DOWKER SPACES REVISITED

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Abstract. In 1951, Dowker proved that a space X is countably paracompact and normal if and only if $X \times \mathbb{I}$ is normal. A normal space X is called a Dowker space if $X \times \mathbb{I}$ is not normal. The main thrust of this article is to extend this work with regards α-normality and β-normality. Characterizations are given for when the product of a space X and $(\omega + 1)$ is α-normal or β-normal. A new definition, *α-countably paracompact*, illustrates what can be said if the product of X with a compact metric space is β-normal. Several examples demonstrate that the product of a Dowker space and a compact metric space may or may not be α-normal or β-normal. A collectionwise Hausdorff Moore space constructed by M. Wage is shown to be α-normal but not β-nornal.

1. Introduction

A topological space X is called *β-normal* (*α-normal*) if for each pair of closed disjoint subsets A, $B \subset X$ there are open sets U, $V \subset X$ such that $\overline{A \cap U} = A$, $\overline{B \cap V} = B$ and $\overline{U} \cap \overline{V} = \emptyset$ ($U \cap V = \emptyset$, respectively). This notion was introduced by Arhangel'skii and Ludwig in 1999 [1] and others have worked on the topic ([2], [6], [7], [9], [10], [11]) . In 1951, Dowker proved that a space X is countably paracompact and normal if and only if $X \times \mathbb{I}$ is normal [4]. A normal space X is called a *Dowker space* if $X \times \mathbb{I}$ is not normal, where $\mathbb{I}$ is the unit interval with the usual topology. The main thrust of this article is to extend this work with regards α-normality and β-normality.

Section 2 is devoted to extending Dowker's characterization of countably paracompact normal spaces to α-normal and β-normal spaces. The two main results of the section, Theorem 2.3 and Theorem 2.9, characterize when $X \times (\omega + 1)$ is α-normal and when this product is β-normal. Corollaries 2.10 and 2.11 show what happens if in the forward supposition of Dowker's theorem, normality is replaced with α-normality and β-normality (respectively). A new definition, *α-countably paracompact*, is introduced in this section and Corollary 2.7 shows that if $X \times (\omega+1)$ is β-normal, then X is β-normal and α-countably paracompact. The converse is an open question.

In Section 3, examples of Dowker spaces whose product with the unit interval are α-normal and β-normal (respectively) are given. Curiously, this section also exhibits Dowker spaces whose product with the unit interval are *not* α-normal and β-normal (respectively).

In Section 4 a collectionwise Hausdorff Moore space constructed by M. Wage is shown to be α-normal but not β-nornal. The article concludes with a list of open

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questions in Section 5. Throughout the paper, unless otherwise stated, a "space" is a T_1, regular, topological space. The ordinals ω, ω_1 are used to denote the first two infinite cardinals. Readers may refer to Engelking [5] for undefined terms.

2. Extending Dowker's result to α-normality and β-normality

To start, we restate Dowker's characterization of countably paracompact normal spaces as a fact for later reference purposes.

Fact 2.1. *A topological space X is countably paracompact and normal if and only if $X \times \mathbb{I}$ is normal.*

In light of Dowker's characterization, it is natural to ask what would happen if one weakened the supposition that $X \times \mathbb{I}$ is normal to α-normal. We begin with a characterization of α-normal spaces. The proof is left to the reader.

Lemma 2.2. *A topological space X is α-normal if and only if for every pair H and K of disjoint closed subsets of X there exists an open set U of X such that $\overline{H \cap U} = H$ and $\overline{U} \cap K$ is nowhere dense in K.*

It should be noted, that in the standard proof of Fact 2.1, the reverse direction only uses the existence of a non-trivial convergent sequence in the space $\mathbb{I}$ [5]. So we can actually say $X \times (\omega + 1)$ is normal if and only if X is normal and countably paracompact. We now have the following.

