Λ stands for the set of all countable limit ordinals. The other parameter is the choice of the fundamental basic open neighborhood V_0^η when η is a successor ordinal. Continuing the suggestive notation above, we define, for all $\alpha \leq \beta < \omega_1$, the triangles

$$\varDelta_\alpha^\beta = \{\langle \xi, \eta \rangle : \langle \xi, \eta \rangle \in \omega_1 \times \omega_1,\ \alpha \leq \eta \leq \xi \leq \beta\}.$$

We employ another suggestive notation for intervals along the diagonal:

$$I_\alpha^\beta = \{\langle \eta, \eta \rangle : \langle \eta, \eta \rangle \in \omega_1 \times \omega_1,\ \alpha \leq \eta \leq \beta\}.$$

Basic open sets are of three kinds: (1) singletons $\{\langle \xi, \eta \rangle\}$ where neither ξ nor η is a limit ordinal; (2) horizontal intervals $(\alpha, \beta] \times \{\xi\}$ where $\alpha \leq \beta < \omega_1$; and (3) sets of the form V_α^γ, where $\alpha < \gamma < \omega_1$, defined by induction, with variations at successor ordinals that influence the later stages in the induction. The two main variations result in 'thick' and 'thin' $\varDelta$'s respectively.

For the thick [*resp.* thin] $\varDelta$, we let $V_i^n = \varDelta_{i+1}^n$ [*resp.* $V_i^n = I_{i+1}^n$] whenever $i < n < \omega$. If V_β^δ has been defined for all $\beta < \delta < \gamma$ and γ is a limit ordinal, let $\alpha_0 = 0$, let $L_\gamma = \{\alpha_{n+1} : n \in \omega\}$, let

$$V_0^\gamma = \{\langle \gamma, \gamma \rangle\} \cup \{\langle \xi, \eta \rangle \in \varDelta_1^\gamma : \eta \in L_\gamma\} \cup \bigcup \{V_{\alpha_n}^{\alpha_{n+1}} : n \in \omega\}$$

and let $V_\alpha^\gamma = V_0^\gamma \cap \varDelta_{\alpha+1}^\gamma$. If η is an infinite nonlimit ordinal, let γ be the greatest limit ordinal that is less than η and, for the thick $\varDelta$'s, let $V_\alpha^\eta = V_\alpha^\gamma \cup \varDelta_{\gamma+1}^\eta$ whenever $\alpha < \gamma$ while letting $V_\alpha^\eta = \varDelta_{\alpha+1}^\eta$ whenever $\gamma \leq \alpha < \eta$. The thin $\varDelta$'s are defined in the same way except that we use $I^\eta_{\gamma+1}$ in place of $\Delta^\eta_{\gamma+1}$ and $I^\eta_{\alpha+1}$ in place of $\Delta^\eta_{\alpha+1}$, respectively.

The other variations in the topology consist of using various sets that are subsets of Δ^η_β and contain I^η_β whenever V^η_α is defined for successor η; the only restriction is that if $\langle \xi, \zeta \rangle \in V^\beta_\alpha$ and $\zeta \le \eta \le \xi$ then $\langle \eta, \zeta \rangle \in V^\beta_\alpha$ as well.

This induction produces a base for a topology on Δ that is finer than the usual topology (Corollary 1.3 below). Each open horizontal ray $(\alpha, \omega_1) \times \{\alpha\}$ is clopen and has its usual topology, and the relative topology on the closed (but not open) diagonal is also the usual topology. If γ is a limit ordinal, and $\alpha_n \nearrow \gamma$, then $\{V^\gamma_{\alpha_n} : n \in \omega\}$ is a nested local base at $\langle \gamma, \gamma \rangle$. Thus Δ is first countable, and it is easy to show by induction that each member of the base we have described just now is compact. In fact, by the induction hypothesis, each $V^{\alpha_{n+1}}_{\alpha_n}$ is compact, and each horizontal interval of the form $(\eta, \alpha] \times \{\xi\}$ is clopen and has the interval topology, hence is compact. So $\langle \alpha, \alpha \rangle$ is the extra point in a one-point compactification.

Like $\gamma\mathbb{N}$ and Ψ, the symbol Δ denotes a large variety of spaces, depending on the way the ladder system is chosen. If it is chosen to witness the axiom ♣ (see Example 3.1 below), then Δ is ω_1-compact and hence (see Corollary 2.6) it is T_5. But it can also be rigged not to be ω_1-compact or even T_5: see Example 3.3 below. On the other hand, as we have remarked, another version (Example 3.4) is T_5 under MA(ω_1). The big unsolved problem is:

Problem 1. Is it consistent that Δ is never T_5?

We now embark on an analysis of the V^γ_α which will make it easy to show that each point of the intersection of two such sets has a basic neighborhood that is a subset of the intersection, and that the sets of the form V^ξ_α form a local base at $\langle \alpha, \alpha \rangle$.

Given a ladder system $\mathcal{L} = \{L_\gamma : \gamma \in \Lambda\}$ and $L_\gamma \in \mathcal{L}$, let $r_\gamma(0) = 0$ and for each $n > 0$, let $r_\gamma(n)$ be the 'rth rung of L_γ':

$r_\gamma(n) =$ the rth element of L_γ

Define L^n_γ by induction as follows:

$L^0_\gamma = \{\gamma\}$; if L^i_γ has been defined, and $\delta \in \Lambda$, $\delta < \gamma$, then $\delta \in L^{i+1}_\gamma$ if, and only if, $\delta + n \in L_\alpha$ for some $n \in \omega$ and $\alpha \in L^i_\gamma$, and $\beta < \delta$ for all $\beta \in L^i_\gamma$ such that $\beta < \alpha$.

An easy induction shows that the sets L^i_γ are disjoint for distinct i. If $\delta \leq \gamma$ and $\delta \in \Lambda$, define

$n_\gamma(\delta) = \ min\{k : \delta \in L^k_\gamma\}$.

It is easy to see that $n_\gamma(\delta)$ exists for all limit ordinals $\delta \leq \gamma$: for each $n \in \omega$, let γ_n be the least member of L^n_γ that is greater than or equal to δ; so $\gamma_0 = \gamma$ and if $\gamma_i > \alpha$, then γ_{i+1} exists and is strictly less than γ_i; so by the well-ordering principle, we arrive at $\gamma_m = \alpha$ after finitely many steps. So we have shown part (1) of the following lemma, while (2) follows by an easy induction on γ, using the sets $V^{\alpha_{n+1}}_{\alpha_n}$ in the definition of V^γ_0.

