THE FINE STRUCTURE OF LOCALLY COMPACT, HEREDITARILY NORMAL SPACES UNDER STRONG CONDITIONS I

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Abstract.

Preamble

This paper builds on a line of research that goes all the way back to 1890, when Peano [1] discovered that the unit square and unit cube were continuous images of the interval $[0, 1]$. When point-set topology blossomed at the beginning of the 20th century, the natural question arose as to how the Hausdorff continuous images of $[0, 1]$ could be characterized internally. This was solved by Hahn [2] and Mazurkiewicz [3] who independently showed the beautiful theorem:

The Hahn-Mazurkiewicz Theorem. *A Hausdorff space is a continuous image of* $[0, 1]$ *if, and only if, it is a locally connected, metrizable continuum.*

[By "continuum" is meant a compact, connected Hausdorff space.]

The search for a natural generalization of this theorem took the better part of a century and occupied some of the best minds in topology. The class of compact, connected, linearly orderable spaces was soon adopted as the canonical generalization of $[0, 1]$. We here adopt the expression, "generalized arcs" for these spaces. The fascinating story of the long search for the best characterization of their continuous images has been ably and thoroughly told by one of its chief participants, the late Sibe Marděsić [4]. Suffice it to say here that the search ended with a generalization due above all to Treybig, Jacek Nikiel, and Mary Ellen Rudin:

The Generalized Hahn-Mazurkiewicz Theorem. *A Hausdorff space is a continuous image of a generalized arc if, and only if, it is a locally connected, monotonically normal continuum.*

The wording only replaces "metrizable" with "monotonically normal," but the full proof is arguably the most difficult in all of point set topology. One of the main purposes of this paper is to make the Generalized Hahn-Mazurkiewicz (GH-M) theorem more accessible by reducing the proof to the case of first countable spaces. It is hoped that this will turn out to be much easier to handle than the general case.

In many ways, the GH-M Theorem is the best generalization that can be had with the usual (ZFC) axioms of set theory. The other main purpose of this paper is to explore what can be done with the help of extra axioms. This exploration was begun in [5], where it was shown that in MM(S)[S] models, the much more general property of hereditary normality

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has many properties in commmon with monotone normality in the class of locally compact, locally connected spaces. It broached the seemingly reckless idea that in some such model, hereditary normality could be substituted for monotone normality in the Generalized Hahn-Mazurkeiwicz Theorem.

While falling far short of being able to make this substitution, this paper will reduce the problem to some special cases while uncovering lots of behavior in common with the monotonically normal case. It is hoped that the structure theory uncovered here will be an aid to better understanding of these spaces.

In this paper, "space" means "Hausdorff space" and hereditarily normal spaces will be referred to as T_5 spaces.

1. Introduction

Monotone normality is a very strong property. Besides being hereditary, it has some striking implications for covering, separation, and cardinal function properties of spaces. For example:

(I) Every monotonically normal space is (hereditarily) collectionwise normal.

(II) Every monotonically normal space is (hereditarily) countably paracompact.

(III) Every monotonically normal space of countable spread is (hereditarily) Lindelöf.

(IV) Every locally compact, monotonically normal space X has *the Strong Balogh-Rudin Property:*

If $\mathcal{U}$ is an open cover of X, then $X = V \cup \bigcup \mathcal{W}$, where $\mathcal{W}$ is a discrete family of copies of regular uncountable cardinals, and V is the union of countably many collections $\mathcal{V}_n$ of disjoint open sets, each of which (partially) refines $\mathcal{U}$.

(V) Every locally compact, monotonically normal space is either hereditarily paracompact or contains a copy of ω_1.

Property (III) extends to all locally compact T_5 spaces in MM(S)[S] models. Moreover, all of (I) - (V) hold for locally compact, locally connected, T_5 spaces in MM(S)[S] models. This is shown in [5] by way of showing that they satisfy a remarkable property in these models:

Definition 1.1. A space X has *the LBS property* if it is locally connected and:

(1) each of its components is the union of an open Lindelöf space L and at most countably many disjoint, closed, connected, countably compact noncompact spaces S_n, and

(2) each S_n includes uncountably many cut points of the component and has exactly one point in the closure of L, and

(3) each S_n is a "string of beads" in which there is a set C_n of cut points of the whole component, such that C_n is homeomorphic to an ordinal of uncountable cofinality, and each "bead" is the connected 2-point compactification of an open subspace whose boundary consists of the two extra points, successive members of C_n, and

(4) the union of these "beads" comprises the entire string.

Of course, the set of bead strings could be empty; so every locally connected Lindelöf space has the LBS Property. Moreover, due to (3) and (4), L is the only subspace of its component that is not locally compact. Whether this slight added generality is of any help is immaterial for this paper, which confines itself to locally compact spaces with the LBS property.

The following is also shown in [5]:

Theorem 1.2. *Let X be a locally compact, locally connected space. If every open subspace of X satisfies (1) of the LBS property, then X satisfies (I) and (II). If in addition X satisfies* (3) *and* (4)*, then X also satisfies (IV) and (V).*

Conversely, if every open subspace of X satisfies (I) through (IV), then X has the LBS Property and witnesses (V).

Note the lack of set-theoretic axioms in Theorem 1.2. As a corollary, every monotonically normal, locally compact, locally connected space has the LBS Property.