Theorem 2.3. *Let X be a T_1 space. The product $X \times (\omega + 1)$ is α-normal if and only if*

(1) *X is α-normal, and*

(2) *if $\{F_n : n \in \omega\}$ is a family of closed sets and $F = \bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} F_k)$, and E is a closed subset of X disjoint from F, then there is a family $\{W_n : n \in \omega\}$ of open sets such that $W_n \cap F_n$ is dense in F_n and $\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} W_k)$ is nowhere dense in E.*

Proof. Let $\{F_n : n \in \omega\}$ be a family of closed sets and $F = \bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} F_k)$, and suppose E is a closed subset of X disjoint from F. Then $A = (\bigcup\{F_n \times \{n\} : n \in \omega\}) \cup (F \times \{\omega\})$ and $B = E \times \{\omega\}$ are disjoint closed subsets of $X \times (\omega + 1)$. Since $X \times (\omega + 1)$ is α-normal, there is an open subset W of $X \times (\omega + 1)$ such that $\overline{W \cap A} = A$ and $\overline{W}$ is nowhere dense in B.

For each $n \in \omega$, define $W_n = \{x \in X : (x, n) \in W\}$. Clearly, $W_n \cap F_n$ is dense in F_n. Since $\overline{W}$ is nowhere dense in B, $\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} W_k)$ is nowhere dense in E.

Conversely, let A and B be disjoint closed subsets of $X \times (\omega + 1)$. Consider the sets

- $A_n = \{x \in X : (x, n) \in A\}$,
- $A_\omega = \{x \in X : (x, \omega) \in A\}$,
- $B_n = \{x \in X : (x, n) \in B\}$, and
- $B_\omega = \{x \in X : (x, \omega) \in B\}$.

Since X is α-normal and A_ω and B_ω are disjoint closed subsets of X, there are disjoint open subsets U_A and U_B of X such that $cl_X(U_A \cap A_\omega) = A_\omega$ and $cl_X(U_B \cap B_\omega) = B_\omega$.

By (2) and the α-normality of X, there are open subsets U_n and V_n of X such that

(a) $U_n \cap A_n$ is dense in A_n for each $n \in \omega$,

(b) $\displaystyle\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} U_k)$ is nowhere dense in B_ω,

(c) V_n is dense in B_n for each $n \in \omega$,

(d) $\displaystyle\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} V_k)$ is nowhere dense in A_ω, and

(e) $U_n \cap V_n = \emptyset$ for each $n \in \omega$.

Let $H = A_\omega \setminus \bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} V_k)$ and $K = B_\omega \setminus \bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} U_k)$. For each $d \in H \cap U_A$, there is an open subset O_d and $n_d \in \omega$ such that

- $O_d \subset U_A$, and
- $O_d \cap \bigcup_{k=n_d}^{\infty} V_k = \emptyset$.

Similarly, for each $d \in K \cap U_B$, there is an open subset O_d and $n_d \in \omega$ such that

- $O_d \subset U_B$, and
- $O_d \cap \bigcup_{k=n_d}^{\infty} U_k = \emptyset$.

Let

$$U = \bigcup\{U_n \times \{n\} : n \in \omega\} \cup \bigcup\{O_d \times [n_d, \omega] : d \in H \cap U_A\}, \text{ and}$$

$$V = \bigcup\{V_n \times \{n\} : n \in \omega\} \cup \bigcup\{O_d \times [n_d, \omega] : d \in K \cap U_B\}.$$

Note that U and V are disjoint subsets of $X \times (\omega + 1)$, U is dense in A, and V is dense in B. Hence $X \times (\omega + 1)$ is α-normal.

□

The β-normal case is similar to the α-normal case, albeit more complicated. For convenience, we break up the theorem into two parts.

Lemma 2.4. *Let X be a T_1 space. If $X \times (\omega + 1)$ is β-normal, then:*

(1) *X is β-normal and*

(2) *if $\{F_n : n \in \omega\}$ is a family of closed sets and $F = \displaystyle\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} F_k)$, and E is a closed subset of X disjoint from F, then there is a family $\{W_n : n \in \omega\}$ of open sets such that $W_n \cap F_n$ is dense in F_n and $\displaystyle\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} W_k)$ is disjoint from E.*

