HEREDITARILY STRONGLY CWH AND WD($\aleph_1$) VIS-A-VIS OTHER SEPARATION AXIOMS

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Abstract. We explore the relation between two general kinds of separation properties. The first kind, which includes the classical separation properties of regularity and normality, has to do with expanding two disjoint closed sets, or dense subsets of each, to disjoint open sets. The second kind has to do with expanding discrete collections of points, or full-cardinality subcollections thereof, to disjoint or discrete collections of open sets. The properties of being collectionwise Hausdorff (cwH), of being strongly cwH, and of being wD($\aleph_1$), fall into the second category. We study the effect on other separation properties if these properties are assumed to hold hereditarily. In the case of scattered spaces, we show that (a) the hereditarily cwH ones are α-normal and (b) a regular one is hereditarily strongly cwH iff it is hereditarily cwH and hereditarily β-normal. Examples are given in ZFC of (1) hereditarily strongly cwH spaces which fail to be regular, including one that also fails to be α-normal; (2) hereditarily strongly cwH regular spaces which fail to be normal and even, in one case, to be β-normal; (3) hereditarily cwH spaces which fail to be α-normal. We characterize those regular spaces X such that $X \times (\omega + 1)$ is hereditarily strongly cwH and, as a corollary, obtain a consistent example of a locally compact, first countable, hereditarily strongly cwH, non-normal space. The ZFC-independence of several statements involving the hereditarily wD($\aleph_1$) property is established. In particular, several purely topological statements involving this property are shown to be equivalent to $\mathfrak{b} = \omega_1$.

1. Introduction

A Hausdorff space is called *strongly collectionwise Hausdorff* (*strongly cwH* for short) if every closed discrete subspace can be expanded to a discrete collection of open sets [Definition 2.1]. This property received a fair amount of attention in the 1970's and early 1980's, but it is only recently that papers have appeared which give some idea of the strength of assuming that a space satisfies it hereditarily, at least in some models of set theory. A remarkable illustration of that strength is the following theorem of the late Zoltán Balogh [B].

Theorem 1.1. *Assume MA(ω_1) and Axiom R. Let X be a locally compact, hereditarily strongly ω_1-cwH space. Then either X is (hereditarily) paracompact or X contains a perfect preimage of ω_1.*

Using another theorem of Balogh, the first author [Ny3, discussion following Corollary 2.8] showed how one can replace "perfect preimage of ω_1" with "copy of ω_1" if one replaces MA(ω_1) with the Proper Forcing Axiom (PFA). The Balogh theorem in question is:

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Theorem 1.2. [D] *The PFA implies every countably compact, first countable space is either compact or contains a copy of ω_1.*

The axioms used in Theorem 1.1 are consistent if it is consistent that there is a supercompact cardinal.

Questions 1.3. *Can large cardinals be eliminated from Theorem 1.1? what if "strongly ω_1-cwH" is replaced by "strongly cwH"?*

Since ω_1 itself is locally compact and hereditarily strongly cwH but not paracompact, one cannot eliminate the second alternative in Theorem 1.1. However, the following theorem, which will appear in a forthcoming paper, suggests that one might still be able to go far without it:

Theorem 1.4. *The PFA implies every locally compact, locally connected, hereditarily normal, hereditarily strongly cwH space is (hereditarily) collectionwise normal and (hereditarily) countably paracompact.*

In this paper, "space" will always mean "Hausdorff space," so there is no ambiguity about the words "regular" and "normal". The following theorem from [Ny4] is related to Theorem 1.4:

Theorem 1.5. *The PFA implies every normal, hereditarily strongly cwH manifold of dimension > 1 is metrizable.*

We do not know whether "locally connected" and/or "hereditarily normal" can be eliminated from the hypothesis of Theorem 1.4, nor whether "normal" can be eliminated from Theorem 1.5. Also we do not know whether "hereditarily strongly cwH" can be eliminated from Theorem 1.4, with or without "locally connected." The following two problems are also open.

Question 1.6. *Is it consistent that every locally compact, hereditarily strongly cwH space is (hereditarily) normal?*

Question 1.7. *Is it consistent that every first countable, hereditarily strongly cwH space is (hereditarily) normal?*

With such grandiose possibilities up in the air, we decided to get a better picture of what can and cannot be done with the property of being hereditarily strongly cwH. This paper gives several examples and theorems, some consistent and some using just ZFC, which we hope will give readers a clearer picture. We also discuss some weakenings of normality, especially pseudonormality, α-normality and β-normality [Definitions 4.3 and 2.3], and some weakenings of the strongly cwH property, especially wD($\aleph_0$) and wD($\aleph_1$) [Definition 2.2].

One of our main results (see Section 4) is a characterization of when $X\times(\omega+1)$ is hereditarily strongly cwH, enabling us to find a consistent example of a hereditarily strongly cwH locally compact, first countable, non-normal space. This explains why Question 1.6 and Question 1.7 ask only for consistency.

In a talk at a 2002 conference in honor of Balogh, the first author noted that Question 1.6 was unsolved even if "locally compact" is weakened to "regular." Example 3.1, defined using only the usual ZFC axioms, answers this variation on Question 1.6 in the negative. It is also a nice illustration of how the strongly cwH property relates to the concept of being β-normal, introduced by Arhangel'skiĭ. Section 2 gives some conditions under which these two concepts are equivalent.

If "normal" is weakened to "pseudonormal" in Question 1.6, the answer is Yes, even if "strongly cwH" is weakened to "wD($\aleph_1$)" as shown by one of us in an earlier paper [JNS, 3.9 and 3.10]. In Section 5 we will give several consistency and independence results that strengthen this one.

In contrast, if "normal" is weakened to "β-normal" in Questions 1.6 and 1.7, the resulting problems are still open. If "normal" is further weakened to "α-normal," we get questions for which we do not even have consistency results:

Question 1.8. *Is every locally compact, hereditarily strongly cwH space (hereditarily) α-normal?*

Question 1.9. *Is every first countable, hereditarily strongly cwH space (hereditarily) α-normal?*

In fact, these questions remain wide open if "strongly" is dropped! Even the following problem is not completely solved:

Question 1.10. *Is there a regular, hereditarily strongly cwH space that is not (hereditarily) α-normal?*

There does exist an example under a very general axiom (see Section 7), but still none is known from ZFC alone.

Question 1.7 is partly inspired by the theorem that every first countable, strongly ω-cwH space is regular: see the comment at the beginning of Section 3. This theorem also motivates:

Questions 1.11. *Is it consistent that every sequential, hereditarily strongly cwH space is regular? What if the space is Fréchet-Urysohn?*

In Section 6 we will give a consistent example of a hereditarily strongly cwH nonregular space that is not even Fréchet-Urysohn, but we know of no ZFC counterexample to either part of Question 1.11.

2. Connections with β-normality and some ZFC examples of hereditarily strongly cwH spaces

Definition 2.1. Given a subset D of a set X, an *expansion* of D is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = \{d\}$ for all $d \in D$. Given an infinite cardinal κ, a space X is *[strongly] κ-collectionwise Hausdorff* if every closed discrete subspace of cardinality $\leq \kappa$ has an expansion to a disjoint [*resp.* discrete] collection of open sets. X is *[strongly] collectionwise Hausdorff* if it is [strongly] κ-collectionwise Hausdorff for all κ.

We use the abbreviation "cwH" for "collectionwise Hausdorff". As with normality and many related properties, one need only check open subspaces to see whether a space is hereditarily [strongly] κ-cwH. Indeed, if D is a discrete subspace of a space X and $W = (X \setminus \overline{D}) \cup D$, then W is an open set, and if $D \subset S \subset X$ and D is closed in S, then $S \subset W$ and we can take the trace on S of the appropriate open expansion in W. This sufficiency of open subspaces for hereditariness also holds true for the properties in our next two definitions.

Definition 2.2. Let κ be a cardinal number. A space X *is weakly κ-collectionwise Hausdorff* [resp. *satisfies property wD(κ)*] if every closed discrete subspace D of cardinality κ has a subset D_0 of cardinality κ which can be expanded to a disjoint [*resp.* discrete] collection of open sets U_d such that $U_d \cap D_0 = \{d\}$ for all $d \in D_0$.

These are obvious weakenings of the properties of being κ-cwH and strongly κ-cwH, respectively. Up to now, strongly ω-cwH spaces have been called "spaces with Property D," a designation introduced by R.L. Moore, while wD($\aleph_0$) spaces have been called "spaces with Property wD." The following concepts were introduced by Arhangel'skiĭ:

Definition 2.3. A space X is *α-normal [*resp. *β-normal]* if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B and $U \cap V = \emptyset$ [*resp.*$\overline{U} \cap \overline{V} = \emptyset$].