The twofold main purpose of this paper is to further explore the structure of locally compact, locally connected, T_5 spaces in MM(S)[S] models and of the monotonically normal ones in ZFC. This is practically synonymous with exploring the structure of the compact ones. That is because the one-point compactification of a locally compact, locally connected, monotonically normal space is also monotonicaly normal [6]. It is therefore the continuous image of a generalized arc.

Researches of Treybig and Nikiel went much deeper into the fine structure of continuous images of generalized arcs than is provided with the wording of the LBS Property. However, this property has a lot of hidden power which will enable us to delve deeply into the fine structure of these spaces; and more generally, all locally connected continua in which every locally compact, locally connected subspace has the LBS property. Our analysis begins with fixing such a space X.

Even with the help of the LBS Property, we will fall far short of being able to replace, even consistently, the "monotononically normal" in the Generalized Hahn-Mazurkiewicz Theorem with "T_5". However, we will reduce this problem to the subclass of first countable, locally connected T_5 continua. We also have some hope of strengthening "first countable" to "perfectly normal" in this reduction. This however will put us up against some of the most difficult and long-standing problems in topology, explained in Section 11.

In Sections 2 through 5, we will "unpack" the beads which are given to us by the LBS property for a locally compact, connected, and locally connected space X. The interior of each bead could have arbitrarily many components, and we repeat the unpacking in the beads of each component. This process can go through arbitrarily many stages, reminiscent of the infinitely many successive magnifications of the Mandelbrot set. The process will temporarily end in Section 5 with us having worked down to beads with only Lindelöf components, all the while building a nice closed monotonically normal subspace. The structure theory at this stage is summed up in:

Theorem 1.3. *Let Y be a locally compact space such that every open subspace has the LBS Property. Then Y is the disjoint union of a closed, rim-finite, monotonically normal subspace R and a family $\mathcal{L}_1$ of disjoint (open) Lindelöf subspaces that are the components of $Y \setminus R$.*

Moreover, Y is monotonically normal if, and only if, each member of $\mathcal{L}_1$ is monotonically normal.

In the original LBS wording, the conditions on R are obviously satisfied by $\bigcup_{n\in\omega} C_n$ in each non-Lindelöf component of Y, but there is no analogue of $\mathcal{L}_1$ because the components of the interiors of the beads can be of any Lindelöf degree whatsoever. So we will build the space R by induction, as the union of subspaces $R(\alpha)$, by a process that will occupy sections 2 through 5.

An optional refinement of the process is explained in Section 3. Its purpose is to make the "nice" subspace R in Theorem 1.3 as big as possible without having to go into a lot of technical details. It uses a modification LBS+ of the LBS property, which puts "a limit ordinal" for "ordinal of uncountable cofinality" in (3) of Definition 1.1.

In Sections 6 through 8 a new phase begins, in which the rim-finite subspace $W \supset R$ that we grow during the induction is not closed. This seems to be the only way we can be sure of getting the components of $X \setminus W$ to be countably tight. At the end of Section 8 we obtain:

Theorem 1.4. *Let X be locally compact, locally connected space such that every locally compact, locally connected, subspace has the LBS Property. Then X has a rim-finite, monotonically normal subspace W, and a family $\mathcal{L}_2$ of disjoint countably tight, Lindelöf subspaces that are the components of $X \setminus W$.*

Moreover, X is monotonically normal if, and only if, each member of $\mathcal{L}_2$ is monotonically normal.

In Section 9, we continue the reduction to go to first countability. With some spaces, we can continue with the earlier techniques to arrive at W^+ with $X \setminus W^+$ first countable. At the opposite extreme, there are some spaces for which it is necessary to leave W as is and to describe a partition of $X \setminus W$ which at least makes the "moreover" part of Theorem 1.4 go through. In any event, we have:

Theorem 1.5. *Let X be a locally compact, locally connected space such that every locally compact, locally connected subspace has the LBS Property. Then there is a family $\mathcal{L}_3$ of disjoint locally compact, connected, locally connected, first countable, Lindelöf subspaces, such that X is monotonically normal if, and only if, each member of $\mathcal{L}_3$ is monotonically normal.*

By working a bit harder, we can arrange that no member of $\mathcal{L}_3$ has any cut points. However, we cannot yet overcome the hurdles in Section 11 that face us in, first, reducing the problem further to the case of perfectly normal, locally connected continua, and second, to solving this case. This second difficulty is so formidable that I have sometimes given the nickname "The Impossible Dream" to the replacement of "monotonically normal" with "T_5" in the Generalized Hahn-Mazurkiewicz Theorem. [In more optimistic moods, I have given it the nickname, "The Holy Grail."]

2. Unpacking the LBS beads

In our analysis, it will be convenient to use some jargon and notation;.

Definition 2.1. Let S be a locally compact, locally connected space. A *bead* [resp. *balloon* (in S) is a compact, connected, locally connected subspace with a two-point [resp. one-point] boundary. The boundary point[s] will be referred to as *ends*. The end [resp. ends] of a balloon [resp. bead] B will be designated p_B [resp. p_B^- and p_B^+] and we will use F_B for both $\{p_B\}$ and $\{p_B^-, p_B^+\}$.

[Mnemonic: F_B is finite and the frontier of B]

The choice of which end of a bead in S is to be called p_B^- can be arbitrary, but if S satisfies the LBS property, a logical choice for each B mentioned there is the end that remains in the same component as the Lindelöf lump L when the other is removed. [Sometimes the choice of of L itself is arbitrary, as when $\overline{L}$ is compact and has a 2-point boundary.]

