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e-13 Generalized Metric Spaces III: Linearly Stratifiable Spaces and Analogous Classes of Spaces

This article is concerned with generalizations of concepts like stratifiability and metrizability to arbitrary infinite cardinalities, in a way that uses linear orders in key places. This has resulted in theories which are remarkably faithful generalizations of the theories of stratifiable, metrizable, etc. spaces. For metrizable spaces, the generalization is to the class of (***Tychonoff***) spaces admitting separated uniformities with totally ordered bases; this class is usually referred to as the class of ω_μ-metrizable spaces of arbitrary cardinality ω_μ, but the term "linearly uniformizable spaces" will be mostly used here, under the convention that "spaces" refers to ***Hausdorff*** spaces. The class of linearly stratifiable spaces is a simultaneous generalization of linearly uniformizable spaces and of stratifiable spaces, and most of the theory of stratifiable spaces carries over, including the basic covering and separation properties of ***paracompactness*** and ***monotone normality***. There are generalizations, along the same lines, of σ-spaces and semistratifiable spaces, as well as classes in between the linearly uniformizable spaces and linearly stratifiable spaces, generalizing M_1 spaces and Nagata spaces. Other generalizations, such as the one of **quasi-metrizable** spaces (quasi-metrics are defined like metrics but without symmetry of the distance function), are less well developed in the literature, and will only be touched on here.

The usual definition of linear stratifiability is based on the definition of stratifiable spaces that says they are monotonically perfectly normal, so to speak; this definition is the case $\omega_\mu = \omega$ of the definition of ω_μ-stratifiable spaces, where ω_μ is an infinite cardinal number. A space (X, τ) is said to be **stratifiable over** ω_μ if it is a T_1 space for which there is a map $S: \omega_\mu \times \tau \to \tau$, called an ω_μ**-stratification** which satisfies the following conditions.

(1) $c\ell(S(\beta, U)) \subset U$ for all $\beta < \omega_\mu$ and all $U \in \tau$.
(2) $\bigcup\{S(\beta, U)\colon \beta < \omega_\mu\} = U$ for all $U \in \tau$.
(3) If $U \subset W$, then $S(\beta, U) \subset S(\beta, W)$ for all $\beta < \omega_\mu$.
(4) If $\gamma < \beta < \omega_\mu$, then $S(\gamma, U) \subset S(\beta, U)$ for all $U \in \tau$.

X is called ω_μ**-stratifiable** if ω_μ is the least cardinal for which X is stratifiable over ω_μ. A space is **linearly stratifiable** if it is ω_μ-stratifiable for some infinite ω_μ, and ***stratifiable*** if it is ω-stratifiable. An ω-stratification is called a **stratification**. If condition (1) is omitted, we get the definition of an ω_μ**-semistratification**. The terms **semistratifiable over** ω_μ, ω_μ**-semistratifiable**, **linearly semistratifiable**, **semistratifiable**, and **semistratification** have the obvious definitions. The key theorem that a space is stratifiable iff it is semistratifiable and monotone normal generalizes easily to arbitrary ω_μ. Condition (4) is unnecessary in the case $\omega_\mu = \omega$ but it is needed to make the theories of stratifiable and semistratifiable spaces generalize to higher cardinals. Similar additions make it possible to generalize two characterizations of (semi-)stratifiable spaces and to make them coincide. One is a pair of Heath–Hodel style characterizations in [27] and [17] with their addition of condition (b), which is unnecessary in case $\omega_\mu = \omega$: A T_1-space (X, τ) is stratifiable over ω_μ if, and only if, there exists a family $\{g_\beta\colon \beta < \omega_\mu\}$ of functions with domain X and range τ such the following hold:

(a) $x \in g_\beta(x)$ for all $\beta < \omega_\mu$;
(b) if $\beta < \gamma < \omega_\mu$, then $g_\beta(x) \supset g_\gamma(x)$ for all x;
(c) if, for every $\beta < \omega_\mu$, $x \in g_\beta(x_\beta)$, then the net $\langle x_\beta\colon \beta < \omega_\mu\rangle$ coverges to x; and
(d) for every $F \subset X$, if $y \in c\ell(\bigcup\{g_\beta\colon x \in F\})$ for all $\beta < \omega_\mu$, then $y \in c\ell(F)$.