For our first theorem, recall that a space is called *scattered* if every subspace has an isolated point in its relative topology; equivalently (because an isolated point of an open subspace is isolated in the whole space) every subspace has a dense set of isolated points in its relative topology. Another characterization is highly revealing of the structure of these spaces. One defines the *ξth Cantor-Bendixson level* X_ξ of any space X by induction as follows: X_0 is the set of isolated points of X; if X_η has been defined for all $\eta < \xi$ then X_ξ is the set of isolated points of $X \setminus \bigcup_{\eta<\xi}\{X_\eta\}$. A space is scattered iff it is the union of all its Cantor-Bendixson levels. It is immediate from this characterization that every point in a scattered space has a neighborhood in which it is the unique point of highest Cantor-Bendixson level—a level which is the same in the neighborhood as it is in the whole space.

Yet another characterization will play a role in Sections 5 and 6: a space is scattered if, and only if, it is "right-separated": that is, there is a well-ordering of the space such that every initial segment is open.

Theorem 2.4. *Every scattered hereditarily cwH space is (hereditarily) α-normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of the scattered hereditarily cwH space X. Let D_1 and D_2 be the (dense) set of relatively isolated points of F_1 and F_2, respectively. Then D_1 and D_2 are closed discrete in the open subspace $U = [X \setminus (\overline{D_1} \cup \overline{D_2})] \cup D_1 \cup D_2$. Let $\mathcal{V}$ be an expansion of $D_1 \cup D_2$ to a family of disjoint open subsets of U and let V_1 and V_2 be the unions of the ones expanding the points of D_1 and D_2 respectively. Then $V_i \cap F_i$ is dense in F_i and $V_1 \cap V_2 = \emptyset$, as desired. □

The next theorem is a natural variation on Theorem 2.4, inasmuch as scattered spaces can also be characterized by all closed subspaces having dense discrete subsets:

Theorem 2.5. *If X is regular, and every closed subspace of X has a dense subspace which is the union of countably many discrete* [resp. *closed discrete] subspaces and X is hereditarily strongly cwH* [resp. *stongly cwH] then X is α-normal.*

Proof. Let $F_i (i = 1, 2)$ be disjoint closed subsets of the strongly cwH space X. Let D_i be dense in F_i and be the countable union of discrete subspaces. If X is hereditarily strongly cwH, or if D_i is the countable union of closed discrete subspaces, then D_i can be covered by countably many open subspaces, each of whose closures miss D_{3-i}. (See Lemma 2.6 below.) The standard proof that every regular Lindelöf space is normal can now be mimicked to put D_1 and D_2 into disjoint open sets. □

The hypotheses of Theorem 2.5 are satisfied by every regular strongly cwH σ-space (in particular, every strongly cwH Moore space and every strongly cwH stratifiable space) and every regular hereditarily strongly cwH quasi-developable space. The latter spaces are characterized by having bases which are the union of countably many open collections, such that each point x of the space has a local base taken from those collections for which x is in exactly one member of the collection [G, proof of Theorem 8.5]. Let $\mathcal{U}_n$ is one of these collections, cut down if necessary so that each $U \in U_n$ has at least one point p such that $ord(p, \mathcal{U}_n) = 1$. If P_n is a set which meets each member of U_n in such a point p, then P_n is discrete and $\bigcup \mathcal{P}_n$ is dense.

Our next theorem establishes an interesting connection between the properties of hereditary β-normality and the hereditary strong cwH property. A general lemma and corollary pave the way.

Lemma 2.6. *A regular space is strongly cwH $\iff$ it is cwH and any two disjoint closed sets, one of which is discrete, can be put into disjoint open sets.*

Proof. Suppose X is regular and strongly cwH, and let F and D be disjoint closed sets, with D discrete. Let $\{U_d : d \in D\}$ be a discrete open expansion of D. For each $d \in D$ let $V_d \subset U_d$ be an open neighborhood of d whose closure misses F. Let $W = \bigcup\{V_d : d \in D\}$; then $X \setminus \overline{W}$ and W are as desired.

The converse is proven just like the familiar theorem that every normal, cwH space is strongly cwH: if D is closed discrete and $\{U_d : d \in D\}$ is a disjoint open expansion of D, let V be an open set containing D whose closure is in $\bigcup\{U_d : d \in D\}$; then $\{U_d \cap V : d \in D\}$ is a discrete open expansion of D. □

Corollary 2.7. *Every cwH β-normal space is strongly cwH.*

Proof. Clearly, every Hausdorff β-normal space is regular. Let A and B be disjoint closed sets, with A discrete. With U and V as in Definition 2.3, we must have $A \subset U$; so U and the complement of $\overline{U}$ are disjoint open sets containing A and B respectively. Now use Lemma 2.6. □

Theorem 2.8. *Let X be a regular scattered space. The following are equivalent.*

(1) *X is hereditarily strongly cwH.*
(2) *X is hereditarily cwH and hereditarily β-normal.*
(3) *Every open subspace of X is cwH and β-normal.*

Proof. The equivalence of (2) and (3) is well known, *cf*. the discussion preceding Definition 2.3; and (2) $\Rightarrow$ (1) is immediate from Corollary 2.7, so it only remains to prove (1) $\Rightarrow$ (3). We use the easy fact [AL] that a space is β-normal iff for every A closed in X and every open $U \supset A$, there is an open $V \subset X$ such that $\overline{A \cap V} = A$ and $\overline{V} \subset U$.

Let W be open in X, let A be closed in W, and let U be open in W and hence in X. Let D be the set of isolated points in the relative topology of A. Then D is closed discrete in the X-open set $(W \setminus A) \cup D$. Expand D to a discrete-in-$(W \setminus A) \cup D$ collection $\mathcal{V}$ of open sets whose individual closures are in $[(W \setminus A) \cup D] \cap U$. Let $V = \bigcup \mathcal{V}$. Then $\overline{V} \cap A = A$ and $\overline{V} \subset U$, as desired. □

In Section 7 we give a ZFC example (Example 7.5) that shows "scattered" cannot be omitted from Theorem 2.8. The following ZFC example shows that "regular" cannot be omitted either:

Example 2.9. *A scattered hereditarily strongly cwH space that is not regular (and hence not β-normal).*

Define a topology on $X = [0, \omega_1]$ as follows. Let the points of $[0, \omega_1)$ have the ordinal topology as a basis. A local basis of ω_1 consist of all sets of the type

$$O_C = \{\omega_1\} \cup \{\alpha + 1 : \alpha \in C\}$$

where $C \subset \omega_1$ is closed and unbounded. E. Murtinová [Mu1] showed X is a Hausdorff α-normal space that is not regular: A club subset C of $[0, \omega_1)$ cannot be separated from ω_1 by disjoint open sets. One can easily show that X is hereditarily strongly cwH and scattered since $[0, \omega_1)$ is hereditarily strongly cwH and scattered. This shows that X is hereditarily α-normal by Lemma 2.4.

Regularity is also needed for the forward implication in Lemma 2.6. Indeed, if X is a strongly cwH non-regular space and p and C are a point and a closed subset of X which cannot be put into disjoint open sets, then the second condition in Lemma 2.6 breaks down for $D = \{d\}$ and $F = C$. In Example 2.9, this is the case with $\{d\} = \{\omega_1\}$, and any club subset of ω_1 will do for C.

In [Mu2], Murtinova gave a countable example of a scattered non-regular α-normal space, denoted Y. Although it is a bit more complicated than Example 2.9, it is easy to see from the description in [Mu2] that Y is obtained by adding one point ∞ to a countable scattered regular space whose set of nonisolated points is denoted X, such that if D is a discrete subspace of X, then D and ∞ can be put into disjoint open sets. From this and regularity of $Y \setminus \{\infty\}$ it readily follows that Y is hereditarily strongly cwH.

The next two examples are not scattered, but they have other nice qualities. In particular, Example 2.11 shows why "scattered" cannot be left out of Theorem 2.4 nor "regular" out of Theorem 2.5, and why "regular" appears also in Question 1.10.