1.2. Lemma. *With L^n_γ and $n_\gamma(\delta)$ defined as above:*

(1) $\bigcup_{i=0}^\infty L^i_\gamma = [0, \gamma] \cap \Lambda$.
(2) *If $\delta \in \Lambda$ and $\delta < \gamma$, then $\langle \delta, \delta \rangle \in V^\gamma_\xi$ if and only if $\xi < \delta < \gamma$.*
(3) *For each $\langle \alpha, \alpha \rangle \in V^\gamma_\xi$, there exists η such that $V^\alpha_\eta \subset V^\gamma_\xi$.*

Proof of (3). If $\alpha = \beta + 1$ then $\eta = \beta$ is as desired since $V^\alpha_\beta = \{\langle \alpha, \alpha \rangle\}$. So let α be a limit ordinal. If $\alpha \in L^1_\gamma$ then let $\eta = \xi$ if $\alpha + n$ is the least member of L_γ for some $n \in \omega$; otherwise, let η be the greatest ordinal less than α in L_γ. For those limit ordinals $\alpha < \gamma$ not in L^1_γ, we define ordinals $\beta_n < \alpha$ by induction as follows. Let $\beta_0 = 0$. If β_n has been defined, let γ_n be as above with α in place of δ and let $\alpha = \gamma_m$. If $n < m$, let ξ_n be the greatest member of L_{γ_n} that is less than α, unless all members of L_{γ_n} are greater than or equal to α, in which case we let $\xi_n = 0$. Let $\beta_{n+1} = max\{\beta_n, \xi_n\}$. A simple induction shows that

$$V^\alpha_{\beta_m} \subset V^{\gamma_{m-1}}_{\beta_{m-1}} \subset \cdots \subset V^{\gamma_1}_{\beta_1} \subset V^\gamma_0.$$

and so $\eta = max\{\xi, \beta_m\}$ is as desired. □

The following is clear from (3):

1.3. Corollary. *The sets V^α_ξ ($\xi < \alpha < \omega_1$) form a base for a topology, and the sets $\{V^\alpha_\xi : \xi < \alpha\}$ form a local base at $\langle \alpha, \alpha \rangle$.* □

A key feature of the basic clopen sets we have chosen is that they go all the way to the left in each row they meet; that is, if $\langle \eta, \alpha \rangle \in V^\gamma_\beta$ then $\langle \xi, \alpha \rangle \in V^\gamma_\beta$ for all

$\xi < \eta$ satisfying $\xi \geq \alpha$. This will help facilitate a number of proofs later in this paper.

2. Some ZFC properties of $\mathit{\Delta}$.

One of the uses of the key feature mentioned just now is to show that $\mathit{\Delta}$ is hereditarily cwn whenever it is normal.

2.1. Definition. A space is *collectionwise normal* (abbreviated *cwn*) if every discrete collection of closed sets expands to a discrete collection of open sets. A space X is *2-fully normal* (or *strongly collectionwise normal*) if each open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that if V and V' are members of $\mathcal{V}$ with nonempty intersection, there exists $U \in \mathcal{U}$ such that $V \cup V' \subset U$. A space X is *ultraparacompact* if every open cover of X can be refined to a partition of X into clopen sets.

An elementary result is that every zero-dimensional Lindelöf space is ultraparacompact: the key is that any countable clopen cover of any space whatsoever can be refined to a partition. Somewhat less elementary is the result that every locally compact, zero-dimensional, paracompact space is ultraparacompact. This is a corollary of the result that every locally compact paracompact space can be partitioned into clopen Lindelöf subsets [E]. Another well known result, which we will not be needing, is that 2-fully normal spaces are precisely the Tychonoff spaces such that the set of all neighborhoods of the diagonal is a uniformity. As their alternative name implies, 2-fully normal spaces are collectionwise normal: the usual proof that fully normal spaces are collectionwise normal (*cf.*[E]) goes through without trouble.

2.2 Lemma. *If X is an open subspace of $\mathit{\Delta}$ which omits a club subset of the diagonal, then X is 2-fully normal.*

Proof. At the risk of confusion with $\mathit{\Delta}$ itself, we will use the usual notation Δ for the diagonal $\{\langle\alpha,\alpha\rangle\}$ of $\mathit{\Delta}$. Let C be a closed unbounded subset of Δ, let $H = \Delta \setminus C$ and let $\mathcal{U}$ be a collection of open subsets of $\mathit{\Delta}$ such that $(\bigcup\mathcal{U}) \cap \Delta = H$. We will show that $\bigcup\mathcal{U}$ can be covered by a family $\mathcal{V}$ as in Definiton 5.2., which will complete the proof. Clearly, H is the disjoint union of countable intervals of Δ of the form

$$I_\xi = [(\alpha_\xi, \alpha_{\xi+1}) \times \omega_1] \cap \Delta, \quad \xi < \omega_1$$

unless H is countable. If H is countable, then it is a subset of some countable triangle Δ_0^α, and $\mathcal{V}$ can easily be defined by using metrizability of this triangle together with the well-known fact that ω_1 is 2-fully normal, proven in a way similar to the use of the R_α and K_α below.

If H is uncountable, we use ultraparacompactness of each I_ξ to define a partition $\mathcal{J}_\xi$ of I_ξ into relatively open, compact intervals, each of which is a subset of some member of $\mathcal{U}$. Now $\mathcal{J}_\xi$ has a natural well-ordering in type ω, as follows. Let J_ξ^0 be the interval in $\mathcal{J}_\xi$ which projects farthest to the left in ω_1. With J_ξ^n defined, let J_ξ^{n+1} be the interval in $\mathcal{J}_\xi \setminus \{J_\xi^0, \ldots, J_\xi^n\}$ with the smallest ordinal in its projection to ω_1. If $\mathcal{J}_\xi$ is infinite and $\langle \beta_n, \beta_n \rangle \in J_\xi^n$ for each n then $\beta_\xi = sup\{\beta_n : n \in \omega\}$ cannot be in I_ξ, otherwise $\mathcal{J}_\xi$ could not be a relatively discrete collection of subsets of I_ξ. Hence $\mathcal{J}_\xi = \{J_\xi^n : n \in \omega\}$, and each point of J_n projects to a smaller ordinal than any point of J_m whenever $n < m$. Since each J_ξ^n $(\xi < \omega_1,\ n \in \omega)$ is compact, it has a least element $\langle \alpha_\xi^n, \alpha_\xi^n \rangle$ and a greatest element $\langle \beta_\xi^n, \beta_\xi^n \rangle$ in its natural order, and if $\alpha_\xi^n < \eta < \beta_\xi^n$ then $\langle \eta, \eta \rangle \in J_\xi^n$ also. In other words, in teh notation we introduced in 1.1, we have $J_\xi^n = I_{\alpha_\xi^n}^{\beta_\xi^n}$. Since I_ξ is open, α_ξ^0 is a successor ordinal, as are all the other α_ξ^n; in fact, $\alpha_\xi^{n+1} = \beta_\xi^n + 1$ for all n because I_ξ is an interval; so let β_ξ^{-1} be the immediate predecessor of α_ξ^0. We may assume without loss of generality that $\mathcal{J}_\xi$ was chosen in such a way that W_ξ^n is a subset of some member of $\mathcal{U}$, where W_ξ^n is the natural expansion of J_ξ^n to a basic clopen set. [Explicitly: $W_\xi^n = V_\mu^\nu$ where $\mu = \beta_\xi^{n-1}$ and $\nu = \beta_\xi^n$.] Then

$$\mathcal{W} = \{W_\xi^n : \xi < \omega_1,\ n \in \omega\}$$

is an expansion of $\mathcal{J} = \bigcup\{\mathcal{J}_\xi : \xi < \omega_1\}$ to a discrete collection of clopen subsets of Δ, each a subset of some member of $\mathcal{U}$. So $W = \bigcup \mathcal{W}$ is a clopen subset of Δ, and it is a routine matter to cover $\Delta \setminus W$ with basic open sets of the form $(\xi_\eta^\alpha, \eta] \times \{\alpha\}$, each missing W. The Pressing Down Lemma (PDL) then gives $\xi(\alpha) \in \omega_1$ such that $\xi(\alpha) = \xi_\eta^\alpha$ for uncountably many $\eta \in \omega_1$. Let

$$\mathcal{H}_\alpha = \{(\xi(\alpha), \eta] \times \{\alpha\} : \xi_\eta^\alpha = \xi(\alpha)\} \text{ and let } \mathcal{H} = \cup\{\mathcal{H}_\alpha : \alpha < \omega_1\}.$$