If condition (d) is omitted, we get a condition equivalent to being semistratifiable over ω_μ.

In [27] there is also a definition of a linearly cushioned pair-base that generalizes that of a σ-cushioned pair-base used in defining M_3 spaces; moreover, the proof that the M_3 concept coincides with stratifiability generalizes in [27] to this more general setting. A collection $\mathcal{P}$ of pairs $P = (P_1, P_2)$ of subsets of a space (X, τ) is said to be a **pair-base** if the members of each pair are open and, for each point x of X and each neighbourhood U of x, there is a pair $(P_1, P_2) \in \mathcal{P}$ such that $x \in P_1$ and $P_2 \subset U$. A collection $\mathcal{C}$ of subsets of a space X is **linearly closure-preserving with respect to** $\leqslant$ if $\leqslant$ is a linear order on $\mathcal{C}$ such that $\bigcup\{c\ell C\colon C \in \mathcal{C}'\} = c\ell(\bigcup \mathcal{C}')$ for any subcollection of $\mathcal{C}' \subset \mathcal{C}$ which has an upper bound w.r.t. $\leqslant$. A collection of pairs $P = (P_1, P_2)$ is **linearly cushioned** with respect to a linear order $\leqslant$ if $c\ell(\bigcup\{P_1\colon P = (P_1, P_2) \in \mathcal{P}'\}) \subset \bigcup\{P_2\colon P = (P_1, P_2) \in \mathcal{P}'\}$ for every subset $\mathcal{P}'$ of $\mathcal{P}$ which has an upper bound with respect to $\leqslant$. Hence in particular, $\mathcal{C}$ is linearly closure-preserving w.r.t. $\leqslant$ if $\{(C, C)\colon C \in \mathcal{C}\}$ is linearly cushioned with respect to $\leqslant$. A regular space X is said to be M_1 **over** ω_μ (respectively M_2 **over** ω_μ) (respectively M_3 **over** ω_μ) if X has a linearly closure-preserving base (respectively a linearly closure-preserving ***quasi-base***) (respectively a linearly cushioned pair-base) with a cofinal set of order type ω_μ. X is **linearly** M_i if it is M_i over ω_μ for some infinite cardinal ω_μ. An ω_μ-M_i space is defined analogously to an ω_μ-stratifiable space.

Clearly, these concepts are numbered in order of increasing generality. More general yet is the concept of having a linearly closure-preserving network of cofinality ω_μ, consisting of closed sets. If $\omega_\mu = \omega$ this gives us the familiar class of σ-spaces. Harris [11], generalizing the Nagata–Siwiec theorem for $\omega_\mu = \omega$, showed that these spaces have a network that is the union of $\leqslant \omega_\mu$ discrete collections. The converse is true if the space is **ω_μ-additive**, meaning that the union of strictly fewer than ω_μ closed sets is closed: this implies that the union of fewer than ω_μ discrete collections is discrete, hence every union of ω_μ discrete collections is linearly closure-preserving with respect to a linear order of cofinality $\mathrm{cf}(\omega_\mu)$. The Heath–Hodel theorem that every stratifiable space is a σ-space [13] generalizes to the theorem that every ω_μ-stratifiable space has a network which is the union of $\leqslant \omega_\mu$ discrete collections, and a linearly closure-preserving network [27]. The theorem that σ-spaces are semistratifiable generalizes to the theorem that a space with a linearly closure-preserving network is linearly semistratifiable [11]. In fact, having a linearly closure-preserving network of cofinality ω_μ consisting of closed sets is equivalent to having a Heath–Hodel function g satisfying (a), (b), and (c) above along with the following condition (e): if $y \in g_\beta(x)$ then $g_\beta(y) \subset g_\beta(x)$. For (c) it is possible to substitute the stronger (c+): if, for every $\beta < \omega_\mu$, $x \in g_\beta(y_\beta)$ and $y_\beta \in g_\beta(x_\beta)$, then the net $\langle x_\beta \colon \beta < \omega_\mu \rangle$ converges to x [11]. Another generalization, that of **elastic spaces**, relaxes the linear order requirement to that of a preorder, but otherwise keeps the pair-base definition of linearly M_3 with the formal restriction that the pair-base is a function; that is, each subset of the space appears as the first element in at most one pair. M. Jeanne Harris showed that this restriction is a mere formality in [11] and [12]: every space with a linearly cushioned pair-base has one which is a function.