Example 2.10. *A hereditarily separable, hereditarily Lindelöf, hereditarily α-normal, hereditarily strongly cwH Baire space that is α-normal but not regular.*

Let X have the plane $\mathbb{R}^2$ as its underlying set, with all points except the origin having their usual base of neighborhoods. A local base at $\mathbf{0} = \langle 0, 0\rangle$ consists of all intersections of Euclidean balls with complements of sets of the form $\bigcup \mathcal{B} \cup (0, +\infty) \times \{0\}$, where $\mathcal{B}$ is a family of closed balls centered on points of $(0, +\infty)$ whose union meets $[0, 1/n) \times \{0\}$ in a set whose one-dimensional measure is $o(1/n)$: that is,

$$(*) \qquad \mu(\bigcup \mathcal{B} \cap [0, \frac{1}{n}) \times \{0\}) \cdot n \to 0 \text{ as n} \to \infty.$$

Let τ denote the resulting topology and E denote the Euclidean topology on $\mathbb{R}^2$. Then $C = (0, +\infty) \times \{0\}$ is τ-closed, but C and $\mathbf{0}$ clearly cannot be put into disjoint τ-open sets. On the other hand, if D is a τ-discrete subspace of X, relatively τ-closed in an open subspace U of X, then D can be expanded to a relatively discrete collection of open subsets of U. This is clear if $\mathbf{0} \notin U$ or if $\mathbf{0} \notin c\ell_E(D)$ since the Euclidean plane, being metrizable, is hereditarily strongly cwH.

So suppose $\mathbf{0}$ is in the Euclidean closure of D and $\mathbf{0} \in U$. It is enough to consider the case when $\mathbf{0} \notin D$. Then there is an open Euclidean ball $B(\mathbf{0}, \epsilon)$ and a family $\mathcal{B}$ of closed balls centered on points of $(0, \epsilon] \times \{0\}$ and satisfying $(*)$, such that $D \cap B(\mathbf{0}, \epsilon) \subset \bigcup \mathcal{B} \cup [\,(0, \epsilon) \times \{0\}]$. We can expand each member of $\mathcal{B}$ slightly while still conforming to $(*)$. Moreover, each discrete subspace of X is countable, so we can put closed balls of positive radius around each point of $D \cap [(0, \epsilon) \times \{0\}]$ and still conform to $(*)$. We can then expand D to a relatively E-discrete collection $\mathcal{U} = \{U_d : d \in D\}$ of E-open subsets of $X \setminus \{\mathbf{0}\}$, while staying inside the new family of closed balls. This way $\mathcal{U}$ will be τ-discrete in X. The argument for X being hereditarily α-normal is similar, using the fact that every countable subspace of $\mathbb{R}$ is of measure 0.

We are indebted to Alan Dow for the suggestion of modifying the above construction by using a remote point. It gives a space with all the properties of Example 2.10 except α-normality, and accounts for the "regular" in Question 1.10.

Example 2.11. *A hereditarily separable, hereditarily Lindelöf, hereditarily strongly cwH Baire space that is neither α-normal nor regular.*

Let p be a remote point of $\beta\mathbb{R}^+ \setminus \mathbb{R}^+$ in the closure of the open unit interval. That is, p is not in the closure of any nowhere dense subset of $(0, 1)$. Let (X, τ) be as in Example 2.10, except that this time $\mathcal{B}$ is any family of closed balls, each centered on points of $(0, +\infty) \times \{0\}$ as before, such that $\{x \in \mathbb{R} : (x, 0) \in \bigcup \mathcal{B}\}$ does not have p in its closure.

As in Example 2.10, (X, τ) is clearly not regular. It is also not α-normal, because any dense subset of the x-axis will have a projection to $\mathbb{R}$ with p in its closure, but the x-axis minus the origin does not have the origin in its closure.

To show (X, τ) is hereditarily strongly cwH, again let D be a relatively τ-closed discrete subset of some open set U containing $\mathbf{0}$. As before, let $\mathcal{B}$ be a family of closed balls centered on points of $(0, \epsilon] \times \{0\}$ such that $D \cap B(\mathbf{0}, \epsilon) \subset \bigcup \mathcal{B} \cup [\,(0, \epsilon) \times \{0\}]$, but this time with p outside the closure of $\{x \in \mathbb{R} : (x, 0) \in \bigcup \mathcal{B}\}$. Let U be the trace on $(0, +\infty)$ of an open neighborhood of p whose closure misses $\{x \in \mathbb{R} : (x, 0) \in \bigcup \mathcal{B}\}$. The complement of this closure is a union of open intervals; for each such open interval (a, b) contained in $B(\mathbf{0}, \epsilon)$, let $B(a, b)$ be the closed ball in $\mathbb{R}^2$ of diameter $b - a$ that meets the x-axis in $[a, b]$. Let $\mathcal{B}'$ be the set of all $B(a, b)$. Each member of $\mathcal{B}$ is in the E-interior of some $B(a, b)$, so we can expand $D \cap \bigcup \mathcal{B}'$ to a τ-discrete family of open balls inside $\bigcup \mathcal{B}'$. Since the rest of D is nowhere dense in the relative topology of the x-axis, its projection to $\mathbb{R}$ does not have p in its closure, and so we can define closed balls of positive radius centered on each of its points without the union of the balls projecting to a set with p in its closure, and finish the argument as in Example 2.10.

By taking the subspace of doubly rational points in Example 2.10 and Example 2.11, we get countable spaces that have all the properties of these examples except for being Baire. None of these four examples is sequential, however. Let A be positive half of the parabola $y = x^2$ (or, in the countable case, its set of rational points). Then A clearly has the origin in its τ-closure, yet A is τ-sequentially closed: every τ-convergent sequence is E-convergent, so we need only check those sequences in A that E-converge to the origin. We can embed an infinite subsequence of each such sequence in a family $\mathcal{C}$ of closed balls that conforms to the recipe in Example

2.10, and if we split this sequence of balls into two infinite subsequences, at most one can have a projection with p in its closure.

Murtinova's countable example Y in [Mu2] is not sequential either. It is clear from her description that $X \times \{0\}$ is sequentially closed but has ∞ in its closure.

3. Some regular non-normal examples

In contrast to the foregoing examples, we have the theorem that every first countable, strongly ω-cwH space is regular. In fact, we can even weaken "strongly ω-cwH" to "wD($\aleph_0$)": see [V], where "wD($\aleph_0$)" is referred to simply as "wD".

The following example shows that first countability cannot be omitted from Question 1.7 even if regularity is added:

Example 3.1. *A regular scattered hereditarily strongly cwH (hence hereditarily β-normal) space that is not normal.*

Let $S = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$, and consider the set

$$X = \{(\alpha, \beta) : \beta \leq \alpha \leq \omega_2 \text{ and } (\alpha, \beta) \neq (\omega_2, \omega_2)\}.$$

Partition X into

$$A = \{(\alpha, \alpha) : \alpha < \omega_2\}$$
$$B = \{(\omega_2, \beta) : \beta < \omega_2\}$$
$$I = \{(\alpha, \beta) : \beta < \alpha < \omega_2\}$$

Topologize X as follows
(i) Let each $(\alpha, \beta) \in I$ be isolated.
(ii) an open basis of $(\alpha, \alpha) \in A$ consist of all sets of the type

$$\{(\gamma, \gamma) : \alpha_0 < \gamma \leq \alpha\} \cup \bigcup \{\{\gamma\} \times C_\gamma : \alpha_0 < \gamma \leq \alpha \text{ and } \gamma \in \mathrm{S}\},$$

where $\alpha_0 < \alpha$ and every C_γ is a closed unbounded (club) subset of γ.
(iii) an open basis of $(\omega_2, \beta) \in B$ consist of all the sets

$$\{(\alpha, \gamma) : \beta_0 < \gamma \leq \beta \text{ and } \alpha_\gamma < \alpha \leq \omega_2\},$$

where $\beta_0 < \beta$ and $\beta \leq \alpha_\gamma < \omega_2$ for each γ.

Below is a summary of known results of X.

(1) All basic open sets defined above are closed, and hence, X is a Tychonov space.
(2) X is β-normal [Mu1] but not normal. The closed sets A and B cannot be separated by disjoint open sets.
(3) $A \cong B \cong \omega_2$ which is hereditarily strongly cwH.