Definition 2.2. A *bridge* is a connected open subspace P of X with a two-point boundary, whose points are also called its *ends*. An *essential cut point* of a bridge P is a point $x \in P$ whose removal from $\overline{P}$ separates the ends of P; in other words, they are in different components of $\overline{P} \setminus \{x\}$.

[Mnemonic: P because of the Latin word "pons" for "bridge."]

In this section and the next two, the bridges we encounter will be components of $B \setminus F_B$ where B is a bead, for which the ends are also the ends of B. Of course, the closures of these bridges also satisfy the definition of a bead, but we will confine the term "bead" to beads that arise in our induction in a special way. Many of them will come directly from the LBS concept applied to X and to various subspaces.

Given a bead B, the components of $B \setminus F_B$ each have an end of B in their closure. If both are in the closure, the component is called a *bridge of* B, otherwise it is a *balloon of* B.

Lemma 2.3. *There is at least one bridge of a bead B, but no more than finitely many.*

Proof. Connectedness and local connectedness of B requires that at least one (open) component of $B \setminus F_B$ have both ends in the closure. On the other hand, every neighborhood of both p_B^- and p_B^+ meets every bridge, but since bridges are connected, each has some points missing from any pair of disjoint open neighborhoods of p_B^- and p_B^+. The left over portions of each bridge together are a discrete family of closed sets, hence there can only be finitely many of them. □

On the other hand, there is no restriction on the number of balloons of a bead.

Our analysis deals with a locally connected continuum X in which every locally compact, locally connected subset has the LBS property. In this section, we need only assume that every open subset has this property. If X is countably tight (equivalently, X does not contain a copy of ω_1) then our analysis can really begin at Section1count after the next two lemmas.

Notation 2.4. *For each bridge P of a bead in X, let $E(P)$ be the set of essential cut points of P.*

In [7] the notation for $E(P)$ would be $E(p_B^-, p_B^+)$ but here this would be ambiguous, because there may be more than one bridge of B.

Lemma 2.5. *For each bridge P of a bead B, the set $E(P) \cup F_B$ is a compact LOTS.*

Proof. Of any pair $\{x, y\}$ of essential cut points, one (say x) will separate p_B^- from the other, while the other separates x from p_B^+. In this way we define $x < y$ and it is an elementary fact (see 28.11 in [7]) that this is a strict total order on the essential cut points. To show compactness, it is enough to consider limit points of ascending sequences $\langle x_\alpha : \alpha < \kappa\rangle$ where κ is a regular cardinal.

For each $\alpha < \kappa$, let C_α be the component of $P \setminus \{x_\alpha\}$ whose closure contains both p_B^- and x_α. Clearly $C_\beta \cup \{x_\beta\} \subset C_\alpha$ whenever $\beta < \alpha$. Moreover, x_β and x_α are together the boundary of an open subset $P(\beta, \alpha)$ of P such that $C_\beta \cup \{x_\beta\} \cup D(\beta, \alpha) = C_\alpha$. [$P(\beta, \alpha)$ contains all bridges from x_β to x_α as well as all balloons attached to x_β, but none attached to x_α since these represent other components of $P \setminus \{x_\alpha\}$ besides C_α.]

We claim $C = \bigcup\{C_\alpha : \alpha < \kappa\}$ has two boundary points, one of which is p_B^-, and the other is an essential cut point x of P, and that C is a bridge between p_B^- and x. Once this is shown, it follows that x is the supremum of $\langle x_\alpha : \alpha < \kappa\rangle$ in the order $<$. Applying this to all possible ascending sequences in $E(P) \cup F_B$, we get that $E(P) \cup F_B$ is compact.

To prove the claim, note that $C \cup \{p_B^-\}$ is not compact since $\{C_\alpha \cup \{p_B^-\} : \alpha < \kappa$ is a relatively open cover of $C \cup \{p_B^-\}$ without a finite subcover.

□

Lemma 2.6. *If $P \setminus E(P) \neq \emptyset$, and G is a component of $P \setminus E(P)$, then G has at most two boundary points. If it has two, then G is a bridge between successive points of the LOTS $E(P) \cup F_B$.*

Proof. Since P is open, connected and locally connected, $P \setminus E(P)$ is an open set whose components must all have at least one point of $E(P)$ in their closure. If G has only one point of $E(P) \cup F_B$ in the closure, then it is a point of $E(P)$ and $\overline{G}$ is a balloon. But there cannot be more than two points of $E(P) \cup F_B$ in the closure, otherwise anything strictly between two of them could not be an essential cut point of P. For the same reason, if x and y are points of $E(P)$ in $\overline{G}$, then one must be the immediate successor in $E(P) \cup F_B$ of the other, and the component is a bridge between them. □

Our inductive construction of the monotonically normal subspace R in Theorem 1.3 begins with picking a point p_0 of X and letting $R(0) = \{p_0\}$. If all components of $X \setminus R(0)$ are Lindelöf, we simply let $R(1) = R(0) = R$, where R is as in Theorem 1.3 and now X is obviously as in this theorem. Otherwise, we fix arbitrary LBS representations for all non-Lindelöf components of $X \setminus R(0)$ and let $R(1) = R(0) \cup C(1) \cup E(1)$ where $C(1)$ is the union of all the countably compact subspaces that are involved in these LBS descriptions, and

$$E(1) = \bigcup\{E(P) : P \text{ is a bridge } P \text{ of some bead } B \text{ with ends in } C(1)\}.$$

Each countably compact subspace used in defining $C(1)$ is a closed copy of an ordinal of uncountable cofinality, and the C_n in the LBS description of a non-Lindelöf component of $X \setminus R(0)$ are a discrete collection of closed sets; and the union of these collections, over all non-Lindelöf componets, is also discrete. Hence $C(1)$ is clearly monotonically normal, locally compact, and scattered, hence 0-dimensional. As for $E(1)$, each $E(P)$ involved is a one-dimensional subspace and while it could even be connected, it has a base of open intervals whose boundaries have ≤ 2 points. It is easy to see that $E(1)$ is the topological direct sum of the individual $E(P)$, and all the points in the closure of $E(1)$ are in $C(1) \cup R(0)$. Thus $R(1)$ is closed in X.