We can easily cover the horizontal interval $(\alpha, \xi(\alpha)) \times \{\alpha\}$ by a countable collection $\mathcal{K}_\alpha$ of disjoint compact open intervals, each of which is a subset of some memeber of $\mathcal{U}$ and each missing W. Let $\mathcal{K} = \bigcup\{\mathcal{K}_\alpha : \alpha < \omega_1\}$. The desired open refinement of

$\mathcal{U}$ is then $\mathcal{V} = \mathcal{H} \cup \mathcal{J} \cup \mathcal{K} \cup \mathcal{W}$. Indeed, two distinct members of $\mathcal{V}$ can only meet if they are members of the same $\mathcal{H}_\alpha$, and their union is the greater of the two, hence is a subset of some member of $\mathcal{U}$. □

2.3. Corollary. *$\mathit{\Delta}$ is hereditarily cwH.*

Proof. We use the elementary fact that a space is hereditarily cwH iff every discrete subspace expands to a disjoint collection of open sets. If D is a discrete subspace of $\mathit{\Delta}$ then there is a club subset C of the diagonal Δ missing D, and the complement V of C is as in the statement of Lemma 2.2. Let $W = V \setminus (\overline{D} \setminus D)$. Then, as in Lemma 2.2, W is an open subspace of $\mathit{\Delta}$ missing C and D is a closed discrete subspace of W. Now use the fact that every 2-fully normal space is cwH. □

2.4. Lemma. *A given version of $\mathit{\Delta}$ is T_5 if, and only if, every open set containing the diagonal contains a closed neighborhood of the diagonal.*

Proof. We will use the elementary fact that a space is hereditarily normal iff every open subspace is normal. Let V be an open subspace of $\mathit{\Delta}$ and let F and K be disjoint relatively closed subsets of V. If neither K nor F meets the diagonal in an uncountable set, say no point above $\langle \alpha, \alpha \rangle$ meets either K or F, then $V \setminus \{\langle \xi, \xi \rangle : \xi > \alpha\}$ is an open subspace of V containing K and F and we can use Lemma 2.2 to conclude that K and F can be put into disjoint open sets. Hence it suffices to prove the claim for the case where one of the sets, say K, meets the diagonal in a club. If F also meets the diagonal, it does so in a countable closed set which can be expanded to a countable clopen set missing K inside some countable clopen $\mathit{\Delta}_0^\alpha$. Hence we may assume that F does not meet the diagonal.

Now we use the hypothesis on the diagonal Δ, as follows. Let $G = \mathit{\Delta} \setminus F$, and suppose there is a closed nbhd N of Δ such that $N \subset G$. Now $K \setminus \Delta$ and F are easy to put into disjoint open subsets of $V \setminus \Delta$: just use the fact that the complement of Δ is the union of relatively open subsets of horizontal rays, each of which is a copy of an open subspace of the hereditarily normal space ω_1. So let H be an open subset of $\mathit{\Delta} \setminus \Delta$ containing $K \setminus \Delta$, such that the closure of H misses F. Then if we let G_1 be the interior of N, let G_2 be the complement of N, let $H_1 = G_1 \cup H$ and let $H_2 = G_2 \setminus \overline{H}$, then H_1 and H_2 are disjoint open sets containing K and F respectively. □

2.5. Corollary. *If $\mathit{\Delta}$ is normal, it is hereditarily collectionwise normal.*

Proof. We will use the elementary fact that a space is hereditarily collectionwise normal (abbreviated 'hereditarily cwn') iff every open subspace is cwn. If $\{C_\alpha : \alpha \in \kappa\}$ is a discrete family of closed subsets in some open subspace V of $\mathit{\Delta}$, then at most one of the C_α meet the diagonal Δ in a club set. If none do, then there is a club subset K of the diagonal that misses $\bigcup\{C_\alpha : \alpha \in \kappa\}$, and we argue as for Lemma 2.2 applied to the open subspace $V \setminus K$ to put the C_α into disjoint open sets. So suppose some C_α meets the diagonal Δ in a club. In that case, we put C_α and $\bigcup\{C_\beta : \beta \neq \alpha\}$ into disjoint open sets G and H, respectively, using the fact that $\mathit{\Delta}$ is T_5, and treat H like we did $V \setminus K$ in the earlier case. □

2.6. Lemma. *If $\mathit{\Delta}$ is not normal, then there is a closed discrete subspace of $\mathit{\Delta} \setminus \Delta$ that cannot be separated from Δ by disjoint open sets.*

Proof by contrapositive. Let F be a closed subspace of $\mathit{\Delta}$ that does not meet the diagonal. For each horizontal ray $R_\xi = (\xi, \omega_1) \times \{\xi\}$ that meets F, let $\langle \eta_\xi, \xi \rangle$ be the leftmost point of $R_\xi \cap A$. Let D be the set of all such points $\langle \eta_\xi, \xi \rangle$. Since $D \subset F$ and D meets each horizontal line in at most one point, D is easily seen to be closed and discrete. Suppose there is an open set U such that $\Delta \subset U$ and $\overline{U} \cap D = \emptyset$. For each $\langle \alpha, \alpha \rangle \in \Delta$, pick a basic nbhd $V^\alpha_{\beta(\alpha)} \subset U$. The closure of $\bigcup\{V^\alpha_{\beta(\alpha)} : \langle \alpha, \alpha \rangle \in \Delta\} = V$ misses each $\langle \eta_\xi, \xi \rangle \in D$, and every point of $\overline{V}$ has all points to the left of it in $\overline{V}$ as well, so no point of F is in $\overline{V}$. So Δ is normal by Lemma 2.4. □

2.7. Corollary. *If $\mathit{\Delta}$ is ω_1-compact, it is T_5.*

Proof. Thanks to Lemma 2.6, it suffices to show that $\mathit{\Delta}$ is always pseudonormal; that is, given two disjoint closed sets D and F, one of which (say D) is countable, there are disjoint open sets U and V such that $D \subset U$, $F \subset V$. Now each countable $D \subset \mathit{\Delta}$ is contained in some countable clopen $\mathit{\Delta}^\alpha_0$ which is first countable, hence metrizable by Urysohn's metrization theorem. Let U and W be disjoint open subsets of $\mathit{\Delta}^\alpha_0$ containing D and $F \cap \mathit{\Delta}^\alpha_0$ respectively, and then U and $V = W \cup (\mathit{\Delta} \setminus \mathit{\Delta}^\alpha_0)$ are as desired. □

The following theorem is also a corollary of Lemma 2.6. Recall that a space is called [*strongly*] collectionwise Hausdorff (abbreviated '[s]cwH') if every closed discrete subspace expands to a disjoint [*resp.* discrete] collection of open sets; "expands" refers to the following concept: if D is a (discrete) subspace of a space X, an *expansion* of D is a family of sets $\{U_d : d \in D\}$ such that $U_d \cap D = d$ for all $d \in D$.