Proof. Clearly X is β-normal. Let $\{F_n : n \in \omega\}$ be a family of closed sets with $F = \displaystyle\bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} F_k)$, and E is a closed subset of X disjoint from F. Note that $A = \bigcup_{n\in\omega}(F_n \times \{n\}) \cup (F \times \{\omega\})$ and $B = E \times \{\omega\}$ are disjoint closed sets in $X \times Y$. Since $X \times Y$ is β-normal, there are open $U, V \subset X \times Y$ such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $\overline{U} \cap \overline{V} = \emptyset$. The sets $W_n = \{x \in X : (x, n) \in U\}$ are open in X and $W_n \cap F_n$ is dense in F_n for each $n \in \omega$. Suppose $C = \bigcap_{n\in\omega}(cl_X(\bigcup_{k=n}^{\infty} W_k)$

is not disjoint from E. Let $z \in C \cap E$. Then every neighborhood of z meets infinitely many W_n and since $W_n \times \{n\} \subset U$, so $(z, \omega) \in \overline{U}$. This is impossible since $\overline{U} \cap B = \emptyset$. □

At this point, it should be noted that Dowker had a useful characterization of countably paracompact.

Fact 2.5. *A topological space X is countably paracompact if and only if for every decreasing sequence $\langle F_n : n \in \omega \rangle$ of closed subsets of X satisfying $\bigcap_{n\in\omega} F_n = \emptyset$ there exists a sequence $\langle W_n : n \in \omega \rangle$ of open subsets of X such that $F_n \subset W_n$ for $n \in \omega$ and $\bigcap_{n\in\omega} \overline{W_n} = \emptyset$.*

This characterization prompted one of the authors to define the concept of α-countably metacompact and α-countably paracompact spaces.

Definition 2.6. A topological space is said to be *α-countably paracompact* (resp., *α-countably metacompact*) if for every decreasing sequence $\langle F_n : n \in \omega \rangle$ of closed subsets of X satisfying $\bigcap_{n\in\omega} F_n = \emptyset$ there exists a sequence $\langle W_n : n \in \omega \rangle$ of open subsets of X such that $W_n \cap F_n$ is dense in F_n for $n \in \omega$ and $\bigcap_{n\in\omega} \overline{W_n} = \emptyset$ (resp., $\bigcap_{n\in\omega} W_n = \emptyset$).

With this new definition and Lemma 2.4, we have the following corollary that exhibits what can be said of a space X if $X \times (\omega + 1)$ is β-normal. In this direction, we can extend the result to the product of X and a compact metric space as all that is needed is a distinct convergent sequence and its limit point. The proofs are left to the reader.

Corollary 2.7. *If $X \times (\omega + 1)$ is β-normal, then X is β-normal and α-countably paracompact.*

Corollary 2.8. *If Y is an infinite compact metric space, and $X \times Y$ is β-normal, then X is β-normal and α-countably paracompact.*

With Lemma 2.4 in hand, we are now ready for the main β-normal result of this section.

Theorem 2.9. *Let X be a T_1 space. The product $X \times (\omega + 1)$ is β-normal if and only if the following three conditions are met:*

(1) *X is β-normal,*
(2) *condition (2) of Lemma 2.4 is satisfied, and*
(3) *for every decreasing sequence $\langle F_n : n \in \omega \rangle$ of closed subsets of X satisfying $\bigcap_{n\in\omega} F_n = \emptyset$, there is a family $\{V_n : n \in \omega\}$ of open sets such that $F_n \subset V_n$ and $\bigcap_{n=0}^{\infty} cl_X(V_n)$ is nowhere dense in the relative topology of F_0.*

Proof. Let C be a closed subset of $X \times (\omega + 1)$ and U an open set of $X \times (\omega + 1)$ containing C. It suffices to find an open set G such that $G \cap C$ is dense in C and $\overline{G} \subset U$. Consider the following sets:

- $C_n = \{x \in X : (x, n) \in C\}$
- $C_\omega = \{x \in X : (x, \omega) \in C\}$
- $U_n = \{x \in X : (x, n) \in U\}$
- $U_\omega = \{x \in X : (x, \omega) \in U\}$

Note that each C_n and C_ω are closed subsets of X, and each U_n and U_ω are open subsets of X. Also, $F = \bigcap_{n\in\omega} cl_X(\bigcup_{k=n}^{\infty} C_k)$ is a closed subset of C_ω. By condition (2) of Lemma 2.4 and β-normality of X, we can find open sets W_n such that

(a) $\bigcap_{n=0}^{\infty} cl_X(\bigcup_{k=n}^{\infty} W_k)$ is a subset of U_ω,
(b) $W_n \cap C_n$ is dense in C_n, and
(c) $cl_X(W_n) \subset U_n$.