Lemma 2.7. *$R(1)$ is rim-finite.*

We could restrict ourselves to non-Lindelöf bridges in defining $E(1)$ and still arrive at Theorem 1.3, but we might as well take advantage of easy opportunities to analyze X deeply. In between any two successive points $x < y$ of $E(P) \cup F_B$ we have a set A such that $A \cup \{x, y\}$ is a bead. As to why $A \cap E(P)$ is empty, that could be either because there is more than one bridge between x and y, or because there is a Lindelöf bridge from x to y. The first possibility will be the focus of the Ladder Lemma, which simplifies the picture that goes with $R(\alpha) = \overline{\bigcup\{R(\beta) : \beta < \alpha\}}$ when α is a limit ordinal.

If $\alpha = \beta + 1$ and $R(\beta)$ has been defined, we use the induction hypothesis that:

(i) $R(\beta)$ is closed and rim-finite; and

(ii) if $\eta < \xi \leq \beta$, then $R(\eta) \subset R(\xi)$, and

(iii) If $\xi = \eta + 1 \leq \beta$, then $R(\xi) = R(\eta) \cup C(\xi) \cup E(\xi)$ where each point of $E(\xi)$ is in a bead with ends in $C(\xi)$.

Clearly the induction hypothesis is satisfied for $\beta = 1$. Let $C(\alpha)$ be the union of all the C_n involved in a choice of LBS representation of all non-Lindelöf components of $X \setminus R(\beta)$. Let For each bridge P of a bead with ends in $C(\alpha)$, let let

$$E(\alpha) = \bigcup\{E(P) : P \text{ is a bridge of a bead with ends in } C(\alpha)\}.$$

Finally let $R(\alpha) = R(\beta) \cup C(\alpha) \cup E(\alpha)$.

Obviously, (ii) and (iii) in the induction hypothesis are satisfied with α in place of β. To show (i),

We continue defining $R(\alpha)$ until $R(\gamma) = R(\gamma + 1)$. In other words, every component of $X \setminus R(\gamma)$ is Lindelöf. So then we let $R = R(\gamma)$ and we let $\mathcal{L}_1$ of Theorem 1.3 be the set of components of $X \setminus R$. In order to show that R is closed and rim-finite, we will need to show that $R(\alpha)$ is rim-finite when α is a limit ordinal. This will be done in Section 4.

But first we have an optional section where we modify the definition of R to make it as big as we can without too much effort. This may not get us any closer to first countability of $X \setminus R$, but for complicated choices of X, it can often significantly aid in understanding the fine structure of the one-point compactification of X as a locally connected continuum. Central to it is the concept of a bead string, which we formalize as follows:

Definition 2.8. Let Y be a connected subspace of X. *A bead string in Y* is a subspace S of Y together with a family C of cut points of Y such that:

(1) there with a well order on C, $\{c_\alpha : \alpha < \theta\}$ where $\theta > 2$, such that c_0 separates S from $Y \setminus S$, and if $\alpha < \beta$ then c_β separates c_α from all c_γ such that $\gamma > \beta$; and

(2) the map which takes α to c_α is a homeomorphism, and

(3) for each $\alpha < \theta$, c_α and $c_{\alpha+1}$ form the ends of a bead B as in Definition 2.1.

The point c_0 will be called *the initial point of S*.

It is easy to see that S is the union of all the beads expressed in (3); and so S is countably compact and noncompact iff C is of uncountable cofinality, and σ-compact, hence Lindelöf otherwise; in particular, S is compact iff it has a greatest element. The following lemma was shown in [5] with a different wording.

Lemma 2.9. *("The Thorn Lemma") Let Y be a connected subspace of X. If (S, C) is a bead string in Y such that C is not closed in X, then $\overline{C} \setminus C$ is a singleton.*

3. Pushing back on the Lindelöf lumps

As explained earlier, we will be working in this section with a modification LBS+ of the formal statement of the LBS Property which puts "limit ordinal" in place of "ordinal of uncountable cofinality." The following simple example helps to motivate this change.

Example 3.1. *("The Dense Comb")* This is a connected modification of the Alexandroff duplicate of $[0, 1]$. Let the underlying set of $A[0, 1]$ be the unit square $[0, 1]^2$, with the following topology. Each set of the form $\{x\} \times (0, 1]$ is homeomorphic to $(0, 1]$ in the natural way, and open in $A[0, 1]$. A base at each point $(x, 0) \in [0, 1] \times \{0\}$ is a set of the form $I \times [0, 1] \setminus [\varepsilon, 1] \times \{x\}$ where I is a basic open interval about x in the relative topology of $[0, 1$. Informally, this is a comb with the vertical line segments $\{x\} \times (0, 1]$ the very densely packed teeth of the comb.