We will now show that X is hereditarily strongly cwH. Let Y be a subspace of X, and let $D \subset Y$ be a closed discrete subset of Y. Without loss of generality, we may assume $D \subset A \cup B$. Consider the following subsets

- $D_A = D \cap A$
- $D_B = D \cap B$
- $N_A = \{\alpha \in S : (\alpha, \alpha) \in D \cap A\}$
- $N_B = \{\alpha \in \omega_2 : (\omega_2, \alpha) \in D \cap B\}$

Notice that D_A and D_B are relatively discrete subspaces of X. Since the closed sets $cl_X(D_A)$ and $cl_X(D_B)$ are disjoint, there are disjoint open sets O_A and O_B such that $D_A \subset O_A$, $D_B \subset O_B$, and $cl_X(O_A) \cap cl_X(O_B) = \emptyset$ by the β-normality of

X. Since A and B are closed, we can choose O_A and O_B such that $A \cap cl_X(O_B) = \emptyset$ and $B \cap cl_X(O_A) = \emptyset$.

Since ω_2 is hereditarily strongly cwH, for each $\alpha \in N_A$, there is a $\alpha_0 \in \omega_2$ such that

(i) $N_A \cap (\alpha_0, \alpha] = \{\alpha\}$,
(ii) $\{(\alpha_0, \alpha] : \alpha \in N_A\}$ is a pairwise disjoint family of open sets in ω_2, and
(iii) $\{(\gamma, \gamma) : \alpha_0 < \gamma \leq \alpha\} \subset O_A$

For each $\gamma \in (\alpha_0, \alpha] \cap S$, choose a club subset C_γ of γ such that $\{\gamma\} \times C_\gamma \subset O_A$. For every $\alpha \in D_A$, set

$$U_\alpha = (\{(\gamma, \gamma) \in Y : \gamma \in (\alpha_0, \alpha]\} \cup \bigcup \{(\{\gamma\} \times C_\gamma) \cap Y : \gamma \in (\alpha_0, \alpha] \cap S\}).$$

For each $\beta \in N_B$, there is a $\beta_0 \in \omega_2$ such that

(iv) $N_B \cap (\beta_0, \beta] = \{\beta\}$,
(v) $\{(\beta_0, \beta] : \beta \in N_B\}$ is a pairwise disjoint family of open sets in ω_2, and
(vi) $\{(\omega_2, \gamma) : \beta_0 < \gamma \leq \beta\} \subset O_B$.

For each $\gamma \in (\beta_0, \beta]$, chose α_γ such that $\{(\alpha, \gamma) : \alpha_\gamma < \alpha \leq \omega_2\} \subset O_B$. For each $\beta \in N_B$, set

$$V_\beta = (\{(\alpha, \gamma) \in Y : \beta_0 < \gamma \leq \beta \text{ and } \alpha_\gamma < \alpha \leq \omega_2\}).$$

It remains to show that

$$\mathcal{U} = \{U_\alpha : \alpha \in N_A\} \cup \{V_\beta : \beta \in N_B\}$$

is a discrete collection of open sets. Clearly, $\mathcal{U}$ is pairwise disjoint.

Let $y \in Y \setminus \bigcup \mathcal{U}$. If $y \in I$, then y is isolated and $\{y\} \cap (\bigcup \mathcal{U}) = \emptyset$. If $y \in A$, then $y = (\alpha, \alpha)$ for some $\alpha \in \omega_2$. Then there is $\alpha_0 < \alpha$ such that $\{(\gamma, \gamma) : \alpha_0 < \gamma \leq \alpha\} \subset Y \setminus \overline{\bigcup \{U_\alpha : \alpha \in N_A\}}$ since D_A is closed. Since $A \cap \overline{O_B} = \emptyset$, one can find an open neighborhood of y which misses $\bigcup \mathcal{U}$. Similarly, if $y \in B$, then $y = (\omega_2, \beta)$ for some $\beta \in \omega_2$. Then there is a $\beta_0 < \beta$ such that $\{(\omega_2, \gamma) : \beta_0 < \gamma \leq \beta\} \subset Y \setminus \overline{\bigcup \{V_\beta : \beta \in N_B\}}$ since D_B is closed. Since $B \cap \overline{O_A} = \emptyset$, one can find an open neighborhood of y which misses $\bigcup \mathcal{U}$. Therefore, Y is strongly cwH and X is hereditarily strongly cwH. □

In [W], M. Wage described a machine that takes any normal, noncollectionwise normal space X and produces a nonnormal space X^*. We will show that if X is hereditarily strongly cwH in addition to the properties above, then X^* will be hereditarily strongly cwH and nonnormal.

We begin by describing Wage's machine. Suppose that X is a normal space that is not cwN. Let $\{H^\alpha : \alpha < \kappa\}$ be a discrete collection of closed sets which witnesses the non-cwN of X. Let $H = \cup \{H^\alpha : \alpha < \lambda\}$ and $C = X - H$. Let

$$X^* = (X \times \{0, 1\}) \cup (C \times \{(\alpha, \beta) : \alpha, \beta < \lambda \text{ and } \alpha \neq \beta\}).$$

If $A \subset X$ and $\delta \in \{0, 1\} \cup \{(\alpha, \beta) : \alpha, \beta < \lambda \text{ and } \alpha \neq \beta\}$, define $A_\delta = (A \times \{\delta\}) \cap X^*$.

Isolate the points of $X^* - (H_0 \cup H_1)$. (That is, let both $\{p\}$ and $\{p\}^c$ be open in X^* if p is not in $H_0 \cup H_1$.) For each open set $U \subset X$ and $\alpha < \lambda$ such that U is contained in $H^\alpha \cup C$, define the following basic open subsets of X^*:

$$\cup \{U_{(\alpha,\beta)} : \alpha \neq \beta < \lambda\} \cup U_0 \text{ and } \cup \{U_{(\beta,\alpha)} : \alpha \neq \beta < \lambda\} \cup U_1.$$

Note that if U is open in X and $U \subset H^\alpha \cup C$, then the two open subsets of X^* derived from U above are disjoint. Give X^* the topology generated by the base described above.

M. Wage showed that X^* is T_1 and regular but not normal. (The closed subsets H_0 and H_1 cannot be separated.) For $A \subset X^*$, define $\pi(A) = \{x \in X : \{x\} \times \{\delta\} \in A$ for some $\delta\}$.

Theorem 3.2. *If X is hereditary strongly cwH, then X^* is hereditary strongly cwH.*

Proof. It suffices to show that each open subset of X^* is strongly cwH. Let O be an open set of X^*, and let D be a relatively discrete subset of O. Without loss of generality, assume $D \subset H_0 \cup H_1$. Note that $\pi(O_0)$ and $\pi(O_1)$ are open subsets of X and $\pi(D_0)$ is a closed discrete subset of $\pi(O_0) \cup \pi(O_1)$. Since X is hereditarily strongly cwH, let $\mathcal{U}_0$ be a family of open sets of X such that

(a) for every $U \in \mathcal{U}_0$, $U \subset \pi(O_0)$
(b) $\mathcal{U}_0$ is a discrete open expansion of $\pi(D_0)$ in $\pi(O_0) \cup \pi(O_1)$, and
(c) for every $U \in \mathcal{U}_0$, $cl_X(U) \subset H^\alpha \cup C$ for some $\alpha < \lambda$,

Note that for each $x \in \pi(D_0)$, there exists a unique $\alpha_x < \lambda$ such that $x \in H^{\alpha_x}$ and there exists a unique $U_x \in \mathcal{U}_0$ such that $x \in U_x$. For $x \in \pi(D_0)$, define

$$B_x = (\cup\{(U_x)_{(\alpha_x,\beta)} : \beta < \lambda \text{ and } \beta \neq \alpha_x\} \cup (U_x)_0) \cap O$$

and let

$$\mathcal{B}_0 = \{B_x : x \in \pi(D_0)\}.$$

<u>claim:</u> $\mathcal{B}_0$ is discrete in O.
Let $y \in O$. We will consider three cases: (1) $y \in X^* - (H_0 \cup H_1)$, (2) $y \in H_0$, and (3) $y \in H_1$.
<u>case (1):</u> Since $\mathcal{B}_0$ is pairwise disjoint and y is isolated, y meets $\mathcal{B}_0$ in at most one set.
<u>case (2):</u> Since $\mathcal{U}_0$ is discrete in $\pi(O_0)$, there is an open subset V of $\pi(O_0)$ such that V meets at most one member of $\mathcal{U}_0$. Let α' be the unique ordinal such that $\pi(y) \in H^{\alpha'}$. Then

$$B_{\pi(y)} = (\cup\{(V)_{(\alpha',\beta)} : \beta < \lambda \text{ and } \beta \neq \alpha'\} \cup V_0) \cap O$$

meets at most one member of $\mathcal{B}_0$.
<u>case (3):</u> There exists a unique $\alpha < \lambda$ such that $\pi(y) \in H^\alpha$. Let V be an open set of X such that V meets $\mathcal{U}_0$ in at most one member. Suppose that $U \cap V \neq \emptyset$ for some $U \in \mathcal{U}_0$. If $cl_X(U) \not\subset H^\alpha \cup C$, then one can find an open set $W \subset X$ such that $U \cap W = \emptyset$ by part (c). Let

$$B_W = (\cup\{W_{(\beta,\alpha)} : \beta < \lambda \text{ and } \beta \neq \alpha\} \cup V_1) \cap O.$$

Then $B_W \cap (\cup\mathcal{B}_0) = \emptyset$. If $cl_X(U) \subset H^\alpha \cup (X \backslash H)$, define

$$B_V = (\cup\{V_{(\beta,\alpha)} : \beta < \lambda \text{ and } \beta \neq \alpha\} \cup V_1) \cap O.$$

Then $B_U \cap B_V = \emptyset$ and $B_V \cap (\cup\mathcal{B}_0) = \emptyset$.