2.8. Theorem. *$\varDelta$ is T_5 if, and only if, it is strongly cwH.*

Proof. If $\varDelta$ is T_5, then Corollary 2.5 shows it is cwH, and every normal cwH space is strongly cwH. Indeed, if D is closed discrete and $\{U_d : d \in D\}$ is a disjoint open expansion of D, let V and W be disjoint open sets containing D and the complement of $\bigcup\{U_d : d \in D\}$ respectively. Then $\{V \cap U_d : d \in D\}$ is a discrete open expansion of D. Conversely, if $\varDelta$ is strongly cwH, we apply Lemmas 2.4 and 2.6. Let D be a closed discrete subspace of $\varDelta$ disjoint from the diagonal Δ. Let $\{U_d : d \in D\}$ be a discrete open expansion of D. Then U_d contains an open interval of the form $[f(d), d] \times \pi_2(d)$ where $\pi_2(d)$, as usual, is the second coordinate of d: if d is an isolated point then $f(d) = d$ works, otherwise $d = \langle \gamma, \xi \rangle$ where γ is a limit ordinal, and there is an open interval $(\eta, \xi] \times \{\gamma\}$ contained in U_d, and letting $f(d) = \eta + 1$ works. Now $\bigcup\{[f(d), d] : d \in D\}$ is an open set containing Δ whose closure misses Δ. Now Lemmas 2.6 and 2.4 show $\varDelta$ is T_5. □

Due to the simplicity of the neighborhoods of the off-diagonal points, the foregoing equivalence provides us with a nice geometrical interpretation of when $\varDelta$ is normal (hence T_5, etc.). It has to do with being able to define a function f_D for each closed discrete subspace D missing the diagonal, like the function f of the proof we have just been through. By the proof of Lemma 2.6, it suffices to take care of those $D \subset (\varDelta \setminus \Delta)$ which meet each horizontal ray $[\xi + 1, \omega_1) \times \{\xi\}$ in at most one point. Given such a closed discrete subspace D, let $D^\#$ be the set of nonisolated points of D. Then if $\varDelta$ is normal, there is a function $g_D : D^\# \longrightarrow \varDelta$ such that $g_D(d)$ is an isolated point strictly to the left of d (in other words, if $d = \langle \gamma, \alpha \rangle$ then $\gamma \in \Lambda$ and $g_D(d) = \langle \xi + 1, \alpha \rangle$ where $\xi < \gamma$) and such that $\{g_D(d) : d \in D^\#\}$ is a closed discrete subspace of $\varDelta$. This much is clear from the proof of Lemma 2.6, and we can let $f_D : D \to \varDelta$ be the function extending g_D which takes each isolated point of D to itself. The existence of an f_D with closed discrete range, sending each nonisolated point d of D to a point strictly to the left of d on the same horizontal line, and each isolated point of D to itself, is necessary and sufficient for $\varDelta$ to be (hereditarily) normal. We will be making use of this geometric criterion in Section 4 to show that certain versions of Δ are T_5 under MA(ω_1). This, coupled with the following lemma, will establish that ω_1-compactness cannot be dropped from the hypotheses of Theorem A, even if "countable" is dropped along with the final sentence.

2.9. Lemma. *If $\mathcal{K}$ is a discrete family of copies of ω_1 in Δ, then $\Delta \setminus \bigcup \mathcal{K}$ contains a copy of ω_1.*

Proof. Let $\mathcal{K}$ be a family of disjoint copies of ω_1 in Δ. If $\Delta \setminus \bigcup \mathcal{K}$ does not contain a copy of ω_1, then $\mathcal{K}$ must include, for each $\alpha \leq \omega_1$, a member K_α which meets the horizontal ray $L_\alpha = (\alpha, \omega_1)$ in a club. Moreover, these members of $\mathcal{K}$ are uniquely determined, and by cutting down K_α if necessary, we may assume $K_\alpha \subset L_\alpha$. For each $\alpha < \omega_1$, let C_α be the image of K_α under the first coordinate projection $\pi : \Delta \to \omega_1$. Let C be the diagonal intersection of $\langle C_\alpha : \alpha < \omega_1 \rangle$. That is,

$$C = \{\gamma \in \omega_1 : \gamma \in C_\xi \text{ for all } \xi < \gamma\}.$$

Since the diagonal intersection of clubs is a club, C is a club, and from its definition it is easy to see that if γ is a limit point of C, then the point $\langle \gamma, \gamma \rangle$ of the diagonal Δ is a limit point of the sets K_ξ satisfying $\xi < \gamma$; in fact, $\langle g, \xi \rangle \in K_\xi$ for all ξ. Hence $\mathcal{K}$ is not a discrete collecton. $\square$

From Lemma 2.9 it follows that it is not possible to cover $\Delta \setminus \bigcup \mathcal{K}$ by a relatively locally finite collection of open sets, each of which is a subset of some basic open set. This leads directly to:

2.10. Corollary. *Δ is not monotone normal.*

Proof. In [BZ], it was shown that in a monotone normal space, every open cover $\mathcal{U}$ can be associated with a discrete family $\mathcal{K}$ of copies of stationary subsets of regular uncountable cardinals in such a way that what is left of the space when $\bigcup \mathcal{K}$ is subtracted can be covered by a family of disjoint open subsets, each of which is a subset of some member of $\mathcal{U}$. However, if Δ is covered by countable open sets, then every discrete family of copies of regular uncountable cardinals (each of which is necessarily ω_1) leaves a copy of ω_1 uncovered, and no collection of countable open sets covering ω_1 can be disjoint. $\square$

Problem 4. Can Δ ever be hereditarily 2-fully normal?

3. Four examples and the T_5 property

The examples in this section are families of versions of $\mathit{\Delta}$. In the first family, everything depends on the ladder system: the spaces can be thin, thick, or anywhere in between. The second family consists of thick spaces, the third of thin spaces. The fourth is a subfamily of the third. The first family depends on the special axiom $\clubsuit$, which is negated by MA(ω_1) and hence by the PFA. The second and third families use only ZFC for construction and verification of properties claimed. The fourth family is a special case of the third in which the axiom $\diamondsuit$ (which is equivalent to $\clubsuit + CH$) is used to get the space to be T_5; on the other hand, this is automatic under MA(ω_1) for all members of the third family, as will be shown in Section 4.