Now suppose that a copy of $A[0, 1]$ is one of the beads B of X with p_B^- the copy of $(0, 0)$ and p_B^- the copy of $(1, 0$. Then the rest of the copy is a bridge P between the two ends of B, and $E(P)$ is the base $[0, 1] \times \{0\}$ of the comb. It would be natural to include the rest of the bead in R, because the bead is rim-finite. But the teeth are all Lindelöf open subsets. However, they are bead strings of cofinality ω, in the following way. For each tooth $\{x\} \times (0, 1]$, let $C_x = \{(x, 1/2^n) : n \in \omega\}$. Then the tooth at x is the union of the beads $\{x\} \times [1/2^{n+1}, 1/2^n]$, each a bridge from one point of C_x to the next.

We can allow such things in $C(1)$ and their $E(P)$ in $E(2)$ without complicating the proof of Theorem 1.3. And we can continue in similar fashion in the rest of the inductive definition of R.

In the LBS description of X or any of the components of open subsets, the Lindelöf bead strings are hidden inside the Lindelöf lump L, but the noncompact ones (i.e., the ones of cofinality ω) can be identified and isolated by the following process. First, if B is a bead of X, a bead string in a component A of $\mathfrak{b}$ will be noncompact iff it has a point p of F_B in its closure. [It will be shown that there is exactly one point of F_B in the closure.] We can then get a description like the LBS property for A, except that some of the C_n could be of cofinality ω. This is made possible by:

Lemma 3.2. *Let B be a bead of a space Y with the LBS property, and let A be a component of $\mathfrak{b}$. Any family of disjoint bead strings of infinite cofinality in A is a discrete collection of subsets of A that are closed in the relative topology of A.*

Proof. It is enough to show that the set of initial points of all the bead strings is closed discrete in A, since $S \setminus \{c_0\}$ is open in Y. If this set had an accumulation point $a \in A$, then there could not be disjoint open sets around a and F_B. This is because $F_B \cup A$ is compact, and every neighborhood of F_B must contain all but finitely many of the bead strings, by an argument like that for Lemma 2.3. Since F_B has at most two points, this would violate the Hausdorff property. $\square$

Corollary 3.3. *With A as in Lemma 3.2, every family of disjoint bead strings of infinite cofinality in A is countable.*

Proof. Any choice function from a disjoint family of bead strings gives a closed discrete subspace of A. Since A is Lindelöf, any closed discrete subspace of A is countable. $\square$

Backing up a bit, we can also get a maximal family of disjoint noncompact bead strings for X by taking the one-point compactification of X and treating it like $A \cup F_B$ was treated above when F_B is a singleton.

However, not every maximal family $\{S_n : n \in \omega\}$ can be used in the LBS+ property, because there is no guarantee that the complement L' of the union of the family is Lindelöf. If $\overline{Y} \setminus S_n$ is Lindelöf we can include S_n in a LBS+ representation of A or X, otherwise we omit it, and leave it in the lump L of the LBS+ representation. Since the S_n are a discrete collection of closed sets, the complement of the union of any subfamily is open, and a LBS+ representation of A [including $A = X$] is provided by:

Lemma 3.4. *If Y has a LBS representation $L_Y \cup \bigcup\{S_n : n \in \omega\}$, and L_Y is locally compact, then a LBS+ representation of Y is provided by $L \cup \bigcup\{T_n : n \in \omega\}$ where for each n there exists m such that $S_n = T_m$, and if $T_k \neq S_n$ for all n, then T_k is a bead string of cofinality ω such that $L_Y \setminus T_k$ is Lindelöf.*

Proof. It remains to show that L is Lindelöf, where $L = L_Y \setminus \bigcup\{T_k : L_Y \setminus T_k \text{ is Lindelöf }\}$.

The LBS representation of Y makes L_Y disjoint from all the S_n, and since $\{T_n : n \in \omega\}$ is a discrete family of closed sets, their initial points (and with them the sets themselves) can be put into disjoint open subsets U_k of Y. Local compactness and Lindelöfness of L_Y makes it σ-compact along with any set of the form $L_Y \setminus \bigcup\{U_k : k \in Z\}$ for any $Z \subset \omega$.

Let $Z = \{j : L_Y \setminus T_j \text{ is Lindelöf }\}$. Then Z includes all j for which T_j coincides with some S_n, of course. Local compactness makes $U_k \setminus T_j$ σ-compact for all $j \in Z$, and so

$$L = L_Y \setminus \bigcup\{T_j : j \in Z\}$$

is σ-compact, hence Lindelöf. $\square$

So, for every locally compact space Y with the LBS property, we can extend a LBS representation of Y to a LBS+ representation as given by Lemma 3.4, shrinking (often properly!) the Lindelöf lump of the LBS representation.

We return now to the case of X and of components of $\mathfrak{b}$ where B is a bead in X. For each such set Y, let $\mathcal{S}(Y)$ be the family of bead strings given by a fixed choice of a LBS+

representation of Y, such that the set of bead strings was maximized by first taking a maximal family of noncompact bead strings, and then using all the strings S for which $L_Y \setminus S$ was Lindelöf.

For each $S \in \mathcal{S}(Y)$, let $C(S)$ be the associated well-ordered family of cut points of S, and let $\mathcal{C}(Y) = \{C(S) : S \in \mathcal{S}(Y)\}$.

Let $R(0) = \bigcup \mathcal{C}(X)$. This time, we let $R(1) = R(0) = R$ only if $\mathcal{S}(A) = \emptyset$ for all components A of $\mathfrak{b}$ where B is a bead in the LBS+ representation of X. In other words, the only noncompact bead strings in any of the components of any of the $\mathfrak{b}$ are those S of countable cofinality for which the complement of S is not Lindelöf.