Note that (3) shows that $D_1 \cap \overline{(\cup\mathcal{B}_0)} = \emptyset$. Similarly, expand $\pi(D_1)$ to a discrete open collection $\mathcal{B}_1$ with $\cup\mathcal{B}_1 \subset O \backslash \overline{\cup\mathcal{B}_0}$. Then $\mathcal{B}_0 \cup \mathcal{B}_1$ is a discrete expansion of D in O. $\square$

Example 3.3. *Another regular hereditary strongly cwH space that is not normal.*

Proof. Fleissner's *George* is hereditarily normal and hereditarily cwH but is not cwN. Hence it is hereditarily strongly cwH. Inputting *George* into M. Wage's machine above produces a hereditarily strongly cwH non-normal space. □

It is easy to see that Wage's machine produces a scattered space from a scattered space. Since Fleissner's *George* is scattered, Example 3.3 is scattered. By Theorem 2.8, Example 3.3 is another example of a β-normal non-normal space in ZFC.

4. Product theorems and a consistent non-normal example

The following theorem sets the stage for our first consistent counterexample. It is a criterion reminiscent of Katětov's theorem that $X \times (\omega+1)$ is hereditarily normal iff X is perfectly normal, and also of Dowker's criteria for when $X \times (\omega+1)$ is normal.

Theorem 4.1. *If X is a regular space, the following are equivalent:*

1. *$X \times (\omega+1)$ is hereditarily strongly cwH*
2. *X is hereditarily strongly cwH, every discrete subspace of X is an F_σ, and for each countable family $\{D_n : n \in \omega\}$ of discrete subspaces of X, there is a choice of open sets $U_n \supset D_n$ such that*

$$(F =) \bigcap_{n=0}^{\infty} c\ell_X(\bigcup_{k=n}^{\infty} D_k) = \bigcap_{n=0}^{\infty} c\ell_X(\bigcup_{k=n}^{\infty} U_k).$$

Proof. (1) ⇒ (2): Since $X \times \{n\}$ is clopen and homeomorphic to X for each $n \in \omega$, X must clearly be hereditarily strongly cwH.

If D is a discrete subspace of X, let $Y = [X \times (\omega+1)] \setminus [(\overline{D} \setminus D) \times \{\omega\}]$. Then $F_0 = (\overline{D} \setminus D) \times \omega$ and $F_1 = D \times \{\omega\}$ are disjoint closed sets in Y, the latter of which is discrete. Using Lemma 2.6, let U and V be disjoint open subsets of Y containing F_0 and F_1, respectively, and let $D_n = \{d \in D : \langle d, n\rangle \in V\}$. Clearly, $D = \bigcup_{n=0}^{\infty} D_n$. Also, D_n is closed in X for all n; indeed, any of its limit points would have to be in $\overline{D} \setminus D$, but $(\overline{D} \setminus D) \times \{n\} \subset U$.

If D_n and F are as in (2), let $Y = \bigcup\{Y_\alpha : \alpha \le \omega\}$ where $Y_\omega = (X \setminus F) \times \{\omega\}$ and $Y_n = [X \setminus (\overline{D_n} \setminus D_n)] \times \{n\}$ for $n \in \omega$. Then $D = \bigcup\{D_n \times \{n\} : n \in \omega\}$ is closed discrete in Y and is disjoint from the closed subset Y_ω of Y. Let U be an open subset of Y containing D whose closure misses Y_ω. Then $U_n = U \cap (X \times \{n\})$ is as desired.

(2) ⇒ (1): Let Y be an open subspace of $X \times (\omega+1)$ and let D be closed discrete in Y. It is enough to expand D to a disjoint family of open subsets of Y whose union is relatively discrete in Y.

Let π be the restriction to Y of the projection of $X \times (\omega+1)$ onto X. Let $D_n = \pi^{\rightarrow}(D \cap (X \times \{n\}))$ and let $D_\omega = \pi^{\rightarrow}(D \cap (X \times \{\omega\}))$. Then each D_α is a discrete subset of X, and $D = \bigcup\{D_n \times \{n\} : n \in \omega\}$. Let $F \subset X$ be as in (2); then $F \times \{\omega\}$ is the derived set in $X \times (\omega+1)$ of $D \setminus (X \times \{\omega\})$, and $(F \times \{\omega\}) \cap Y = \emptyset$ because D is relatively closed in Y. Expand each $D \cap (X \times \{n\})$ to a discrete family of open sets contained in $U_n \times \{n\}$. The union of these families has closure meeting $X \times \{\omega\}$ in $F \times \{\omega\}$ and is thus discrete in Y.

It remains to expand $D_\omega \times \{\omega\}$ to a discrete collection of open sets. First, expand D_ω to a disjoint family $\{V_d : d \in D_\omega\}$ of open subsets of $X_\omega = X \setminus (\overline{D_\omega} \setminus D_\omega)$ that is relatively discrete in X_ω. Next we bring the fact that discrete sets are F_σ in X into play. Let D_ω be the ascending union of closed sets $D_\omega^n (n \in \omega)$, and let

$$C_n = D_\omega^n \setminus c\ell_X(\bigcup_{k=n}^{\infty} U_k).$$

Then D_ω is also the ascending union of the closed sets C_n.

In $Y \subset X \times (\omega+1)$, expand $(C_n \setminus C_{n-1}) \times \omega$ to a family $\mathcal{G}_n$ of open sets of the form

$$G_d = Y \cap (W_d \times (\omega \setminus n))$$

such that $W_d \subset V_d$. Then $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ is a discrete family of open subsets of Y. □

We will now use the criterion (2) to provide a consistent example of a locally compact, locally countable (hence first countable) hereditarily strongly cwH space which is not normal. The following lemma paves the way.

Lemma 4.2. *If X is a locally countable, hereditarily normal space and Q is a countable subset of X, then there is an open subset U of X containing Q such that $\overline{U} \setminus \overline{Q}$ is countable.*

Proof. Let V be a countable open set containing Q, and let $Y = (X \setminus \overline{Q}) \cup V$. Using normality of Y and the fact that $c\ell_Y Q \subset V$, let U be an open subset of V containing Q and satisfying $c\ell_Y U \subset V$. Then $c\ell_X U = c\ell_Y U \cup c\ell_X Q$. □

We can clearly weaken "hereditarily normal" in Lemma 4.2 to "hereditarily pseudonormal":

Definition 4.3. A space is *pseudonormal* if every pair of disjoint closed sets, one of which is countable, can be put into disjoint open sets.

Theorem 4.4. *If X is a hereditarily pseudonormal, hereditarily separable, locally countable space, then $X \times (\omega + 1)$ is hereditarily strongly cwH.*

Proof. X is hereditarily strongly cwH because it is hereditarily pseudonormal and every discrete subspace is countable. Obviously, every discrete subspace of X is an F_σ. Finally, suppose $\{D_n : n \in \omega\}$ is a family of discrete subspaces of X. For each $n \in \omega$, let $Q_n = \bigcup_{k=n}^{\infty} D_k$ and let V_n be an open subset of X containing Q_n such that $\overline{V_n} \setminus \overline{Q_n}$ is countable, and such that $V_{n+1} \subset V_n$ for all n.

