Some screenable anti-Dowker spaces

Normal spaces that are not countably paracompact, and countably paracompact, regular spaces that are not normal, are called Dowker and anti-Dowker spaces, respectively. The twin questions of whether there is a screenable Dowker space or a screenable anti-Dowker space are of special interest due to the 1955 theorem of Nagami:

Theorem A. *A space is paracompact if, and only if, it is screenable, normal, and countably paracompact.*

Most of the research surrounding this theorem has had to do with whether countable paracompactness could be dispensed with. This was a very demanding problem, and was first solved by Mary Ellen Rudin using the extra set-theoretic axiom $\diamondsuit^{++}$ [1], and fifteen years later in ZFC by Zoltán Balogh [2].

I have not seen much attention paid to the complementary problem of whether there can be a screenable anti-Dowker space — equivalently, by Nagami's theorem, a screenable, regular, countably paracompact space that is not paracompact. The purpose of this note is to describe two examples: a simple one obtained under the set-theoretic hypothesis $\mathfrak{q}_1 > \omega_1$ and a more complicated ZFC example which is also paralindelöf (that is, every open cover has a locally countable open refinement). Both examples are metacompact and subparacompact, so just about every "slight" weakening of paracompactness is realized.

Definition. A *countably paracompact* space is a space such that every countable open cover has a locally finite open refinement. A *screenable* [*metacompact*] [*subparacompact*] space is one such that every open cover has a σ-disjoint open [point-finite open] [σ-locally finite closed] refinement.

Our examples apply a simplification of the Wage machine [see [3] for a description] to two well-known examples of normal spaces that are not collectionwise normal. Each one has a discrete family $\mathcal{H}$ of closed sets which cannot be put into disjoint open sets. In both spaces, each $H \in \mathcal{H}$ is equipped with a pair of "wings," and our machine replaces $\bigcup \mathcal{H}$ with two disjoint copies of $\bigcup \mathcal{H}$ and gives each of the two copies of each H one of the "wings" associated with H.

The set-theoretic cardinal $\mathfrak{q}_0$ [$\mathfrak{q}_1$] is the least cardinal λ such that some [every] subset of the real line of cardinality λ fails to be a Q-set (*i.e.,* a subset Q of $\mathbb{R}$ such that every subset of A is a G_δ — equivalently, an F_σ in the relative topology of Q). A well-known consequence of Martin's Axiom is that $\mathfrak{q}_0 = \mathfrak{c}$.

Example 1. [$\mathfrak{q}_1 \geq \omega_2$] Let X be Heath's tangent V space. The underlying set is the closed upper half plane. Points not on the x-axis are isolated, points on the x-axis have "tangent V" basic neighborhoods. Given $p = (x, 0)$, these V's are formed by line segments of length $1/n$ ($n \in \omega \setminus \{0\} = \mathbb{N}$) at 45 degree angle to the

x-axis beginning at p. As is well known, this space is a metacompact Moore (hence subparacompact) space that is neither normal nor collectionwise Hausdorff (cwH).

Let Y be a subspace of X consisting of the upper half plane and a Q-set of cardinality ω_1 on the x-axis. This is a standard example of a metacompact Moore space that is normal but not cwH, obtained under extra set-theoretic hypotheses. In normal spaces, countable paracompactness is equivalent to countable metacompactness, so Y is countably paracompact. Y is not cwH because the points on the x-axis cannot be expanded to a disjoint collection of open sets. This is a simple cardinality argument using the usual topology on the x-axis, which is hereditarily Lindelöf.

Let $Y^\dagger$ be the space obtained by replacing the Q-set Q on the x-axis by two copies, $Q_0 = \{\langle p, 0\rangle : p \in Q\}$ and $Q_1 = \{\langle p, 1\rangle : p \in Q\}$, and giving the points of Q_0 the "left wing" of each of the basic tangent V's, with the "right wings" going to the corresponding points of Q_1. That is, the neighborhoods of $\langle(r, 0), 0\rangle$ are the sets that contain some $V_n(r, 0) = \{\langle(r, 0), 0\rangle\} \cup \{(x, y) : 0 < y < 1/n, x = r - y\}$, while the neighborhoods of $\langle(r, 0), 1\rangle$ are the sets that contain some $V_n(r, 1) = \{\langle(r, 0), 1\rangle\} \cup \{(x, y) : 0 < y < 1/n, x = r + y\}$.

Given an open cover $\mathcal{U}$ of $Y^\dagger$, we can refine it to $\mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$, where, for $i = 0, 1$, is a family of "wings" one apiece for the points of Q_i, left for Q_0 and right for Q_1; and $\mathcal{W}_2$ is the set of all singletons of $\mathbb{R} \times (0, \infty)$ that are not covered by $\mathcal{W}_0 \times \mathcal{W}_1$. This is a cover of order 2.

The projection map $\pi : Y^\dagger \to Y$ is clearly closed and at most 2-to-1, so countable paracompactness is easily seen to be inversely preserved. Normality fails because if Q_0 and Q_1 could be put into disjoint open sets U_i, then the whole of $Q_0 \cup Q_1$ could be expanded to a disjoint collection of open sets in $Y^\dagger$, and hence Q would also have such an expansion in Y, a contradiction.

Example 2. Caryn Navy's space N is described in [4], which includes the proofs that N is paralindelöf and normal but not collectionwise normal. Navy's space has Baire's zero-dimensional space $F = D^\omega$ of weight $\aleph_1$ (where D is the discrete space with underlying set ω_1) playing the role Q played in Example 1. However, rather than a single V at each point, the basic neighborhoods are the sets $[\sigma] = \{f \in F : \sigma \subset f\}$ together with a pair of wings attached to each one. Each σ is a finite sequence of elements of ω_1. The wings reach into a subspace of isolated points in a counterpart of the open upper half plane in Example 1.

This counterpart is a swarm of copies of $G = 2^{\mathcal{T}}$ where $\mathcal{T}$ is the family of open subsets of F. These are indexed by entwined pairs $\langle\rho, \tau\rangle$, and the right wing of $[\sigma]$ reaches into the copies of G indexed by the ρ's extending σ, while the left wing into the ones indexed by the τ's that extend σ.

One detail missing from [4] is that N is a σ-space, *i.e.,* it has a σ-discrete network. This follows from the easy facts that the isolated points form an F_σ and that the $[\sigma]$

form a σ-discrete base of clopen sets for the relative topology on the closed subspace F. Thus N is perfectly normal and subparacompact.

With $N^\dagger$ defined analogously to $Y^\dagger$, one proves analogously that $N^\dagger$ is screenable, metacompact, subparacompact, and countably paracompact, but not normal.

[1] M. E. Rudin, “A normal, screenable, non-paracompact space,” Topology Appl. **15** (1983) 313–322.

[2] Z. Balogh, “A normal screenable nonparacompact space in ZFC,” Proc. Amer. Math. Soc. **126** (1998) no. 6, 1835–1844.

[3] P. Nyikos and T. Porter, “Hereditarily strongly cwH and wD($\aleph_1$) vis-a-vis other separation axioms,” Top. Appl. 156 (2008) no. 2, 151–164.

[4] W. G. Fleissner, “The normal Moore space conjecture,” in: *Handbook of Set-theoretic Topology,* K. Kunen and J. E. Vaughan, ed., Elsevier Science Publishers B.V., 1984, pp. 733–760.