As before, we let

$$E(1) = \{x : x \text{ is an essential cut point of some bridge } P \text{ of some bead } B \text{ of } X\}.$$

and now we let

$$C(1) = \bigcup \{\cup \mathcal{C}(A) : A \text{ is a component of } X \setminus R(0)\}.$$

And, as before, $R(1) = R(0) \cup E(1) \cup C(1)$.

The rest of the inductive definition of R goes through with the obvious changes from the preceding section. At the end of Section 8 it will be shown how to handle those bead strings which were omitted due to the complement not being Lindelöf.

4. The Ladder Lemma and resulting simplifications

5. Completing the proof of Theorem 1.3

6. Eversions

In order to show Theorems 1.4 and 1.5 for general X, we will use a process which we call eversion, a sort of turning of the space inside out, informally speaking. Starting with members of $\mathcal{L}_1$ and continuing with connected pieces as these are broken down in a transfinite induction, the process is focused on points p that do not have countable neighborhood bases in a piece that contains them. In this section and the next two, we will focus on the following kinds of points.

Definition 6.1. Let L be a connected Lindelöf subspace of X. A point p of L is a *t-eversion [resp. t_0-eversion] point of* L if $L \setminus \{p\}$ contains a bead string of uncountable cofinality [resp. of cofinality ω] with p in its closure.

For those L that are locally compact and locally connected as well [as they will be all through the analysis], the t-eversion points coincide with those where $L \setminus \{p\}$ contains a relatively closed copy of an ordinal of uncountable cofinality. The use of t_0-eversion points is optional and only comes into play in the optional Section 10. Another kind of eversion point will come into play in Section 9:

Definition 6.2. Let L be a connected, locally connected, Lindelöf subspace of X. A point p of L is a *χ-eversion point* if $L \setminus \{p\}$ has uncountably many components, and every component of $L \setminus \{p\}$ is Lindelöf.

The relevance of χ-eversion points is that they are precisely the non-t-eversion points without a countable base in the relative topology of L. In particular, if L is countably tight, they are the points of L without countable neighborhood bases.

If p is a t-eversion point, then any bead string of uncountable cofinality with p in its closure will be part of some non-Lindelöf component of $X \setminus \{p\}$. As in the earlier sections, we add designated cut points of such components to a growing rim-finite subspace. The difference is that in the earlier sections, p came from an earlier stage in the building of the subspace. Here, p could be buried deep inside some Lindelöf lump from the previous stage. The following example shows why this can make a big difference; in particular, it helps explain why the rim-finite subspace W in Theorem 1.4 is not closed.

Example 6.3. ("The Long-haired Sphere"). Let J be the long line, expressed as the set of all $\alpha + r$ and all $-(\alpha + r)$ such that $\alpha \in \omega_1$ and $r \in [0, 1)$ with the obvious order and resulting topology making it a nonmetrizable countably compact 1-manifold. Let J^* be the two-point compactification $J \cup \{-\infty, \infty\}$ with the obvious order. The underlying set for the long haired sphere is $S^2 \times J^*$, and its topology is analogous to that of $A[0, 1]$.

Each point $(x, -\infty)$ of $S^2 \times \{-\infty\}$ has a complement in which one of the components is a copy of $J \cup \infty$. In analogy with the preceding sections, we might try to add $(x, -\infty)$ to R and in the next step to add the copy that converges to it, but if we do this for all such points, we wind up adding the whole long haired sphere, and thereby lose rim-finiteness. Some new distinctions are in order.

Definition 6.4. A t-eversion point p of L is *free-floating in L* if the only injective sequences that converge to it are in bead strings of $L \setminus \{p\}$ with p in the closure. Otherwise p is *attached in L*.

In Example 6.3, (x, ∞) is free-floating in the whole space, but none of the $(x, -\infty)$ are. In the first case, the subspace $\{x\} \times (J \cup \{-\infty\})$ can be looked upon as a bead string converging to (x, ∞); however, it would not be good to include the initial point in the rim-finite subspace we will be building. But $\{x\} \times J$ can be added for all $x \in S^2$.

Some t-eversion points of L may be cutpoints of L, and this includes both free-floating and attached t-eversion points. A trivial way is for p to be the boundary point of a balloon in L. But at least one component of $L \setminus \{p\}$ will also have boundary points of L in its closure, and when L is open in X these will be distinct from p itself. If there is more than one such component, then p is a cut point of L. Unlike in the case of the bridges that were encountered in earlier sections, these components may have more than one boundary point in the closure, and perhaps even denumerably many. But as with bridges, there can only be finitely many with p in the closure, and the proof of Theorem 2.3 goes through with hardly any change.

Definition 6.5. Let p be a point of a locally compact, connected, locally connected subspace L of X, such that $\overline{L} \setminus L$ is closed discrete. Let $q \in \overline{L} \setminus L$. A *meta-bridge in L from p to q* is a component of $L \setminus \{p\}$ with q in its closure.

Of course, since L is connected and locally connected, every component of $L \setminus \{p\}$ has p in its closure, and local compactness ensures that there will be only finitely many meta-bridges from p to points of $\overline{L} \setminus L$.