With F as in Theorem 2.1 (2), let $A = \bigcap\overline{\{V_n : n \in \omega\}}$ and let $A \setminus F = \{a_n : n \in \omega\}$. [Clearly, $A \setminus F$ is countable!] For each $k \in \omega$ pick $m(k) \geq k$ such that $a_k \notin Q_{m(k)}$, pick an open nbhd W_k of a_k whose closure misses $Q_{m(k)}$, and let $U_n = V_n \setminus \bigcup\{W_k : m(k) \leq n\}$. Then U_n is as in (2). □

Example 4.5. [Assuming $\diamondsuit$] *A hereditarily strongly cwH, locally compact, locally countable, non-normal space.*

The example is $X \times (\omega + 1)$ where X is a locally compact, locally countable, hereditarily separable, hereditarily normal Dowker space. Such a space X was constructed in [Ny2] using the axiom $\diamondsuit$. It is immediate from Theorem 4.4 that $X \times (\omega+1)$ is hereditarily strongly cwH, but since X is Dowker, $X \times (\omega+1)$ is not normal.

5. Some related independence results involving wD($\aleph_1$)

In Section 7, we will give a ZFC example (7.1) to show that "locally Lindelöf" cannot be dropped from (1) in the following theorem.

Theorem 5.1. *The following statements are ZFC-independent:*

(1) *Every regular, locally Lindelöf, hereditarily wD($\aleph_1$) space is pseudonormal.*
(2) *Every locally compact, hereditarily wD($\aleph_1$) space is pseudonormal.*
(3) *Every locally compact, locally countable, hereditarily wD($\aleph_1$) space is wD($\aleph_0$).*

Moreover, (3) *is equivalent to* $\mathfrak{b} > \omega_1$.

Regularity is needed in Theorem 5.1 (1) since every second countable, non-regular space (of which there are many easy examples) is vacuously (hereditarily) wD($\aleph_1$), but is not even wD($\aleph_0$), thanks to the theorem mentioned at the beginning of Section 3.

The first step in proving Theorem 5.1 is an easy known result which we prove for the sake of self-containment:

Lemma 5.2. *In a locally Lindelöf regular space, every point has an open Lindelöf neighborhood.*

Proof. Each point has a closed Lindelöf, hence normal neighborhood, so the space is Tychonoff and has a base of cozero sets. A cozero subset of a Lindelöf space is Lindelöf. □

The following two lemmas have the same proof as Lemma 3.1 and Lemma 3.9 in [JNySz]:

Lemma 5.3. *In a locally Lindelöf, hereditarily wD*($\aleph_1$) *space, the boundary of any open Lindelöf subset has countable spread.*

Lemma 5.4. *Let X be a locally Lindelöf regular space such that every Lindelöf subset has Lindelöf closure. Then any two disjoint closed subsets of X, one of which is Lindelöf, can be put into disjoint open sets. Hence X is pseudonormal.*

Proof of Theorem 5.1. To show the consistency of (1), use the well-known facts that every regular space of countable spread is hereditarily Lindelöf iff there are no S-spaces, and that the PFA implies there are no S-spaces [R]. By Lemma 5.3, the former fact implies that in a locally Lindelöf regular space, every Lindelöf subset has Lindelöf closure. Now use Lemma 5.4. This obviously establishes the consistency of (2) and (3) as well, but note that the foregoing arguments can obviously be modified to show that (2) holds in any model in which every locally compact space of countable spread is hereditarily Lindelöf, and (3) holds in any model in which there are no locally compact, locally countable S-spaces. Models of (2) thus include any model of MA($\aleph_1$) and also the model used in the solution of Katětov's problem [LT]. There are also models of (3) in which $2^{\aleph_0} < 2^{\aleph_1}$ [ENyS].

To complete the proof of Theorem 5.1, it is enough to show the "Moreover" part. We will do even better:

Theorem 5.5. *The following statements are equivalent:*

(1) $\mathfrak{b} > \omega_1$.
(2) *Every locally hereditarily Lindelöf, first countable, regular, hereditarily weakly ω_1-cwH space is pseudonormal.*

(3) *Every locally compact, locally countable, hereditarily wD($\aleph_1$) space is pseudonormal.*
(4) *Every locally compact, locally countable, hereditarily wD($\aleph_1$) space is wD($\aleph_0$).*

Remark 5.6. Of course, many other equivalent statements could be included in Theorem 5.5, intermediate between (2) and (3) or between (3) and (4). In fact, the only reason (3) was included was for easy comparison with the following remark.

Remark 5.7. In forthcoming papers, it will be shown that the following variations on (2) and (3) are also ZFC-independent, though not equivalent to $\mathfrak{b} > \omega_1$:

($2'$)*: Every first countable, hereditarily strongly ω-cwH space is pseudonormal.*
($3'$)*: Every locally compact, locally countable, hereditarily strongly cwH space is pseudonormal.*

Specifically, Alan Dow has shown that in Cohen's original model, every first countable, strongly ω-cwH ("Property D") space is pseudonormal, solving a problem posed in [vDW], where a first countable, strongly ω-cwH space was constructed under $\mathfrak{p} = \mathfrak{c}$. The first author has constructed a counterexample to ($2'$) under the same set-theoretic axiom, and a counterexample to ($3'$) under CH.

Remark 5.8. Dow's result does not generalize to higher cardinals. In [vD1] there is a locally compact, locally countable space in which the nonisolated points form a countably compact subspace; thus the space is clearly strongly cwH. On the other hand, the space has a pair of disjoint closed subspaces of cardinality ω_1 which cannot be put into disjoint open sets. This space is not hereditarily cwH, however (it is separable and has an uncountable discrete subspace), so it does not answer Question 1.6.

Proof of Theorem 5.5. It is enough to show (1) implies (2) and (4) implies (1), inasmuch as every locally compact, locally countable space is regular, first countable, and locally Lindelöf, while every pseudonormal space is wD($\aleph_0$).

To show (1) implies (2) we use Theorem 3.7 of [Ny1]:

Theorem. *Every regular, first countable space of Lindelöf number $< \mathfrak{b}$ is pseudonormal.*

Let X be locally hereditarily Lindelöf, first countable, and regular. Given a countable closed subset C of X, let U be an open Lindelöf neighborhood of C. It is clearly enough to show that C and $\overline{U} \setminus U$ can be put into disjoint open sets. First we show that $\overline{U}$ is of countable spread. If there were an uncountable discrete subset of $\overline{U}$, the hereditarily weakly ω_1-cwH property would give us an uncountable disjoint collection of open subsets of $\overline{U}$ and hence of U; but any family of disjoint open subsets of a hereditarily Lindelöf space is countable.

Next we show that $\overline{U}$ is of Lindelöf number $\leq \omega_1$; this and the theorem just cited will finish the proof that (1) implies (2).

Suppose $\overline{U}$ has Lindelöf number $> \omega_1$. By induction, let V_α be defined for each $\alpha < \omega_2$ so that each V_α is a relatively open subset of $\overline{U}$ and contains some point y_α not in any V_β ($\beta < \alpha$). Then $Y = \{y_\alpha : \alpha < \omega_2\}$ is a scattered subspace of $\overline{U}$. But each Cantor-Bendixson level of Y is countable because $\overline{U}$ is of countable spread. And every point of Y is on some countable level because Y is locally hereditarily Lindelöf. This contradiction completes the proof that (1) implies (2).

We will show (4) implies (1) by contrapositive. One main ingredient is the powerful result of Todorčević that $\mathfrak{b} = \omega_1$ implies there is a locally compact, locally countable, hereditarily separable space of cardinality ω_1. The other is the following theorem:

Theorem 5.9. *Every locally compact, locally countable space of cardinality $\mathfrak{b}$ can be embedded as a co-countable subspace in one that is separable and does not satisfy wD($\aleph_0$).*

Once this theorem is proven, one need only note that every hereditarily separable space is vacuously wD($\aleph_1$) to complete the proof that (4) implies (1) and hence of Theorems 5.1 and 5.5.

Proof of Theorem 5.9. Let X have underlying set $\mathfrak{b}$ and be given a locally compact, locally countable topology. This makes it scattered, so the topology can be chosen so that $[0, \xi)$ is open for all $\xi \in \mathfrak{b}$.

An elementary fact about $\mathfrak{b}$ is that there is a $<^*$-well-ordered set $\{f_\alpha : \alpha < \mathfrak{b}\}$ of increasing functions that is $<^*$-unbounded [vD2]. Another is that any such family is $<^*$-unbounded on every infinite set; that is, if A is an infinite subset of ω then $\{f_\alpha \restriction A : \alpha < \mathfrak{b}\}$ is $<^*$-unbounded on A.

So let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be as above. We define a locally compact, locally countable topology on $X \cup [\omega \times (\omega + 1)]$ by induction, giving $\omega \times (\omega + 1)$ its usual topology, making $\omega \times \omega$ the set of isolated points and making $\omega \times \{\omega\}$ and X into disjoint closed subsets of our space Z.

