Diagonalizable and related spaces

In the first issue of the new journal *Applied General Topology,* Arhangel'skiĭ [1] called a space with a binary operation a *semitopoid* if the operation is separately continuous and a *topoid* if the operation is jointly continuous. Thus the [semi]topological semigroups are the associative [semi]topoids.

Arhangel'skiĭ also introduced the concept of a diagonalizable space:

Definition 1. A space X is *diagonalizable at* e *[*resp. *continuously diagonalizable at* e*]* if there is a binary operation on X with identity element e, such that the operation is separately [*resp.* jointly] continuous at e. X is *diagonalizable [*resp. *continuously diagonalizable]* if it is [continously] diagonalizable at every point.

Separate continuity at e means that the maps $\ell_x : \{x\} \times X \to X$ and $r_x : X \times \{x\} \to X$ are both continous at e for all $x \in X$, while "joint continuity at e" refers to continuity at each point on the X*-cross at* e, the set $\{e\} \times X \cup X \times \{e\}$.

Diagonalizability at some point thus generalizes the property of admitting a semitopoid with identity, while continuous diagonalizability at some point bears the same relation to being a topoid with identity.

Theorem 1. *If* X *is a space with a singleton* $\{e\}$ *which is the intersection of a countable collection of clopen subsets of* X*, then* X *can be made into a topological monoid (i.e., a topological semigroup with identity).*

Proof. Let $\{e\} = \bigcap_{n=0}^{\infty} U_n$ where each U_n is clopen and $U_{n+1} \subset U_n$ for all n. If $x \in X$ and $x \neq e$, then $x \in U_n \setminus U_{n+1}$ for some unique n. If then $y \in U_{n+1}$, let $xy = yx = x$.. If $y \notin U_n$, then switching y with x and altering the subscript on U gives $yx = xy = y$. If $z \in U_n \setminus U_{n+1}$, let $xz = x$ and $zx = z$. Finally $ee = e$. Intuitively, if x and y are at different "distances" from e, then the factor further out takes precedence, otherwise the first factor takes precedence. It is easy to see that this operation is associative: if one member of a threefold product is further out than the rest, it predominates; if one member is closer in than the rest, it is absorbed; and of three elements equally far away, the leftmost factor predominates.

To see that the operation is continuous, note that any net converging to $x \in U_n \setminus U_{n+1}$ is eventually in $U_n \setminus U_{n+1}$, while any net converging to e is eventually in every U_m. Hence if $\langle x_\alpha \rangle \to x$ and $\langle y_\alpha \rangle \to y$ then the products eventually mimic the behavior of the products of the points they are converging to. For example, if $y \in U_{n+1}$ (this includes the case $y = e$) then eventually y_α is in U_{n+1} while x_α is eventually in $U_n \setminus U_{n+1}$, so eventually $x_\alpha y_\alpha = x_\alpha \to x = xy$, etc. □

Theorem 2. *If* X *is a space with a singleton* $\{e\}$ *which is the intersection of a chain of closed neighborhoods of* e*, then* X *is continuously diagonalizable at* e*.*

Proof. Let $\{N_\xi : \xi < \kappa\}$ be a well-ordered family of closed neighborhoods of e and define the operation similarly to the above, with $xy = yx = x$ whenever $y \in N_\xi$ and $x \notin N_\xi$ for some ξ, and $xz = x$, $zx = z$ if x and z are in all the same N_η's. Associativity is clear as before. The operation is jointly continuous at (x, e) and (e, x): if $x_\alpha \to x \neq e$ while $y_\alpha \to e$, then eventually x_α is in the complement of some N_ξ while y_α is eventually in N_ξ, so $x_\alpha y_\alpha = y_\alpha x_\alpha = x_\alpha \to x = xe = ex$; while if x_α also converges to e, then since the product $x_\alpha y_\alpha$ is always one or the other of x_α, y_α, it too will converge to e. □

Corollary. *If X is a regular lob-space (meaning: every point has a linearly ordered local base) then X is continously diagonalizable.* □

Remark. There is a problem with joint continuity, indeed separate continuity of the above operation if $x \in N_\xi \setminus intN_\xi$ and there exists $z \in N_\xi \setminus N_{\xi+1}$, because if $x_\alpha \to x$ and $x_\alpha \notin N_\xi$ for all α while $z_\alpha = z$ for all α, then $z_\alpha x_\alpha = x \to x \neq zx$. In fact, while the long ray can be given an operation making it a topological monoid, the long line cannot even be made into a semitopoid with identity.

Problem. Can S^2 be made into a topoid with identity?

As is well known, S^2 cannot be made into a topoid which is a loop—a set with a binary operation with identity, in which the equation $xy = z$ has a unique solution y for each given x and z, and a unique solution x for each given y and z. On the other hand, S^2 can be made into a semitopological monoid in a natural way, by extending addition on $\mathbb{R}^2$ to the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$, letting $x + \infty = \infty + x = \infty$ for all x. This operation is separately continuous at ∞, but not jointly continuous since $-n$ and n both converge to ∞ but their sum stays at 0.

In [2], Arhangelskiĭ defined an even weaker concept than diagonalizability, involving:

Definition 2. A *partial product* on a set X is a function from a subset Y of $X \times X$ to X. We use the notation ab for the image of $\langle a, b\rangle$ whenever $\langle a, b\rangle \in Y$, and call Y the *domain* of the partial product. The partial product *has identity* e if $\langle e, x\rangle$ and $\langle x, e\rangle$ are in Y and $ex = xe = x$ for all $x \in X$.

Definition 3. A space X is *partially diagonalizable* if there is a partial product on X with identity e and domain Y, and an open set V whose closure is a neighborhood of e, such that:

(a) the product operation xb is left continuous on Y at $b = e$ for all $x \in X$; that is, the restriction to Y of each map $\ell_x : \{x\} \times X \to X$ is continuous at e; and

(b) for every $x \in V$ there is a G_δ-subset Q_x of X containing e such that the product qx is defined for each $q \in Q_x$ and is (right) continuous on $Q_x \times \{x\}$ at $q = e$, with respect to the G_δ-topology on X and its subspaces.

If the partial product can be defined so that $e \in V$ then X is said to be *strictly partially diagonalizable.*

Partial diagonalizability is still strong enough to put significant restrictions on X. For example, the one-point compactification of an uncountable discrete space is not partially diagonalizable at the nonisolated point [2]. Also:

Example. Let X be the union of the right and top edges of $\omega_1 + 1 \times \omega + 1$. This product is diagonalizable, but X itself is not diagonalizable at $\langle \omega_1, \omega\rangle$.

The example illustrates:

Theorem 3. *Let X be a suborderable space and let $e \in X$. The following are equivalent:*

1. *X is partially diagonalizable at e*
2. *e has a totally ordered local base*
3. *X is continuously diagonalizable at e.*