7. The double induction that gives W

The inductive construction of W as in Theorem 1.4 begins with a choice of a t-eversion point $p_0(L)$ for each $L \in \mathcal{L}_1$ where there is such a point. [This happens iff L is not countably tight.] The construction of W is a double induction beginning with:

$$W_0(0) = R \cup \{p_0(L) : p_0(L) \text{ is free-floating in } L \text{ and } L \in \mathcal{L}_1.$$

We could make the analogy between $W_0(0)$ and $R(0)$ stronger by putting all $p_0(L)$ into $W_0(0)$, thus treating the relevant L the way X was treated earlier. But this could only be continued for finitely many steps as the example of the long-haired sphere indicates. So we start setting the pattern already in this initial step. The next step is to define $V_0(1)$ in analogy with $C(1)$, but with one difference that is explained in the following paragraph.

Let $\mathcal{V}_0(L) = \{A : A$ is a non-Lindelöf component of $L \setminus \{p_0(L)\}$, and let $\mathcal{V}_0(X) = \bigcup\{\mathcal{V}(L) : L \in \mathcal{L}_1\}$. For each $A \in \mathcal{V}_0(X)$ let $V_0^1(A)$ be the set of ends of beads in a LBS-representation of A *except for initial points of bead strings.*

The italicized portion refers to the bead strings that converge on $p_0(L)$ in L and begin at the Lindelöf lump $L(A)$ given by the chosen LBS representation of $A \subset L$. But since we are adding the initial points of bead strings to $L(A)$, the resulting Lindelöf set is not open. Also if $p_0(L)$ is not free floating, it is added to the Lindelöf meta-bridges to whose closure it belongs. In the process, finitely many components of $L \setminus \{p_0(L)\}$ may be fused into one.

For each $L \in \mathcal{L}_1$, let $G(L) = \bigcup\{A \setminus \overline{L(A)} : A \in \mathcal{V}_0(L)\}$ If $p_0(L)$ is free-floating, let $G^+(L) = G(L) \cup \{p_0(L)\}$, otherwise let $G^+(L) = G(L)$. Because $\bigcup \mathcal{V}_0(L)$ is open, so are $G(L)$ and $G^+(L)$. These facts will help in showing rim-finiteness of W.

The analogue of $C(1)$ is $V_0(1) = \bigcup\{V_0^1(A) : A \in \mathcal{V}_0(X)\}$. [We use V rather than C because the analogous sets are no longer closed in X.] Except for those exceptional initial points, each $A \in V_0(X)$ is being treated like the components of $X \setminus R(0)$ were in defining $C(1)$.

The analogy between $E_0(1)$ and $E(1)$ is even closer. For all $A \in \mathcal{V}_0(X)$, let

$$Pons(A) = \{P : \text{ is a bridge of a bead of the chosen LBS-representation of } A.$$

If $A \in \mathcal{V}_0(X)$, each P in $Pons(A)$ is a bridge of a bead with an end in $V_0^1(A)$, even if the other end is an initial point that is missing from $V_0^1(A)$. And now,

$$E_0(1) = \bigcup\{E(P) : P \in Pons(A), A \in \mathcal{V}_0(X)\}.$$

Finally, $W_0(1) = W_0(0) \cup V_0(1) \cup E_0(1)$.

It is $W_0(2)$ that really sets the pattern for the general $W_0(\alpha)$ by tightening up the analogy with $R(\alpha)$. All end points of beads of non-Lindelöf components of $L \setminus W_0(1)$ are within the open set $G+(L)$, as are end points of beads that arise between successive points of the $E(P)$

making up $E_0(1)$. These endpoints make up $V_0(2)$, and none of them are initial points of bead strings that were the exception in forming $V_0(1)$.

And so, in forming $W_0(2)$, the components of $X \setminus W_0(1)$ can be treated exactly like those of $X \setminus R(1)$ were treated in forming $R(2)$, with just a change in notation. This continues to hold true for all α in place of 2, until we arrive at θ_1 such that all components of $X \setminus W_0(\theta_1)$ are Lindelöf, when we let $W_1 = W_0(\theta_1)$.

Once W_α has been defined, the process in going from it to $W_{\alpha+1}$ is essentially a repeat of the process that we have used in defining W_1.

When α is a limit ordinal, we simply let W_α be the union of the earlier W_β, without taking the closure. The induction ends when we arrive at θ such that all components of $X \setminus W_\theta$ are countgably tight; in other words, there are no t-eversion points in $X \setminus W_\theta$. We let $W = W_\theta$.

8. Completing the proof of Theorem 1.4

Lemma 8.1. *Each component of $X \setminus W$ is locally connected and locally compact.*

9. Reduction to the first countable case

The reduction in this section involves locating χ-eversion points in the various $L \in \mathcal{L}_2$, which are countably tight. So for each point $p \in L \in \mathcal{L}_2$, $L \setminus \{p\}$ is the topological direct sum of Lindelöf components. A χ-eversion point of L is one where there are uncountably many components. The following example shows why we treat all these the way we treated attached t-eversion points in arriving at Theorem 1.4 and why we may not be constructing any more rim-finite subspaces.

Example 9.1. *("Disc Trees").* We will be using the set-theoretic definition of a *tree* as a partially ordered set T such that the set of predecessors of every element of T is well-ordered. Every tree can be used to construct a locally compact, connected, locally connected, monotonically normal space $D(T)$ by replacing the elements of the tree by copies of the closed unit disc D_2, and adding points at the ends of branches with no maximal element. We illustrate one way of doing this when T is the full $\mathfrak{c}$-ary tree of height ω. Elements of this tree are the finite sequences σ of real numbers ordered by end extension, including the empty sequence. We replace each σ with a copy $D(\sigma)$ of D_2 in the following fashion.