Let $G_0 = \bigcup_{n=0}^{\infty}(W_n \times \{n\})$. Note that $\overline{G_0} \cap (X \times \{\omega\})$ is a subset of $X \times (\omega+1)$, while $E = X \setminus U_\omega$ is a subset of X. Hence $\overline{G_0} \subset U$.

By the β-normality [1] of X, we can find an open set $V \subset X$ such that $V \cap C_\omega$ is dense in C_ω and $cl_X(V) \subset U_\omega$. Let $E_k = X \setminus U_k$ and let $F_n = cl_X(\bigcup_{k=n}^{\infty} E_k) \cap C_\omega$. Then $\langle F_n : n \in \omega\rangle$ is a decreasing sequence of closed subsets of X such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$ and by condition (3) of Theorem 2.9 we can find open sets $V_n \supset F_n$ such that $\bigcap_{n\in\omega} \overline{V_n}$ meets F_0 in a nowhere dense set.

Now $C_\omega \setminus V_n$ is a closed set disjoint from $cl_X(\bigcup_{k=n}^{\infty} E_k)$. So $W_n = (V \setminus \overline{V_n}) \times [n, \omega]$ is an open set whose closure is disjoint from E_k for all k and so, by definition of V and E_k, the closure of W_n is a subset of U. Let $G_1 = \bigcup_{n=0}^{\infty} W_n$. Then $G_1 \cap C_\omega = C_\omega \setminus \bigcap_{n=0}^{\infty} \overline{V_n}$ is dense in C_ω, and the closure of G_1 is easily seen to be a subset of U. Thus $G = G_0 \cup G_1$ is the desired open set.

Conversely, let $Y = X \times (\omega+1)$ and suppose Y is It remains to verify condition (3) of Theorem 2.9.

Consider a decreasing sequence $\langle F_n : n \in \omega\rangle$ of closed subsets of X satisfying $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Let $C = F_0 \times \{\omega\}$ and $E = \bigcup_{n\in\omega} F_n \times \{n\}$. Then C and E are disjoint closed subsets of Y. By β-normality of Y, there is an open subset W of Y whose intersection with C is dense in C, and whose closure is a subset of $Y \setminus E$. Let $V_n = \{x \in X : (x, n) \notin \overline{W}\}$. Clearly, V_n is an open subset of X, and $F_n \subset V_n$. If $\bigcap_{n\in\omega} \overline{V_n} \neq \emptyset$, let z be in the intersection. Every neighborhood of (z, ω) meets all but finitely many of the sets $\overline{V_n} \times \{n\}$. Each of these sets is a subset of the closure of $Y \setminus \overline{W}$ and so it misses W. Therefore, $(z, \omega) \notin W$, and so $\bigcap_{n\in\omega} \overline{V_n}$ meets F_0 in a nowhere dense subset of F_0. □

We now consider what happens if the supposition in Fact 2.1 is changed from X normal to X α-normal or β-normal. Notice that countably paracompact spaces satisfy condition (2) of Theorem 2.3 and conditions (2) and (3) of Theorem 2.9. Hence, we have the following corollaries.

Corollary 2.10. *Let X be α-normal and countably paracompact. Then $X \times (\omega+1)$ is α-normal.*

Corollary 2.11. *Let X be β-normal and countably paracompact. Then $X \times (\omega+1)$ is β-normal.*

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[1]Here we use the equivalent definition of β-normality: A space X is β-normal if for each closed $A \subseteq X$ and for every open $U \subseteq X$ that contains A, there exists and open $V \subseteq X$ such that $\overline{V \cap A} = A \subseteq \overline{V} \subseteq U$.

3. Motivating Examples

After Dowker characterized countably paracompact normal spaces (Fact 2.1), he asked whether every normal space is countably paracompact or not. That is, does there exist a normal space X such that $X \times \mathbb{I}$ is not normal (i.e., a Dowker space). For several decades, the Dowker problem has fueled a great deal of research. In 1971, M.E. Rudin constructed a Dowker space [13]. In light of α-normality and β-normality, it is natural to ask whether the product of a Dowker space and the unit interval can be α-normal or β-normal.