Linearly stratifiable spaces enjoy many of the nice properties of the subclass of stratifiable spaces; for example, they are ***monotonically normal*** and (hereditarily) ***paracompact***. There is a subtle hole in the proof of the latter fact in [26] and [27], which is repaired by Harris's theorem. It is also possible to show, more simply, that every open cover in a linearly stratifiable space has an open refinement which is linearly cushioned in it [28]. This refinement condition is equivalent to paracompactness, and "linearly cushioned" can be weakened to "elastic" [26]. Linearly stratifiable spaces have most of the nice preservation properties possessed by stratifiable spaces. For example, the class is closed under the taking of subspaces and closed images, and finite unions of closed subspaces. This also applies to the class of linearly M_2-spaces. The best known of the (much weaker) known preservation properties of M_1 spaces also carries over: if f is a closed irreducible continuous map from a space X that is M_1 over ω_μ, onto a space Y such that for every $y \in Y$, $f^{-1}(y)$ is ω_μ-compact, then Y is linearly M_1 [11]. Finite products of spaces that are ω_μ-stratifiable over the same ω_μ are also ω_μ-stratifiable, as are box products of fewer than ω_μ of them. Both of these results are generalized by the fact that if ω_μ is regular, then the ω_μ-box product of ω_μ or fewer ω_μ-stratifiable spaces is ω_μ-stratifiable: the ω_μ-box product is defined like the box product except that one restricts fewer than ω_μ-many coordinates [3]. (The restriction on agreement in ω_μ is important: $\omega + 1$ and the one-point Lindelöfization of a discrete space of cardinality ω_1 constitute a pair of spaces, one stratifiable and the other ω_1-stratifiable, whose product is not linearly stratifiable – it is not even hereditarily normal.) If a space X is dominated by a collection of closed subsets, each of which is stratifiable over ω_μ, then X is stratifiable over ω_μ. If X and Y are stratifiable over ω_μ and A is a closed subset of X and $f : A \to Y$ is continuous, then $X \cup_f Y$ (the adjunction space) is stratifiable over ω_μ [27].

The celebrated Gruenhage–Junnila theorem that all M_3 spaces are M_2 has been generalized within the class of **ω_μ-additive spaces** (also known as **P_{ω_μ}-spaces**); that is, spaces in which the intersection of strictly fewer than ω_μ open sets is open. The theorem is that every P_{ω_μ} space which is ω_μ-M_3 is also ω_μ-M_2. The problem of whether the P_{ω_μ} condition can be dropped is still open. The notorious problem of whether all three classes are the same also generalizes to linearly M_i spaces; in fact, it is open for all infinite cardinalities ω_μ, even for P_{ω_μ}-spaces. Moreover, where uncountable ω_μ are concerned, we even have a fourth class, the class of spaces M_0 over ω_μ, to add to this coincidence problem. Spaces that are M_0 over ω_μ are defined like spaces M_1 over ω_μ but with "open" replaced by "clopen". That is, a space is **M_0 over ω_μ** if it is a regular space with a linearly closure-preserving base $\mathcal{B}$ of clopen sets, where the linear order on $\mathcal{B}$ has cofinality ω_μ. As might be expected, **linearly** M_0" and "**ω_μ-M_0**" are defined analogously to the same concepts for higher subscripts.