3.1 Example. [$\clubsuit$] If the ladder system used to define the topology on $\mathit{\Delta}$ witnesses $\clubsuit$, then the space is ω_1-compact and hence T_5. To see this, let D be a closed discrete subspace of $\mathit{\Delta}$; then D meets each horizontal line in a finite set. If D is uncountable, then by cutting D down if necessary, we may assume that $D = \{\langle \alpha_\xi, \beta_\xi \rangle : \xi < \omega_1\}$ where $max\{\alpha_\xi, \beta_\xi\} < min\{\alpha_\eta, \beta_\eta\}$ whenever $\xi < \eta$. Let S be the set of 2nd coordinates of the points of D. Let γ be such that $L_\gamma \subset S$. Then $\{\langle \alpha_\xi, \beta_\xi \rangle : \beta_\xi \in L_\gamma\} \subset \mathit{\Delta}_0^\gamma$, because $\beta_\xi < \gamma$ for all ξ. Moreover, since the β_ξ in L_γ are of order type ω with supremum γ, every V_ζ^γ contains all but finitely many of the points $\{\langle \alpha_\xi, \beta_\xi \rangle : \beta_\xi \in L_\gamma\}$. Since each V_ζ^γ is compact, these points converge to $\langle \gamma, \gamma \rangle$.

On the other hand, as noted in the introduction, it is compatible with CH for $\mathit{\Delta}$ never to be ω_1-compact. This is also the case with the set-theoretic hypotheses of Theorem A:

Corollary 3.2. [PFA + Axiom F] *There is an uncountable closed discrete subspace in $\mathit{\Delta}$.*

Proof. If $\mathit{\Delta}$ were ω_1-compact, it would satisfy all the hypotheses of Theorem A. In particular, it would be T_5 by Corollary 2.7. However, it is immediate from Lemma 2.9 that $\mathit{\Delta}$ can never satisfy the conclusion of Theorem A. $\square$

Actually, a close study of the proof of Lemma 5.1 in [Ny1] shows that the PFA is already enough to produce an uncountable closed discrete subspace of $\mathit{\Delta}$. The next example, done in ZFC, is not ω_1-compact and is not even normal. We will explicitly define an uncountable closed discrete subspace D which meets each horizontal line in at most one point, and which cannot be separated from the diagonal by disjoint open sets. As we saw in Lemma 2.6, any non-normal frayed octant contains such a D.

3.3. Example. This example, which is thick, uses the triangles $\mathit{\Delta}_\gamma^{\gamma+n}$ where $\gamma \in \Lambda$ to approach the points $\langle \gamma+\omega, \gamma+1 \rangle$ from the left. It will have the property that the union of all the line segments $[\gamma+1, \gamma+\omega) \times \{\gamma+1\}$ is ω_1-compact and yet the subspace $D = \{\langle \gamma+\omega, \gamma+1 \rangle : \gamma \in \Lambda\}$ is closed discrete. This makes it impossible to put D in an open set whose closure misses the diagonal: if U is an open set containing D, then each $\langle \gamma+\omega, \gamma+1 \rangle$ has a neighborhood $[\gamma+n, \gamma+\omega) \times \{\gamma+1\}$ in U; and the uncountably many $\langle \gamma+n, \gamma+1 \rangle$ have an accumulation point, which can only be in Δ.

The construction will inductively be made to satisfy four conditions (let γ be any limit ordinal):

(1) $L_\gamma \subset (\omega_1 \setminus \Lambda)$.
(2) For all $\delta \in \Lambda$, we have $\gamma+1 \notin L_\delta$.
(3) For all $\delta \in \Lambda$, there is at most one $\gamma+n$ in L_δ.
(4) If $\langle \gamma_n : n \in \omega \rangle$ is a strictly increasing sequence in Λ with supremum γ, and $k(\gamma_n, \gamma)$ is defined as $max\{k : \langle \gamma_n + k, \gamma_n + 1 \rangle \in V_0^\gamma\}$, then $k(\gamma_n, \gamma) \to \infty$.

Condition (4) insures that $Z_k = \Delta \cup \{\langle \gamma+k, \gamma+1 \rangle : \gamma \in \Lambda\}$ is countably compact for all k and hence that the union of all the line segments $[\gamma+1, \gamma+\omega) \times \{\gamma+1\}$ is ω_1-compact. Condition (2) insures that D is closed discrete: an easy induction shows that $\langle \gamma+\omega, \gamma+1 \rangle \notin V_0^\delta$ for all $\delta \in \Lambda$, and D meets each horizontal line in at most one point while the sets V_0^δ cover the diagonal. Condition (3) is done for bookkeeping purposes, while (1) is in anticipation of Example 3.4: the same identical construction we have below, but with a thin $\mathit{\Delta}$, will give us an example in which the sub-diagonal $s\Delta = \{\langle \xi+1, \xi \rangle : \xi \in \omega_1\}$ is a countable union of closed discrete subspaces. Example 3.5, using $\diamondsuit$, will be a T_5 version of Example 3.4.

For complicated limit ordinals, (4) is achieved by induction. First, we give an alternative characterization of $k(\lambda, \gamma)$ for any pair $\lambda < \gamma$ of limit ordinals. Let $\beta_0 = \gamma$ and let β_1 be the least member of L_γ that is above λ. If $\beta_1 \geq \lambda + \omega$ we continue, letting β_2 be the least member of L_{β_1} above λ; continue like this until

$\beta_j < \lambda + \omega$; in other words, $\beta_j = \lambda + i$ for some $i \geq 2$. Next, an easy induction shows that V_0^γ meets the horizontal line $[\beta_i, \omega_1) \times \{\beta_i\}$ in the (infinite) interval $[\beta_i, \beta_{i-1}] \times \{\beta_i\}$ whenever $0 < i \leq j$. But when we reach β_j, there are no members of any of the ladders associated with any of the L_γ^n between $\lambda + 1$ and $\beta_j = \lambda + i$; so in a thick Δ, the basic open set V_0^γ meets the i lines immediately above $[\lambda, \omega_1) \times \{\lambda\}$ in $\Delta_{\lambda+1}^{\lambda+i} \setminus \{\langle \lambda + i, \lambda + i \rangle\}$. It is then clear that $k(\lambda, \gamma) = i$. So we can characterize $k(\lambda, \gamma)$ as the first integer k for which $V_0^\gamma \cap (\omega_1 \times \{\lambda + k\}$ is infinite.

Let Λ' denote the derived set of Λ; that is, Λ' is the set of all countable ordinals that are limits of limit ordinals. If $\gamma \in (\Lambda \setminus \Lambda')$ — in other words, γ is of the form $\beta + \omega$ where $\beta \in \Lambda \cup \{0\}$, we simply let $L_\gamma = \{\beta + n + 2 : n \in \omega\}$. If γ is of the form $\beta + \omega^2$ for some $\beta \in \Lambda \cup \{0\}$, let

$$L_\gamma = \{\beta + \omega \cdot n + (n + 2) : n \in \omega\}.$$

All limit ordinals between β and γ are of the form $\beta + \omega \cdot n$; so the only horizontal lines between β and γ which meet V_0^γ in an infinite set are those which are either in L_γ or in $L_{\beta+n}$ and at or above the line $\omega_1 \times \{\beta + \omega \cdot n + n + 2\}$ The part of V_0^γ between the line $\omega_1 \times \{\beta + \omega \cdot n + n + 2\}$ and β consists of the triangle $\Delta_{\beta+\omega\cdot n+1}^{\beta+\omega\cdot n+n+2}$ except for its apex point $\langle \beta + \omega \cdot n + n + 2, \beta + \omega \cdot n + n + 2 \rangle$. Thus, in particular, it meets the crucial line $\omega_1 \times \{\beta + \omega \cdot n + 1\}$ in the interval $[\beta + \omega \cdot n + 1, \beta + \omega \cdot n + n] \times \{\beta + \omega \cdot n + 1\}$.