To specify the topology on a locally compact scattered space S, it is enough to specify a neighbornet—a function V assigning to each $x \in S$ a neighborhood $V(x)$—such that each $V(x)$ is compact and open, and x is the unique point of maximal rank in $V(x)$. If x is isolated, this constrains the choice $V(x) = \{x\}$. Otherwise the collection of all sets of the form $V(x) \setminus [V(x_0) \cup \cdots \cup V(x_n)]$, with $x_i \in V(x)$ for all i, is a base for the neighborhoods of x, inasmuch as its intersection is $\{x\}$, and any filterbase of closed neighborhoods of a point p in a compact space whose intersection is $\{p\}$ is automatically a base for the neighborhoods of $\{p\}$.

So now let V be such a neighbornet for X. We will define a neighbornet U for $X \cup (\omega \times \omega)$ such that $U(\xi) \cap X = V(\xi)$ for all $\xi \in \omega_1 = X$. This will be done by attaching the graph of f_ξ as a "whisker" to ξ and trimming the whiskers attached to the points of $V(\xi) \setminus \{\xi\}$ by clipping off finitely many points in ways to be specified by induction. In this way the union of ξ with the graph of f_ξ will be a topological copy of $\omega + 1$.

If $(i, j) \in \omega \times \omega$, let $U(i, j) = \{(i, j)\}$, so that (i, j) is isolated in $X \cup (\omega \times \omega)$. If $n \in \omega$, let $U(n)$ be the graph of f_n, together with n. If $\xi \in X \setminus \omega$, assume U has been defined for all $\eta < \xi$ so that $(\omega \times \omega) \cup [0, \xi)$ is locally compact and all points of $U(\eta) \cap (\omega \times \omega)$ are in $f_\eta^\downarrow$, where $f^\downarrow$ stands for the set of all points on or below the graph of f.

Next recall Theorem 3.7 of [Ny1] cited above. Since $V(\xi)$ is countable and compact, $V(\xi) \setminus \{\xi\}$ is closed in $[0, \xi) \cup [\omega \times (\omega + 1)]$ and there are disjoint clopen sets U and W in $[0, \xi) \cup [\omega \times (\omega + 1)]$ containing $V(\xi) \setminus \{\xi\}$ and its complement. Trim U if necessary so that it meets $\omega \times \omega$ in a subset of $f_\xi^\downarrow$. Also, have it contain the graph of f_ξ, and let $U(\xi) = U \cup \{\xi\}$. This extends the neighbornet to $[0, \xi] \cup [\omega \times (\omega + 1)$

so that $U(\xi)$ is the one-point compactification of U, and so that the induction hypothesis for $\xi+1$ is satisfied.

Clearly, the space Z is locally compact, locally countable, separable, and has X as a co-countable subspace. To see that Z is not wD($\aleph_0$), we use the closed discrete subspace $\omega \times \{\omega\}$. Let

$$D = \{(i_n, \omega) : n \in \omega\},$$

with $i_n < i_{n+1}$ for all n. If U_n is an open set containing (i_n, ω), there exists k_n such that every point above (i_n, k_n) is in U_n. Let $f(j) = k_n$ for the least integer n such that $j < i_n$. Then $f : \omega \to \omega$ is a nondecreasing function, and is below the graph of all but countably many f_α in infinitely many places. Since f_α is increasing, this implies $f_\alpha(i_n) > f(i_n)$ for infinitely many n, so that $\bigcup\{U_n : n \in \omega\}$ has uncountably many points of X in its closure. □

When X is hereditarily separable, the space constructed in proving Theorem 5.9 is scattered of height and cardinality ω_1: it is scattered and each point is on a countable level because it is locally compact and locally countable, and each level is countable because of hereditary separability, while $\mathfrak{b} = |X|$ is uncountable.

By making a careful choice of hereditarily separable X, we can use a quotient map under a variety of set-theoretic hypotheses to produce a non-regular, hereditarily strongly cwH, Fréchet-Urysohn space. In the next section we will use the hypothesis "$\mathfrak{b} = \omega_1 + \exists$ an Ostaszewski space." In a forthcoming paper, the restriction $\mathfrak{b} = \omega_1$ will be eliminated by using a different choice of Y.

6. A consistent construction using Ostaszewski spaces

Definition 6.1. An *Ostaszewski space* is a locally compact, locally countable, countably compact uncountable space in which every open subset is either countable or co-countable.

If we omit "countably compact" in the foregoing definition, we get the definition of a *sub-Ostaszewski space.* All sub-Ostaszewski spaces are hereditarily separable and scattered of height ω_1.

Example 6.2. [Assume "$\mathfrak{b} = \omega_1 + \exists$ an Ostaszewski space"] *A locally countable, hereditarily separable, hereditarily strongly cwH space which is Fréchet-Urysohn but not regular.*

Let X be an Ostaszewski space, so that X can be used in the construction of Z in the proof of Theorem 5.9. Let S be the quotient space of Z obtained by identifying the points of $\omega \times \{\omega\}$ to a single point p. Clearly S is both locally countable and hereditarily separable.

S is not regular: If U is a neighborhood of p, it includes all points of $\omega \times \omega$ above the graph of some function; now argue as in the proof that Z is not $wD(\aleph_0)$.

S is Fréchet-Urysohn: Every point of S has a countable base of nbhds except for p, which has a countable open neighborhood $U = (\omega \times \omega) \cup \{p\}$. This neighborhood is homeomorphic to the well-known Fréchet-Urysohn fan $S(\omega)$, and it is easy to see that a subset of $S \setminus \{p\}$ has p in its closure if, and only if, it meets some column $\{n\} \times \omega$ in an infinite set. Any sequence listing the points of this set converges to p.

S is hereditarily strongly cwH: First note that S is Hausdorff, since every point of X has a neighborhood below the graph of some f_α, and p is outside its closure.

Let D be a discrete subspace of S; by hereditary separability, D is countable.

Case 1: $D \cap X$ has co-countable closure (in X, hence in S).

Then $W = (S \setminus \overline{D}) \cup D$ is countable, and clearly regular, so we can expand $(D \cap X) \cup \{p\}$ to a collection $\mathcal{U}$ of open sets that is discrete in W, and then $\mathcal{U} \cup \{\{d\} : d \in D \cap \omega \times \omega\}$ is an expansion of D to a discrete collection of open sets in W.

Case 2: $D \cap X$ has compact closure.

Then $\overline{D \cap X}$ and p can be put into disjoint open countable sets U and V. Because $X \cup (\omega \times \omega)$ is regular, we can expand $D \cap X$ to a countable collection $\mathcal{U}$ of disjoint open subsets of U, relatively discrete in $W = [(S \setminus \overline{D}) \cup D] \setminus \{p\}$, all of which are below the graph of some f_α. Then

$$\mathcal{U} \cup \{V \setminus \bigcup \mathcal{U}\} \cup \{\{d\} : d \in D \cap \omega \times \omega\}$$

is an expansion of D to a discrete collection of open sets in $(S \setminus \overline{D}) \cup D$.

In both cases, we are done—see the comments preceding Definition 2.3. □

In a forthcoming paper, other examples of hereditarily strongly cwH, Fréchet-Urysohn, nonregular spaces will be constructed assuming the Continuum Hypothesis (CH). This complements Example 6.2 somewhat because there are models of CH in which there are no Ostaszewski spaces [ER] and also Ostaszewski spaces in models where CH fails. For instance, V.I. Malykhin showed long ago [Ma] that Ostaszewski spaces cannot be destroyed by the usual method of adding Cohen reals, while J. Tatch Moore showed recently [Mo] that adding uncountably many random reals to any ground model in the usual way gives an Ostaszewski space. A generalization of this latter fact can be found in [DHM].

7. Some ZFC examples based on the Tychonoff plank

In this section we present an assortment of variations on the Tychonoff plank that help clarify relationships between the properties we have studied here. The following example shows that "locally Lindelöf" cannot be dropped from Theorem 5.1(1).