A big advantage of linearly M_0-spaces over the more general linearly M_1-spaces is that they are easily seen to be hereditary; their perfect images are linearly M_1 [11], but not necessarily linearly M_0, at least not when the domain is simply M_0: the closed unit interval is a non-M_0 perfect image of the Cantor set, which is clearly M_0, as is any **strongly zero-dimensional** metrizable space. The strongly zero-dimensional spaces can be characterized as those Tychonoff spaces in which disjoint zero sets can be put into disjoint clopen sets [6, 16.17], [E, 6.2.4] or those which have totally disconnected Stone–Čech compactifications [E, 6.2.12]. All ω_μ-M_0-spaces are strongly zero-dimensional, even in the case $\omega_\mu = \omega$ [14]. Also, every Tychonoff space which is a ***P-space*** [that is, a P_{ω_1}-space] is strongly zero-dimensional; indeed, every zero set is clopen in such spaces since it is a G_δ-set. Remarkably enough, it is not known whether every strongly zero-dimensional ω_μ-stratifiable space is ω_μ-M_0, whatever the value of ω_μ; nor whether every ω_μ-stratifiable space (or every space stratifiable over ω_μ) is strongly zero-dimensional when ω_μ is uncountable. Since stratifiability over ω_μ is preserved on collapsing a closed set to a point, the latter problem is equivalent to whether all ω_μ-stratifiable spaces (or all spaces stratifiable over ω_μ) are **zero-dimensional**, i.e., have a base consisting sets that are both open and closed.

Various well-known equivalences of the M_2-M_1 problem also carry over, some with the addition of ω_μ-additivity. Two generalizations by Harris [11] of a well-known theorem of Heath and Junnila [14] account for several of them, including the problems of whether every closed subspace, or every closed image of an M_1 space is M_1. One generalization says that every linearly M_2-space is the image of a linearly M_1-space under a retraction. The other says that if ω_μ is regular, and if the P_{ω_μ}-space X is stratifiable over ω_μ, then X is the image of a linearly M_1 space under a closed retraction with ω_μ-compact fibers. Some quite general classes of linearly stratifiable spaces are linearly M_1. For instance, if ω_μ is a regular cardinal and X is an ω_μ-stratifiable P_{ω_μ}-space in which every closed subset of X has a linearly closure-preserving neighbourhood base of open sets in which ω_μ is cofinal, then X is linearly M_1 [11]. The condition that X is ω_μ-stratifiable can be formally relaxed to the condition that X is paracompact and has a network which is the union of $\leqslant \omega_\mu$ discrete collections [11]. This generalizes an old result [2] for the case $\omega_\mu = \omega$, while the following generalizes one of Ito [16]: if X is a P_{ω_μ}-space that is M_3 over ω_μ, and every point of X has a closure-preserving open base, then every closed subset of X has a closure-preserving base of open sets [11] (and hence X is linearly M_1).