The actual induction will cover the derived set $\Lambda^{(2)}$ of Λ' (that is, $\Lambda^{(2)}$ is the set of all countable ordinals that are limits of ordinals in Λ'); we have already taken care of the ordinals in $\gamma \in (\Lambda \setminus \Lambda^{(2)})$. Let δ be in $\Lambda^{(2)}$ and suppose (4) holds for all $\gamma \in \Lambda'$ such that $\gamma \leq \delta$. Let $\langle \gamma_n : n \in \omega \rangle$ be a strictly increasing sequence of limit ordinals whose supremum is δ. If γ_n were always close enough to γ_{n+1} for $V_{\gamma_n}^{\gamma_{n+1}}$ to meet all horizontal lines of the form $\omega_1 \times \{\lambda + 1\}$ ($\gamma_n \leq \lambda \leq \gamma_n + 1, \lambda \in \Lambda \cup \{0\}$) in a set of size n or greater, then we could simply let $L_\delta = \{\gamma_n + n : n \in \omega$, and (4) would hold with δ in place of γ. Our goal is to insert, if necessary, finitely many ordinals between γ_n and γ_{n+1} for each n to get the elements of L_δ close enough together in the sense just mentioned.

For each $n \in \omega$, let $\beta_n^0 = \gamma_n$ and let β_n^1 be the least ordinal $\beta < \gamma_n$ such that, for all limit $\alpha \in [\beta, \gamma_n)$, we have $\langle \alpha + n, \alpha + 1 \rangle \in V_0^{\gamma_n}$. If β_n^1 is finite (equivalently, $\beta_n^1 = 0$) or if $\beta_n^1 < \gamma_{n-1}$ (in case $n > 0$) then we let $A_n = \{\gamma_n + n\}$. Otherwise, β_n^1 is a limit ordinal, and it is strictly less than $\beta_n^0 = \gamma_n$ by the induction hypothesis.

If $k > 0$ and β_n^k has been defined and is either finite in case $n = 0$, or $\beta_n^k \leq \gamma_{n-1}+j$ for some finite j in case $n > 0$, then we let

$$A_n = \{\beta_n^{k-1} + n, \ldots, \beta_n^0 + n\}.$$

Otherwise, let β_n^{k+1} be the least ordinal $\beta < \beta_n^k$ such that, for all limit $\alpha \in [\beta, \beta_n^k)$, $\langle \alpha + n, \alpha + 1\rangle \in V_0^{\beta_n^k}$. Because of the induction hypothesis on δ, the descending sequence

$$\beta_n^0 > \beta_n^1 > \cdots > \beta_n^i \cdots$$

eventually terminates for each $n \in \omega$, and hence the finite A_n becomes well defined. In particular, every ordinal in A_n will be strictly less than every ordinal in A_{n+1}. Thus if we let

$$L_\delta = \bigcup_{n=0}^{\infty} A_n,$$

then L_δ will be of order type ω with supremum δ. In the notation of 1.1, each α_n with $n > 0$ is some $\beta_j^i + j$; and if λ is a limit ordinal and $\alpha_n \leq \lambda < \alpha_{n+1}$, then

$$V_{\alpha_n}^{\alpha_{n+1}} \cap (\omega_1 \times \{\lambda + 1\}) \quad \supset \quad [\lambda + 1, \lambda + j] \times \lambda + 1;$$

moreover, $n \to \infty$ implies that $j \to \infty$ as well. This completes the induction.

3.4. Example. We follow the definition of the L_δ in the inductive construction of Example 3.3, except that where we chose β_n^1 and β_n^{k+1} we use the alternative characterization of $k(\lambda, \gamma)$ as the first integer k for which $V_0^\gamma \cap (\omega_1 \times \{\lambda + k\})$ is infinite. The key to the transition is that V_0^γ meets exactly the same horizontal lines in finite sets in a thin $\varDelta$ as it does in a thick $\varDelta$. In thin $\varDelta$, the intersection is the single point of the diagonal Δ on the horizontal line. Because of this, the effect of Condition (4) this time is to insure that the following subsets of $\varDelta$ are closed discrete:

$$C_n = \{\langle \gamma + n + 1, \gamma + n\rangle : \gamma \in \Lambda \cup \{0\}\}.$$

It follows from the discussion following Theorem 2.8 that any set D that meets the same horizontal lines as C_n in a single point or even finitely many points is closed discrete.

3.5. Example [$\diamondsuit$] Recall that $\diamondsuit$ is the axiom that there is a family $\{S_\alpha : \alpha < \omega_1\}$ of subsets of ω_1 such that (i) $S_\alpha \subset \alpha$ for all $\alpha < \omega_1$ and (ii) for each subset X

of ω_1 the set of all α such that $X \cap \alpha = S_\alpha$ is stationary. We will use this axiom to construct a special case of Example 3.4 in which every closed discrete subset of $s\Delta$ is the union of a countable set and finitely many sets of the form C_n. More generally, every closed discrete subset will be the union of a countable set and a subset of some $X_0 \cup \cdots \cup X_n$ where $X_k = (\omega_1 \times \Omega_k) \cap \Delta$ where $\Omega_k = \{\gamma + k : \gamma \in \Lambda\}$. From this it will be easy to deduce that this example is strongly cwH and hence T_5.

The strategy is in the contrapositive direction: given any subset X of $\mathit{\Delta}$ that has uncountable intersection with infinitely many distinct X_n, we will insure that X will have a limit point. This we do with a careful choice of the ladders, taking care that each one meets infinitely many Ω_k whenever it is possible for it to do so. Explicitly: if S_α meets infinitely many Ω_k, then we pick $L_\alpha = \{\alpha_n : n \in \omega\}$ so that whenever S_α meets Ω_k in a cofinal subset of α, then L_α meets Ω_k also. To see that this works, suppose D is a subspace of $\mathit{\Delta}$ that meets infinitely many X_k in an uncountable set and also meets each horizontal line in a finite set. Every discrete subspace of $\mathit{\Delta}$ must satisfy the latter property, but we will show that the former property rules out D being closed.

A routine transfinite induction gives a subspace $T = \{t_\xi : \xi \in \omega_1\}$ of D such that (1) T meets infinitely many X_k in an uncountable set and (2) if $\xi < \eta$ then both coordinates of t_ξ are less than either coordinate of t_η. In other words, the transfinite sequence is the graph of a strictly increasing function from the set of first coordinates of members of T to the second coordinates. The key to the induction is that if $\eta < \omega_1$ and $\{t_\xi : \xi < \eta\}$ have been chosen, all but countably many points of $D \cap X_k$ have second coordinates greater than both coordinates of any earlier t_ξ; and, of course, $\pi_1(x) \geq \pi_2(x)$ for all $x \in \mathit{\Delta}$.