Example 7.1. *A regular, scattered, hereditarily wD($\aleph_1$) space which is not wD($\aleph_0$).*

Let X be the space with underlying set $\omega_1 + 1$ in which all points except ω_1 are isolated, while a set containing ω_1 is open iff its complement is nonstationary. Let Z be the subspace of $X \times (\omega + 1)$ obtained by removing the corner point (ω_1, ω). In Z, no infinite subset of the countable closed discrete subspace $E = \{\omega_1\} \times \omega$ can be expanded to a discrete collection of open sets. Hence Z is not wD($\aleph_0$). On the other hand, if D is a closed discrete subset of Z of cardinality $\aleph_1$, the non-isolated points of $D \setminus E$ are contained in $\omega_1 \times \{\omega\}$, and if there are uncountably many of them, there is a nonstationary subset N of ω_1 such that $(N \times \{\omega\}) \cap D$ is uncountable. Then $(N \times \{\omega\}) \cap D$ can easily be expanded to a discrete collection of open sets. Since the non-isolated points of Y form a closed discrete subspace, the wD($\aleph_1$) property is clearly inherited by all subspaces of Y.

Alan Dow came up with the following modification of 7.1, included here with his permission.

Example 7.2. *A hereditarily wD($\aleph_1$), hereditarily strongly ω-cwH regular space that is neither β-normal nor pseudonormal.*

Use the same space for X as in 7.1, but instead of $\omega+1$ use $Y = \mathbb{Q} \cup \{y\}$ where y is a remote point of $\beta\mathbb{Q} \setminus \mathbb{Q}$. Let Z be the subspace of $X \times Y$ obtained by removing the corner point (ω_1, y). The argument that Z is hereditarily wD($\aleph_1$) is essentially unchanged. Since y is not in the closure of any nowhere dense subset of Y, the product of any discrete subspace of $\mathbb{Q}$ with X fails to have any point of $(X \setminus \{\omega_1\}) \times \{y\}$ in its closure. On the other hand, if U is a neighborhood of any dense subset of $\mathbb{Q}$, its complement is nowhere dense, so U has y in its closure; consequently, the subspaces $(X \setminus \{\omega_1\}) \times \{y\}$ and $\mathbb{Q} \times \{\omega_1\}$ witness the failure of β-normality as well as of pseudonormality.

In [vDW], a ZFC example, due to Eric van Douwen, was announced of a strongly ω-cwH ("Property D") regular space which is not pseudonormal, but van Douwen does not seem to have ever published this example, so Example 7.2 may be the first example of such a space in print.

Example 7.3. *A pair of hereditarily cwH regular spaces that are not α-normal*

Let Z be the product space $(\omega_1 + 1) \times (\mathbb{Q} \cup \{y\})$ with the corner point (ω_1, y) removed, and let S be the space obtained from Z by isolating the points of $\omega_1 \times \mathbb{Q}$. Discrete subspaces of $\{\omega_1\} \times (\mathbb{Q} \cup \{y\})$ expand to relatively discrete collections of open sets just as they do in Example 7.2. From this, and from the fact that closed discrete subspaces of both spaces are countable, it follows that both Z and S are strongly cwH. If D is a discrete subspace of S contained in $\omega_1 \times \{y\}$, then there is a nonstationary subset N of ω_1 such that $D \subset N \times \{y\}$ and so D can be expanded to a disjoint collection of open sets. So any discrete subspace of S can be expanded to a disjoint collection of open sets. As for Z, the discrete subspaces of $\omega_1 \times (\mathbb{Q} \times \{y\})$ omit a set of the form $C \times (\mathbb{Q} \times \{y\})$ where C is club in ω_1, so

If A is a dense subset of $\{\omega_1\} \times \mathbb{Q}$, and I is the set of successor ordinals in ω_1, then A and $B = I \times \{y\}$ cannot be put into disjoint open sets in either S or Z, just as in the proof that Example 7.2 is not β-normal.

Neither Z nor S is strongly cwH: no infinite subset of $\{\omega_1\} \times (\mathbb{Q} \cup \{y\})$ can be expanded to a discrete collection of open sets. There are ZFC examples of strongly cwH, hereditarily cwH regular spaces that are not α-normal, but the problem (see Question 1.10) of finding one that is hereditarily strongly cwH is still not completely solved. In a forthcoming paper, however, one will be constructed under the following very general axiom:

Axiom 7.4. *There is an uncountable cardinal λ such that $2^\lambda = \lambda^+$.*

To negate this axiom requires one to assume the consistency of some very large cardinals; some inkling of how large they have to be is provided towards the end of [F].

On the other hand, it takes only a minor modification of the above examples to produce:

Example 7.5. *A hereditarily strongly cwH regular space that is neither β-normal nor pseudonormal.*

This space S is coarser than Example 7.2 and finer than the spaces of Example 7.3. We use the same underlying set $[(\omega_1+1)\times(\mathbb{Q}\cup\{y\})]\setminus\{(\omega_1, y)\}$ for S, refining the product topology by letting the points of $\omega_1 \times \mathbb{Q}$ be isolated, while neighborhoods of any point of $\{\omega_1\} \times \mathbb{Q}$ are of the form $W \times U$, where U is open in $\mathbb{Q}$ while $W = C \cup \{\omega_1\}$ with C a club subset of ω_1. Neighborhoods of points in $\omega_1 \times \{y\}$ are their usual product neighborhoods. Informally, S is Example 7.2 with the closed discrete subspace $(X \setminus \{\omega_1\}) \times \{y\}$ replaced by ω_1.

To show that S is neither pseudonormal nor β-normal, argue just as in Example 7.2. The following facts show S is hereditarily strongly cwH: (1) discrete subsets of $\{\omega_1\} \times \mathbb{Q}$ expand just as in Example 7.2; and (2) if D is a discrete subspace of ω_1, then D omits a club subset C of ω_1, and so the limit points of $D \times (\mathbb{Q} \cup \{y\})$ are just the limit points of $D \times \{y\}$.

Example 7.1 can be modified in the same way as Example 7.2 was modified to Example 7.5. If Y is a subspace of the resulting space, then Y satisfies a strengthening of wD($\aleph_1$): given any closed discrete subspace D of Y such that $|D| = A_1$, there is a subspace D_0 of D such that $D \setminus D_0$ is countable, and such that D_0 can be expanded to a discrete collection of open subsets of Y. In fact, we can take D_0 to be $D \setminus (\{\omega_1\} \times \omega)$.

8. The flip side

This paper would not be complete without some mention of what happens when "normal," etc. are switched with "[strongly] cwH" in the questions we have posed. On the one hand, it has long been known that $\mathbf{V} = \mathbf{L}$ implies that every first countable normal space, and every locally compact normal space is cwH, hence strongly cwH. The arguments for these facts in [F] and [W2] readily generalize to show:

Theorem 8.1. [Assume $\mathbf{V} = \mathbf{L}$] *Every first countable α-normal space, and every locally compact α-normal space is cwH.*

Theorem 8.2. [Assume $\mathbf{V} = \mathbf{L}$] *Every first countable β-normal space, and every locally compact β-normal space is strongly cwH.*

On the other hand, it has also long been known that MA(ω_1) implies the existence of a locally compact, locally countable, perfectly normal (hence hereditarily normal) space which is not cwH. This is just the reverse of the situation with the directions we explored in the earlier sections. As shown in Section 4, $\mathbf{V} = \mathbf{L}$ implies the existence of a locally compact, locally countable, hereditarily strongly cwH space that is not normal, while strengthenings of MA(ω_1) seem to give the best chance of affirmative answers to Questions 1.6 and 1.7.

In other models of set theory, the contrast is not so pronounced. Adding enough Cohen reals makes every first countable normal space and every locally compact normal space (strongly) collectionwise Hausdorff; if we assume the consistency of supercompact cardinals, we can even get collectionwise normality [F]. As alluded to in Remark 5.7, adding $\aleph_2$ Cohen reals to a model of $\mathbf{V} = \mathbf{L}$ also makes every first countable strongly ω-cwH space pseudonormal. However, if uncountably many

Cohen reals are added to any model of set theory, the forcing extension satisfies $\mathfrak{b} = \omega_1$ and thus gives us a locally compact, locally countable space that is wD($\aleph_1$) but not wD($\aleph_0$).

Random reals do not change $\mathfrak{b}$, and the status of our questions is unknown in models where random reals are added to a model of MA(ω_1), where $\mathfrak{b} > \omega_1$; also the status of the statements in Theorem 5.1 (1) and (2) is unknown. But the best chance of going both ways—from normality-like properties to cwH-variations and back—seems to be the family of forcings which includes the one used in solving Katětov's problem. In these iterated forcings, generic sets are added at initial stages to all posets of a certain kind (e.g., ccc) that do not destroy a carefully crafted Souslin tree, and the last step consists of forcing with the tree itself. In some of these models, every first countable normal space is (strongly) cwH, there are no locally compact S or L spaces, and many other consequences of $\mathbf{V} = \mathbf{L}$ and of MA(ω_1) hold simultaneously.

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