An important class of linearly stratifiable spaces might be called **linearly Nagata**: these are the ω_μ-Nagata spaces as ω_μ varies over all infinite regular cardinals. The ω_μ-Nagata spaces can be simply characterized as the ω_μ-stratifiable spaces in which each point has a totally ordered neighbourhood base. Of necessity, this base will have cofinality ω_μ if the point is nonisolated. By the foregoing theorems, and the elementary fact that every ω_μ-Nagata space is a P_{ω_μ}-space, it follows every linearly Nagata space is linearly M_1. There are other characterizations of ω_μ-Nagata spaces, including one based on the Nagata general metrization theorem [10]: an ω_μ-Nagata space is a T_1 space with a system $\langle \mathfrak{U}, \mathfrak{S} \rangle$ where $\mathfrak{U}$ and $\mathfrak{S}$ are collections of functions U_β and S_β ($\beta < \omega_\mu$), each with domain X, and such that (1) for each $x \in X$, $\{U_\beta(x)\colon \beta < \omega_\mu\}$ is a base for the neighbourhoods of x, and so is $\{S_\beta(x)\colon \beta < \omega_\mu\}$; (2) for every $x, y \in X$, $S_\beta(x) \cap S_\beta(y) \neq \emptyset$ implies that $x \in U_\beta(y)$; and (3) If $\beta < \gamma < \omega_\mu$, then $S_\beta(x) \supset S_\gamma(x)$ for all x. As usual, (3) is superfluous if $\omega_\mu = \omega$, and we simply have the class of ***Nagata spaces*** then. Another characterization [27] dispenses with $\mathfrak{U}$, requires that each $S_\beta(x)$ be open, and substitutes for (2) the condition that if U is a neighbourhood of x, there exists $\beta < \omega_\mu$ such that $S_\beta(x) \cap S_\beta(y) \neq \emptyset$ implies that $y \in U$. Clearly, any subspace of an ω_μ-Nagata space is ω_μ-Nagata, and any ω_μ-box product of ω_μ-Nagata spaces over the same ω_μ is again ω_μ-Nagata. The closed continuous image X of an ω_μ-Nagata space is likewise an ω_μ-Nagata space provided that, for each point $x \in X$, there exists a totally ordered neighbourhood base. If X is ω_μ-Nagata over an uncountable regular ω_μ, then X a P_{ω_μ}-space and hence is strongly zero-dimensional. As is well known, a space X satisfies $\dim(X) = 0$ iff X is normal and strongly zero-dimensional, and X is **ultraparacompact** iff it is paracompact and strongly zero-dimensional. Since linearly stratifiable spaces are paracompact and hence normal, the ω_μ-Nagata spaces have both of these other properties if ω_μ is uncountable. (And so too, of course, do all linearly M_0 spaces and all linearly stratifiable P-spaces.) This gives the theory of these kinds of linearly Nagata a different flavor from that of Nagata spaces (the countable case $\omega_\mu = \omega$).

An easy example of a space that is M_0 over a regular cardinal ω_μ and is ω-M_0 at the same time is obtained by isolating all but the last point of $\omega_\mu + 1$, taking the product of the resulting space with $\omega + 1$, and removing every nonisolated point except $\langle \omega, \omega_\mu \rangle$. The set of all open sets containing this point is a closure-preserving clopen base for the point, and the isolated points can be grouped either horizontally or vertically, with initial segments being clopen in either case. This is also an example a space that is M_0 over ω_μ but is not linearly Nagata. The converse problem, whether an ω_μ-Nagata space is necessarily ω_μ-M_0 if ω_μ is regular uncountable, is unsolved.

Linearly uniformizable spaces have a long history, due to the fact that they can be characterized by distance functions that satisfy the usual definition of a metric, except that the distances are not necessarily real numbers, but rather take on their values in an ordered Abelian group (often the additive group of an ordered field). Hausdorff [8, p. 285] introduced the use of such distance functions to general topology, and it was shown that a space is linearly uniformizable iff it admits such a generalized metric. Important examples of such generalized metrics are valuations, which play an important role in algebraic number theory [24]. Many well-known metrization theorems have generalizations that say when a space is linearly uniformizable: The Urysohn Metrization Theorem [23]; the Nagata–Smirnov Theorem [29]; Frink's Metrization Theorem, Bing's Metrization Theorem, Nagata's Generalized Metrization Theorem (the one on which the definition of a Nagata space is based) and several other [20]. The Morita–Hanai–Stone Theorem generalizes to the theorem that a closed map from a ω_μ-metrizable space to another space has ω_μ-metrizable image iff the boundary of each point-inverse is ω_μ-compact [20].