Let K be the set of all indices k such that $T \cap X_k$ is uncountable. The following subset of ω_1 is a club:

$$C = \{\alpha : (\pi_2^{\rightarrow} T) \cap (\alpha \times \Omega_k) \text{ is cofinal in } \alpha \text{ for all } k \in K\}.$$

As usual, π_2 denotes the projection to the second coordinate. C is trivially closed, while unboundedness is established by a standard leapfrog argument. Using the "stationary" aspect of $\diamondsuit$, let $\alpha \in C$ be such that $S_\alpha = \pi_2^{\rightarrow}(T \cap \alpha \times \omega_1)$. Then, given any $\alpha_n \in L_\alpha$, there is $\xi_n < \alpha$ such that $t_{\xi_n} = \langle \beta_n, \alpha_n \rangle$ for some $\beta < \alpha$. Since V_0^α contains all of $\{t_{\xi_n} : n \in \omega\}$ it follows that T is not closed discrete, and so neither is D.

4. Applying Martin's Axiom

We will now show how MA(ω_1) implies that Example 3.4 is T_5, completing the proof that Theorem A cannot dispense with ω_1-compactness. Our proof extends to any version of $\mathit{\Delta}$ in which the subdiagonal $s\Delta$ is a countable union of closed subspaces. In example 3.4, it is the sets of the form C_n that are the most convenient choice for the countably many closed (discrete) subspaces in which $s\Delta$ is divided.

Most of proof will probably seem almost mechanical to people familiar with the uses of Martin's Axiom in [Ru] or [K, Chapter 2]. This includes the posets we set up and the proof that they are ccc. We introduce two related classes of ccc posets. The first kind will be used in showing that, in Example 3.4, $s\Delta$ contains a closed discrete subspace that meets all of the discrete subspaces C_n in an uncountable set. This shows that some extra set-theoretic hypothesis was necessary in Example 3.5 to maintain tight control over the closed discrete subspaces.

Our posets will be subsets of a poset P which is not, itself, ccc. Elements of P are ordered pairs $p = \langle [p], \mathcal{U}_p \rangle$ where $[p]$ is a finite collection of points, meeting each horizontal line at most once, off the diagonal, and $\mathcal{U}_p$ is a finite collection of basic open sets V^α_ξ. The partial order is given by:

$$p \leq q \iff [q] \subset [p], \mathcal{U}_q \subset \mathcal{U}_p, \text{ and if } d \in [p] \setminus [q], \text{ then } d \notin U \text{ for all } U \in \mathcal{U}_q.$$

It is easy to check that this is a partial order on P. To see that P itself is not ccc, note that the following is an uncountable antichain of P: $\{(\langle \alpha, 0 \rangle, \emptyset) : \alpha \in \omega_1\}$

4.1. Two kinds of ccc posets. Our ccc subsets of P will each involve (discrete) subsets of $\mathit{\Delta} \setminus \Delta$ that meet each horizontal line in at most one point. In the first kind we let D be any such subset and define

$$Q_D = \{q \in P : [q] \subset D\}.$$

In the second kind, we let C be any such *closed* subset, and define

$$P_C = \{p \in P : (\cup \mathcal{U}_p) \cap C = \emptyset \text{ and every point of } [p] \text{ is strictly to the left of some point of } C\}.$$

Without loss of generality, we need only consider those C which are uncountable and consist only of nonisolated points. In both kinds of posets, the following sets are obviously dense for each $\xi \in \omega_1$:

$$D_\xi = \{p \in P : \langle \xi, \xi \rangle \in U \text{ for some } U \in \mathcal{U}_p\}$$

In the posets Q_D, the following sets are clearly dense for each ξ in ω_1 and each $n \in \omega$ for which X_n meets D in an uncountable set:

$$E_\xi^n = \{p \in P : [p] \cap X_n \cap (\omega_1 \times \{\alpha\}) \neq \emptyset \text{ for some } \alpha > \xi\}.$$

In the posets P_C, the following are dense for each $c = \langle \xi, \alpha \rangle \in C$:

$$D_c = \{p \in P : \langle \eta, \alpha \rangle \in [p] \text{ for some } \eta < \xi\}.$$

This is because all the $\bigcup \mathcal{U}_p$ miss C, and they are clopen, and points of C have limit ordinals for their first coordinates. Thus there is ample room to the left of each point of C to extend any $[p]$.

If G is generic for the given $\aleph_1$ dense sets in the respective posets, then $Z = \bigcup\{[p] : p \in G\}$ is a closed discrete subspace of $\mathit{\Delta}$, because at most finitely many points of Z are in any of the U in $\bigcup\{\mathcal{U}_p : p \in G\}$, and these U's cover the diagonal because of the density of each D_ξ. But if a subset of $\mathit{\Delta}$ meets each horizontal line in a finite set, its only accumulation points can be on the diagonal. In $Q_{s\Delta}$, this closed discrete subspace Z meets every C_n in an uncountable set, as stated earlier. In the posets P_C, we can expand the set C to a discrete collection of clopen intervals $(\eta, \xi] \times \{\alpha\}$ where $\langle \xi, \alpha \rangle \in C$ and $\langle \eta, \alpha \rangle \in Z$. The union of these is a clopen set containing C and missing the diagonal.

4.2. Proof of ccc property. The proof that these posets are ccc follows familiar lines. Let A be any uncountable subset of either poset. If there is a pair p_1, p_2 of distinct elements of A such that $[p_1] = [p_2]$, then p_1 and p_2 are compatible and we are done. So assume all the $[p]$ with $p \in A$ are distinct. Let A_0 be an uncountable subset of A in which all the $[p]$ are of the same (finite) cardinality n. Let A_1 be an uncountable subset of A_0, in which the $[p]$ form a Δ-system with root r. In A_1, the sets $[p] \setminus [r]$ are all disjoint, and so only countably many of them can meet any portion of $\mathit{\Delta}$ whose second coordinates are bounded; this is obvious in case of the Q_D and follows for the P_C from the fact that there are only countably many points directly to the left of any point in C. Hence A_1 has an uncountable subset A_2 in which, given any two members p_1 and p_2 of A_2, all points of $[p_1]$ outside the root r have smaller second coordinate than any of the points of $[p_2]$ outside the root r (or vice versa). If we designate the former alternative by $p_1 \prec p_2$, then A_2 is strictly well-ordered in order type ω_1 by $\prec$. Hence $\bigcup\{[p] : p \in A_2\}$ meets each horizontal line at most once, and the only thing impeding compatibility of the members of A_2 is the interplay of the various $[p]$'s and $\mathcal{U}_q$'s.