Example 3.1. The product of a normal space and a compact metric space can be α-normal without being normal.

Proof. Consider a hereditarily separable Dowker space X and a compact metric space Y. Hereditarily separable Dowker spaces have been constructed under a variety of axioms independent of ZFC (see [12], [14], [15], and [16]). Since these spaces are hereditarily separable and Y is second countable, $X \times Y$ is hereditarily separable. A hereditarily separable regular space is α-normal [1]. $\square$

We will see that the properties of the Dowker space dictate the outcome of the product. In Example 3.1, the product of a *hereditarily separable* Dowker space with a compact metric space resulted in an α-normal product space. If this condition is dropped, as the next example demonstrates, the product may fail to be α-normal.

Example 3.2. (ZFC) The product of a normal space and a compact metric space need not be α-normal.

Proof. Recall that a topological space X is called a *P-space* if the intersection of countably many open sets is open. If X is a Dowker P-space and is *extremally disconnected*, that is if the closure of an open set is open, then X fails condition (2) of Theorem 2.3. Dow and van Mill [3] have constructed such a space in ZFC. $\square$

Remark 3.3. Note that in extremally disconnected spaces, α-countably paracompactness is equivalent to countable paracompactness. Thus, the normal space in Example 3.2 is not α-countably paracompact.

Although β-normality seems a much stronger condition than α-normality, it is not enough to determine the Dowker situation as the next example illustrate.

Example 3.4. (V=L) A normal space whose product with a compact metric space is β-normal but not normal.

Proof. P. Nyikos [12] constructed a scattered hereditarily strongly collectionwise (scwH) Hausdorff Dowker space X under the axiom **V=L**. Recently, it was shown that $X \times (\omega + 1)$ is scattered and hereditarily scwH, and therefore $X \times (\omega + 1)$ is (hereditarily) β-normal by Nyikos and Porter's Theorem 2.8 [11]. $\square$

4. Moore space results

In light of Theorem 2.9, one may consider how close α-normal and β-normal are in the presence of condition (2) of Theorem 2.3 and conditions (2) and (3) of Theorem 2.9. The next example gives some insight on this.

Example 4.1. A first countable Tychonov space that is α-normal, collectionwise Hausdorff, and α-countably paracompact but not β-normal.

Proof. In [17], Wage produced an example of a collectionwise Hausdorff first countable Tychonoff space that is not normal. We state the following lemma used by Wage.

Lemma 4.2. *There exist subsets of the real line A and B such that $B \subset A$ and every countable subset of B is contained in a G_δ that does not meet $A - B$, yet every G_δ containing B does meet $A - B$.*

To create Wage's example, topologize A by letting the points of B have the usual neighborhoods and each point of $A - B$ be isolated. Let $X = A \times (\omega + 1) - B \times \{\omega\}$. Wage showed this space to be first countable, Tychonoff, non-normal, pseudonormal, and collectionwise Hausdorff. A similar argument to the one Wage used to show that X is collectionwise Hausdorff will be used to show that X is α-normal.

Since the points of $(A - B) \times \omega$ are isolated and $B \times \omega$ is hereditary separable, it suffices to show that every countable subset of $B \times \omega$ is contained in an open set whose closure misses $(A - B) \times \{\omega\}$. Let $C \subset B \times \omega$ be countable. Since every countable subset of B is contained in a G_δ that misses $A - B$, there exist open sets $U_n \subset A$ with $U_{n+1} \subset U_n$ such that $\{x \in B : (\exists n \in \omega)(x, n) \in C\} \subset \bigcap\{U_n : n \in \omega\}$ and $\bigcap\{U_n : n \in \omega\} \cap (A - B) = \emptyset$. Note that $C \subset \bigcup\{U_n \times \{n\} : n \in \omega\}$, and the closure of $\bigcup\{U_n \times \{n\} : n \in \omega\}$ misses $(A - B) \times \{\omega\}$. That is, X is α-normal.