Linearly uniformizable spaces with bases of uncountable cofinality (in other words, ω_μ-metrizable, nonmetrizable spaces) are both linearly Nagata and linearly M_0. In a uniform space, the intersection of every descending sequence of entourgages with no last element is an equivalence relation. Hence, any uniform space with a linearly ordered base of uncountable cofinality has a (linearly ordered) base of equivalence relations; these partition the space into clopen sets. Well-ordering the members of the partitions, with members of coarser partitions preceding the members of the finer partitions, gives a linearly closure-preserving base of clopen sets – the linearly M_0 property. Bases like these are well suited for showing that ω_μ-box product of ω_μ-many ω_μ-metrizable spaces is ω_μ-metrizable and that a space is ω_μ-metrizable for uncountable regular ω_μ iff it embeds in a ω_μ-box product of ω_μ-many discrete spaces. Monotone normality and ultraparacompactness of linearly uniformizable nonmetrizable spaces follow easily from the fact that the base given by these partitions is a tree by reverse inclusion. For ultraparacompactness, the $\supset$-minimal members of a tree base $\mathcal{B}$ which can be put in some member of the open cover $\mathcal{U}$ constitute a partition into clopen sets refining $\mathcal{U}$. For a point x and an open set U containing x, one can let U_x be any member B whatsoever of $\mathcal{B}$ that satisfies $x \in B \subset U$, and then the Borges definition of monotone normality follows from the fact that if U_x meets V_y, then either $U_x \subset V_y$ or $V_y \subset U_x$. Indeed, every tree base for a space is a **base of rank** 1, which means that any two members are either disjoint or related by $\subset$. Spaces with rank 1 bases are called **non-Archimedean spaces**, and actually coincide with spaces with tree bases [19]. The natural common generalization of non-Archimedean and metrizable spaces is that of **proto-metrizable** spaces. These are the spaces with rank 1 pair-bases [7]; $\mathcal{P}$ is a **pair-base of rank** 1 if whenever $\langle P_1, P_2\rangle$ and $\langle P_1', P_2'$ are in $\mathcal{P}$ and $P_1 \cap P_1' \neq \emptyset$, then either $P_1 \subset P_2'$ or $P_1' \subset P_2$. These spaces share many of the nice properties common to metrizable and non-Archimedean spaces, including paracompactness and monotone normality.

Non-Archimedean spaces are suborderable but not all orderable – the Michael line is a standard example [15, 21] of a non-orderable non-Archimedean space. There even exist examples of non-orderable ω_μ-metrizable spaces for all uncountable cofinality ω_μ. This is in contrast to the case of strongly zero-dimensional metrizable spaces (the cofinality $= \omega$ case), all of which are linearly orderable. In fact, a space is metrizable and strongly zero-dimensional iff it is metrizable, linearly orderable, and totally disconnected [9]. Another characterization is that these are the spaces that can be given a compatible **non-Archimedean metric**, one that satisfies the **strong triangle inequality**: given any three points x, y, z, one has $d(x,z) \leqslant \max\{d(x,y), d(x,z)\}$ [5]. If ω_μ is uncountable regular, then every ω_μ-metrizable space can be given a distance function satisfying this property, with values an ordered Abelian group.

There are a few aspects of the theory of metrizable spaces that do not carry over to linearly uniformizable spaces without modification. One is that, for a ω_μ-metric space to be ω_μ-compact (meaning: every open cover has a subcover of cardinality $< \omega_\mu$) it is not enough for it to be complete and totally bounded. For completeness one must substitute the stronger concept of supercompleteness [1]; the two concepts coincide for metric spaces. Sometimes one must use extra qualities of the cardinal ω to have a really satisfactory extension of some classical result. For example, the elementary fact that ${}^{\omega}2$ with the product topology is compact only generalizes to ***weakly compact cardinal***s ω_μ in place of ω when the ω_μ-box product topology is used. Classical characterizations of the Cantor set (the only totally disconnected, compact, dense-in-itself metrizable space) and the irrationals (the only zero-dimensional, ***nowhere locally compact***, ***completely metrizable***, ***separable*** space) only generalize for weakly compact cardinals and ***strongly inaccessible cardinal***s, respectively [19], and one must substitute spherical completeness for ordinary completeness.

In principle, almost every "generalized metric" property can be effectively generalized with judicious uses of total orderings. Sometimes, as with metrizable and non-Archimedeanly metrizable spaces, two or more distinct classes coalesce for uncountable regular ω_μ. One such example is that of quasi-metrizable and ***non-Archimedeanly quasi-metrizable*** spaces [22]. The argument in [22] can be easily modified to show that the uncountable analogues of γ***-spaces*** also coincide with those of quasi-metrizable spaces.

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