To take care of that, let k be the least integer such that infinitely many $[p] \in A_2$ have the property that all points of $[p]\setminus[r]$ have second coordinates of the form $\gamma+j$, $\gamma \in \Lambda$, $j \leq k$. Let $\langle p_n : n \in \omega\rangle$ be a $\prec$-increasing sequence of such points and let $U = \bigcup\{\bigcup\mathcal{U}_{p_n} : n \in \omega\}$. Let $q \in A_2$ be such that $([q]\setminus[r])\cap U = \emptyset$ — this is possible because U is countable. Now $\bigcup\{[p_n] \setminus [r] : n \in \omega\}$ is closed discrete, being a subset of $X_0 \cup \cdots \cup X_k$ that meets each horizontal line at most once. Hence all but finitely many $[p_n] \setminus [r]$ miss the compact set $\bigcup\mathcal{U}_q$. Let m be such that $[p_m] \cap \bigcup\mathcal{U}_q \subset [r]$; then p_m and q are compatible. □

Clearly, the foregoing proof of the ccc property extends to all cases where $s\mathit{\Delta}$ or D is a countable union of closed (discrete) subspaces. More generally, we have:

4.3. Theorem. *Let D be a subset of $\mathit{\Delta} \setminus \Delta$ which meets each horizontal line in at most one point. If each uncountable subset E of D has an uncountable subset F of its own that is closed in $\mathit{\Delta}$, then:*

(1) *Q_D is ccc and*
(2) *P_C is ccc for all closed subspaces C of D and*
(3) MA(ω_1) *implies that each closed subspace C of D can be separated from Δ by disjoint open sets and*
(4) MA(ω_1) *implies that D is a countable union of closed discrete subspaces.*

Proof. We extend the cutting-down process of 4.2 as follows. Let $A_3^0 = A_2$ and inductively let A_3^i be an uncountable subset of A_3^{i-1} such that the set of all ith elements of $[p] \setminus [r]$ is closed (discrete) as p ranges over A_3^i. If $[r]$ has m elements, let A_4 be any subset of A_3^{n-m} of order type ω with respect to $\prec$ and proceed as above, with $q \in A_3^{n-m}$, to show items (1) and (2). Then (3) follows from (2) as indicated earlier. As for (4), that follows from (1) and the well-known fact that MA(ω_1) implies every ccc poset of cardinality $\aleph_1$ is σ-centered: each uncountable centered subcollection of $\{p \in Q_D : p = \langle\{d\}, \{V_0^{\pi_2(d)}\}\rangle\}$ clearly corresponds to a closed discrete subspace of $\varDelta$ since the sets $V_0^{\pi_2(d)}$ involved cover the diagonal. $\square$

By employing the axiom TOP* we can also simplify the situation that occurs when the hypothesis of 4.3. fails. The axiom TOP*, denoted "Axiom 0*" in [NP], has to do with ideals of subsets of ω_1:

4.4. Definition TOP* is the following axiom:

Let $\mathcal{I}$ be an ideal on ω_1 with A_1 generators. Then either there is an uncountable subset A of ω_1 such that every countable subset of A is in $\mathcal{I}$, or ω_1 is the union of countably many sets B_n such that $B_n \cap I$ is finite for all $I \in \mathcal{I}$.

4.5 Lemma. [TOP*] *If S is a subset of $\varDelta$ missing the diagonal and containing at most finitely many points of each horizontal line, then either:*

1. *There is an uncountable subset A of S such that every countable subset of A has compact closure, or*
2. *S is the union of countably many closed discrete subsets of $\varDelta$.*

The following is immediate from Theorem 4.3 and Lemma 4.5.

4.6. Corollary. [TOP* + MA(ω_1)] *If a version of $\varDelta$ is not normal, then the sub-diagonal $s\Delta$ has an uncountable subset S such that every countable subset of S has compact closure.*

5. Other topological properties of $\varDelta$

In this section, we show that $\varDelta$ is always countably metacompact, and even satisfies a number of stronger properties, sometimes under added conditions. In particular, if it is normal, it is countably paracompact. The converse is also true: this is immediate from Lemma 2.6 and an interesting general theorem.

5.1. Theorem. *Let X be a countably paracompact space. If C is a countably compact closed subspace and D is a closed discrete subspace smaller than the first uncountable measurable cardinal, then C and D can be put into disjoint open sets.*

Proof. Let $\mathcal{F}$ be the collection of all subsets F of D such that the closure of every open set containing F meets C. Suppose every member of $\mathcal{F}$ could be split into two disjoint subsets in $\mathcal{F}$. In that case, we could produce an infinite discrete subfamily $\{F_n : n \in \omega\}$ of $\mathcal{F}$. But if $\{U_n : n \in \omega\}$ were a locally finite open expansion of $\{F_n : n \in \omega\}$, then we could use the countable compactness of C to arrive at a contradiction, as follows. For each point x of C let V_x be an open nbhd of x which misses all but finitely many of the sets U_n, and let W_n be the union of all the V_x that miss all U_m such that $m \geq n$. Then the cover $\{W_n : n \in \omega\}$ of C has a finite subcover, whose union misses all but finitely many of the U_n, contradicting the membership of all F_n in $\mathcal{F}$. Hence $\{F_n : n \in \omega\}$ does not have a locally finite open expansion, contradicting the countable paracompactness of X.

So $\mathcal{F}$ has a member F whose subsets in $\mathcal{F}$ form a linked system. In fact, these subsets form a filterbase $\mathcal{B}$ since if $F_1 \cap F_2$ were not in $\mathcal{F}$ then the complement of this intersection in F_i would be in $\mathcal{F}$ for $i = 1, 2$ contradicting the way F was chosen. *Claim. $\mathcal{B}$ has the countable intersection property.* Once the claim is proved, we are done as soon as we show $\mathcal{B}$ is a base for an ultrafilter. Were it not, we could write F as a union of two sets, neither of which is in $\mathcal{F}$; but clearly, if both could be put into open sets whose closure misses C then so can their union.

Proof of Claim If $\mathcal{B}$ did not have the countable intersection property, we could produce a descending sequence $\{B_n : n \in \omega\}$ of subsets of F in $\mathcal{F}$ whose intersection is not in $\mathcal{F}$. But then the sets $B_n \setminus B_{n+1}$ would be a partition of F into countably many sets, none of which is in $\mathcal{F}$. Expanding these to a locally finite collection of open sets whose closures miss C, and taking the union of these open sets, gives an open nbhd of F whose closure misses C, again giving a contradiction. □

Definition a poset P is *totally proper* if it is proper and forcing with P does not add reals. The following gives a useful criterion for when a poset is totally proper.

Lemma A. [ER] *A poset P is totally proper if, and only if, given a countable elementary submodel M of $\mathbb{H}_\theta$ for some large enough regular θ, every $p \in M \cap P$ has a totally (M, P)-generic extension q. That is, for every dense open subset D of P that is in M, there exists $p \in M \cap D$ satisfying $q < p$.*

The following lemma suggests why so many totally proper posets use elements with countably infinite working parts:

Lemma B. [E] *If P is totally proper, then any generic subset G of P is countably closed, i.e., given $\{p_n : n \in \omega\} \subset G$, there exists a $q \in G$ that extends each p_n.*

I conjecture that some such poset as the following is totally proper and can be iterated with countable supports to make any space of the form 3.4 normal.

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