To show that X is α-countably paracompact, let $\{F_n : n \in \omega\}$ be a sequence of decreasing closed sets such that $\bigcap F_n = \emptyset$. For each $n \in \omega$ let

(i) $G_n = F_n \cap (B \times \omega)$,
(ii) $H_n = F_n \cap (A - B) \times \{\omega\}$, and
(iii) $I_n = F_n \cap (A - B) \times \omega$

Note that I_n is open in X. For each $(x, \omega) \in H_n$, let $U_{(x,\omega)} = \{(x, k) : k \geq n\}$. Note that $\overline{\bigcup\{U_{(x,\omega)} : (x, \omega) \in H_n\}} \cap (A - B) \times \{\omega\} = H_n$. Since $A \times \omega$ is paracompact open subset of X and $\{G_n : n \in \omega\}$ is a nested sequence of closed sets, we can find open sets V_n such that $G_n \subset V_n$ and $\bigcap \overline{V_n} \cap (A \times \omega) = \emptyset$. Since G_n is closed, we can find an open set O_n such that $\overline{G_n \cap O_n} = G_n$ and $\overline{O_n} \cap (A - B) \times \{\omega\} = \emptyset$ by the above arguments. Let $U_n = V_n \cap O_n$, and let

$$W_n = I_n \cup (\bigcup\{U_{(x,\omega)} : (x, \omega) \in H_n\}) \cup U_n.$$

Note that $\bigcap \overline{W_n} = \emptyset$, and X is α-countably paracompact.

To show that X is not β-normal, we show that the closed sets $B \times \omega$ and $(A-B) \times \{\omega\}$ cannot be β-separated. Suppose U and V are open sets such that $(B \times \omega) \cap U$ is dense in $B \times \omega$ and $(A-B) \times \{\omega\} \cap V$ is dense in $(A-B) \times \{\omega\}$. Since $(A-B) \times \{\omega\}$ is discrete, for every $x \in A - B$ there is an $n_x \in \omega$ such that $\{(x, n) : n \geq n_x\} \subset V$. We claim there exists $x' \in B$ and a sequence $\{x_k\}$ in $A - B$ and an $m \in \omega$ such that $x_k \to x'$ and $n_{x_k} = m$. Since $\overline{U} \supset B \times \omega$, $\overline{U}$ must contain $\{(x', n) : n \in \omega\}$. This shows that $\overline{U} \cap \overline{V} \neq \emptyset$.

To prove the claim, let $E_m = \{x \in A - B : n_x = m\}$. If the claim were not true, then for every $x \in B$ there is a neighborhood O_x of x such that $O_x \cap E_m = \emptyset$. Let $O_m = \bigcup_{x \in X} O_x$. Note that $O_m \cap E_m = \emptyset$, and $\bigcap_{m \in \omega} O_m$ is a G_δ set which contains B but misses $A - B$, a contradiction. This completes the proof. □

Wage used this space to construct a collectionwise Hausdorff non-normal Moore space. This gives the following interesting result.

Example 4.3. There exists a collectionwise Hausdorff, non-β-normal, α-normal Moore space, X'.

Proof. Let X' be the set of all non-isolated points of X. The space

$$Y = X' \times \{\omega\} \cup (X - X') \times \omega$$

as a subspace of $X \times (\omega + 1)$ is a Moore space which is α-normal but not β-normal by the previous arguments. $\square$

5. Questions

The authors close the paper by listing some open questions that the authors were unable to answer.

Question 5.1. Is there a Dowker space whose product with a compact metric space is β-normal in ZFC?

Question 5.2. Is there a Dowker space whose product with a compact metric space is α-normal, but not β-normal?

Question 5.3. If X is β-normal and α-countably paracompact, is $X \times (\omega + 1)$ β-normal? α-normal?

Question 5.4. If X is α-normal and α-countably paracompact, is $X \times (\omega + 1)$ α-normal?

Question 5.5. If $X \times (\omega + 1)$ is α-normal, is X α-countably metacompact? α-countably paracompact?

Question 5.6. Are β-normal α-countably metacompact spaces α-countably paracompact?

Question 5.7. Is there a β-normal non-normal Moore space?

It follows from condition (3) of Theorem 2.9 that the product of a Dowker P-space with $\omega + 1$ is not β-normal. Recall that Rudin's Dowker [13] space is a P-space.

Question 5.8. Is Rudin's Dowker space α-countably paracompact?

Question 5.9. Is the product of Rudin's Dowker space with $\omega + 1$ α-normal?

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