MOVING-OFF COLLECTIONS AND SPACES OF CONTINUOUS FUNCTIONS

by Peter Nyikos

Definition 1. Let $\mathcal{C}$ be a collection of subsets of a space X. A collection $\mathcal{K}$ of nonempty subsets of X is said to *move off* $\mathcal{C}$ if for each $C \in \mathcal{C}$ there exists $K \in \mathcal{K}$ such that $C \cap K = \emptyset$.

Let X be a Tychonoff space. As usual, we use $C_p(X)$ [*resp.* $C_k(X)$] to denote the space of continuous real-valued functions on X with the product topology (a.k.a. the topology of pointwise convergence) [*resp.* with the compact-open topology].

• $C_p(X)$ is countably tight iff every collection of **closed** subsets of X which moves off the **finite** sets has a countable subcollection which also moves off the finite sets [AP].

• $C_p(X)$ is Fréchet-Urysohn iff every collection of **closed** subsets of X which moves off the **finite** sets has an infinite point-finite subcollection [GN].

• $C_p(X)$ is Baire iff every countably infinite collection of **finite** sets which moves off the **finite** sets has an infinite subcollection that expands to a discrete collection of open sets [GN].

Let X be a locally compact space. $C_0(X)$ stand for the set of continuous functions that vanish at infinity, with the compact-open topology. ["Vanish at infinity" is the quaint old way of describing functions that extend to a continuous function on the one-point compactification of X and send the extra point to 0.]

• $C_k(X)$ is Baire iff $C_0(X)$ is Fréchet-Urysohn iff every collection of **compact** sets which moves off the **compact** sets has an infinite discrete subcollection. [For the Baire part, see [GM]; for the other, see Theorem 1 below.]

• $C_0(X)$ is countably tight iff every collection of **compact** subsets of X which moves off the **compact** sets has a countable subcollection which also moves off the compact sets. See Theorem 1 below.

Some of the properties listed above have special names attached to them:

Definition 2. A space X has *the Moving Off Property*, abbreviated *MOP*, if every collection $\mathcal{K}$ of compact subsets of X that moves off the compact sets has an infinite subcollection $\mathcal{L}$ that has a discrete open expansion in $\mathcal{X}$.

For locally compact spaces and normal spaces, the MOP is equivalent to the simpler condition that says $\mathcal{L}$ is simply discrete.

Definition 3. A space X has *Property* γ or *the* γ*-property* if every collection of closed sets which moves off the finite sets has an infinite point-finite subcollection. A γ*-set* is a subset of the real line $\mathbb{R}$ with the γ-property.

In the literature, the γ-property is formulated using the complements of closed sets as above. An open cover $\mathcal{U}$ is called an ω*-cover* if every finite subset of the space is contained in some member $\mathcal{U}$. Definition 3 is clearly equivalent to every open ω-cover having a subcover $\mathcal{V}$ such that every element is contained in all but finitely many members of $\mathcal{V}$.

Lemma 1. *Let $\mathfrak{P}$ be one of the following properties of a collection $\mathcal{L}$: point-finite, locally finite, discrete. If every member of $\mathcal{C}$ is countably compact, and X is not countably compact, and $\mathcal{D}$ includes all singletons and is closed under finite union, then the following are equivalent.*

(1) *Every subfamily of $\mathcal{D}$ that moves off $\mathcal{C}$ has an infinite subfamily satisfying $\mathfrak{P}$ [*resp. *which has an open expansion satisfying $\mathfrak{P}$.]*
(2) *Given a sequence of subfamilies $\mathcal{K}_0, \mathcal{K}_1, \dots$ of $\mathcal{D}$ which move off $\mathcal{C}$, there are $K_i \in \mathcal{K}_i$ such that $\{K_i : i \in \omega\}$ satisfies $\mathfrak{P}$ [*resp. *has an open expansion satisfying $\mathfrak{P}$.]*

The special case of Lemma 1 where $\mathcal{C}$ is the family of compact sets and $\mathfrak{P}$ is "discrete" was Theorem 2.3 of [GM], and the proof there works almost word for word for this general case. The special case where $\mathcal{C}$ is the family of finite sets and $\mathfrak{P}$ is "discrete" was $(\gamma) \implies (\gamma')$ in [GN].

Theorem 1. *If X is locally compact, then:*

(1) *$C_0(X)$ is countably tight $\iff$ every collection of compact sets that moves off the compact sets in X has a countable moving-off subcollection.*
(2) *$C_0(X)$ is Fréchet-Urysohn $\iff$ X has the MOP.*

To show the converse in (1) and (2), let $\mathcal{F}$ be a subset of $C_0(X)$ with 0 in its closure. Let

$$\mathcal{K}_n = \{(f^{\leftarrow}(-\frac{1}{n+1}, \frac{1}{n+1}))^c : f \in \mathcal{F}\}.$$

Then $\mathcal{K}_n$ is moving-off for all n and since the members of $\mathcal{F}$ are all in $C_0(X)$, each member of each $\mathcal{K}_n$ is compact. Pick a countable moving-off subfamily $\mathcal{L}_n$ of $\mathcal{K}_n$ for each n and take the union of these subfamilies; the corresponding functions are easily seen to have 0 in the closure. If in addition X has the MOP, we use Lemma 1 to choose $K_n \in \mathcal{K}_n$ so that $\{K_n : n \in \omega\}$ has a discrete open expansion. The corresponding functions are then easily seen to converge to 0. $\square$

A γ-set coincidence

Daniel K. Ma [M] showed an interesting connection between the parallel theories on C_p and C_k involving two different topologies on subspaces of the Cantor tree. Identifying the Cantor set with the top level of the full binary tree of height $\omega + 1$, we let T be the full binary tree of height ω. If A is a subset of the Cantor set we put the interval ("tree") topology on $X = T \cup A$, so that T is a countable dense set of isolated points, A is a closed discrete subspace, and X is locally compact.

The following are equivalent (i) $C_k(T \cup A)$ is Baire.
(ii) $C_0(T \cup A)$ is Fréchet-Urysohn.
(iii)$C_p(T \cup A)$ is Fréchet-Urysohn.

(iv) $T \cup A$ has the MOP; equivalently, every collection of **compact** sets which moves off the **compact** sets has an infinite discrete subcollection.

(v) In the Euclidean topology, A is a γ-set; that is, every collection of **closed** subsets of A which moves off the **finite** sets has an infinite point-finite subcollection.

We can even replace A by $T \cup A$ in (v) if we make use of a nice geometric embedding of the Cantor tree in the Euclidean plane [Ny] and put the relative topology on $T \cup A$. This is because of the fact, easily proven using Lemma 1, that a space has the γ-property iff some co-countable subset has it. In Figure 1, taken from [Ny], the thin guide lines are not actually part of the Cantor tree but do help with understanding the basic neighborhoods of the points of the tree in the both the interval topology and the Euclidean topology. Basic neighborhoods in the interval topology are like neighborhoods in the Sorgenfrey plane, rotated 45 degrees counterclockwise. Thus the interval topology is finer than the Euclidean topology. The whole Cantor tree is compact in the Euclidean topology but its subspaces $T \cup A$ are compact only if A is closed, which is not the case if A is an uncountable γ-set. In fact it is ZFC-independent whether there is an uncountable γ-set.

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