Let $\phi_\sigma : \mathbb{R} \to D(\sigma)$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ be bijections. Given $(x, y) \in \mathbb{R}^2$, let $z = \psi(x, y)$ and identify $\phi_\sigma(x) \in D(\sigma)$ with the copy of the point $(1, 0)$ in $D(\sigma\hat{}z)$, where $\hat{}$ denotes concatenation. So, beginning with $D(\emptyset) = \{\phi_\emptyset(x) : x \in \mathbb{R}$, we are building a disc tree where the various points in one $D(\sigma)$ are identified with the points $(1, 0)_\tau$ on the rims of $\mathfrak{c}$-many $D(\tau)$ where τ is an immediate successor of σ in T, and each such τ is assigned to a unique point of $D(\sigma)$.

The underlying set of $D(T)$ is the union of the D_σ with points on the last level of the full $\mathfrak{c}$-ary tree of height $\omega + 1$: this level consists of all ordinary sequences $\rho : \omega \to \mathbb{R}$, and $\sigma < \rho$ iff σ is an initial segment of ρ. The topology on $D(T)$ is adapted from the coarse wedge topology on T. For trees of height $\leq \omega$ this is the one where a base for the neighborhoods of $t \in T$ consists of all subsets of the form $V_t \setminus (V_{t_1} \cup \cdots \cup V_{t_n}\}$ where each t_i is an immediate successor of t and $V_p = \{q \in T : p \leq q\}$ The adaptation to $D(T)$ has a base where each point $p \in D(\sigma)$ has neighborhoods which include:

(1) all points in the closed ε neighborhood of p in $D(\sigma)$;

(2) all points in all but finitely many of the other discs where p is identified with a point of that disc;

(3) all points in the closed ε neighborhood of p in the finite set of exceptions for (2); and

(4) all points of $D(T)$ above every point described in (1), (2) and (3), other than p itself.

[Formally, a point ρ on the top level of $D(T)$ is *above* any point of $D(\sigma)$ whenever ρ extends σ; and if $r \in D_{\tau_1}$ then r is *above* a point $q \in D(\tau)$ if τ_1 extends τ and either
(i) $\tau_1 = \zeta$ and $(1, 0)_\zeta$ is identified with q, or
(ii) τ_1 extends a ζ as in (i).]
If ρ is on the top level of $D(T)$, then has a base of compact neighborhoods consists of all $[(1, 0)_\sigma]^\uparrow$ such that ρ extends σ. Here $[(1, 0)_\sigma]^\uparrow$ refers to all points that are equal to or above $(1, 0)_\sigma$ in $D(T)$. It is easy to check that the interior of $[(1, 0)_\sigma]^\uparrow$ is all of it except the bottom point $(1, 0)_\sigma$ itself.

A special role is played by the copies of $(1, 0)$ in each $D(\sigma)$. Except for $\sigma = \emptyset$, they are identified not just with points of the same form in the immediate successors of $D(\sigma)$, but also with some point in the immediate predecessor of $D(\sigma)$, characterized by omitting the last term in σ.

Clearly, each point of this particular $D(T)$ is of character $\mathfrak{c}$. In other words, every point is a χ-eversion point, and there seems to be nothing gained by isolating any rim-finite subspace of $D(T)$. In fact, if $X = D(T)$ then Theorem 1.5 is satisfied by letting

$$\mathcal{L}_3 = \{D(\emptyset)\} \cup \{D(\sigma) \setminus \{(1, 0)_\sigma\} : \sigma \in T \setminus \{\emptyset\}\} .$$

making $D(T) = \bigcup \mathcal{L}_3$.

10. Minimizing the cyclic elements of the leftover pieces

11. Obstacles in the way of further progress

Already with Theorem 1.5, the usefulness of the LBS property seems to have reached a natural end as far as topological properties of families like $\mathcal{L}_3$ are concerned. This is illustrated by two examples of first countable, locally compact, locally connected T_5 spaces, the first of which is an elementary ZFC example.

Example 11.1. The lexicographically ordered unit square is a well known first countable, compact, connected LOTS (and hence locally connected as well). It is not perfectly normal, as shown by the top and bottom edges. These form a closed copy of the Alexandroff Double Arrow space, which is not a G_δ subset. The strategy of the preceding section would have us do something with the components of the complement. These are the individual vertical copies of $(0,1)$, and are ideal for addition to W (or W^+) except for one thing: the closed set that we removed is not locally connected; in fact, it is totally disconnected.

Now, it is true that this subspace is perfectly normal and thus is in line with our general strategy. More importantly, the entire lexicographically ordered square can be incorporated into R^{++} as is, if it shows up in our analysis. But in a "generic" example, the first countable Lindelöf subsets at the end of the Section 9 analysis are *terra incognita*. There is no assurance that we can find a perfectly normal closed non-G_δ subset so easily, nor that the components of its complement are so well behaved as in this example. And the situation could be just as bad as with Example 11.1 where locally connected closed non-G_δ subsets are concerned: there are none in Example 11.1 at all!

Lemma 11.2. *Let L be a compact, connected, first countable LOTS. Every closed, locally connected subset of L is a G_δ.*

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NOTE: It has been decided to split this paper. The first of the new papers will be titled, "The structure of locally compact, locally connected, monotonically normal spaces," and the second will be titled "The structure of T_5 and related locally compact, locally connected in $MM(S)[S]$ models."

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