STRONG α-FAVORABILITY OF THE (GENERALIZED) COMPACT-OPEN TOPOLOGY

Peter J. Nyikos and László Zsilinszky

Abstract. Strong α-favorability of the compact-open topology on the space of continuous functions, as well as of the generalized compact-open topology on continuous partial functions with closed domains is studied.

1. Introduction

Spaces of partial maps have been studied for various applications throughout the century ([Ku1-2], [AB], [BB], [Ba], [DN1-2], [Fi], [KS], [La], [Se], [St], [Wh], [Za]). In particular, the so-called *generalized compact-open topology* on the space of continuous partial functions with closed domains proved to be a useful tool in mathematical economics ([Ba]), in convergence of dynamic programming models ([La], [Wh]) or more recently in the theory of differential equations ([BC]). This topology was also scrutinized from purely topological point of view e.g. in [BCH], [Ho], [HZ1-2], where among others, separation axioms and some completeness properties (such as Baireness, weak α-favorability, Čech-completeness, complete metrizability) of the generalized compact-open topology have been investigated.

Our paper continues in this research by looking at strong α-favorability in this setting. Section 3 contains our results on strong α-favorability of τ_C as well as a short proof of a recent theorem of *Holá* on complete metrizability of τ_C.

We will rely on the close connection that exists between the generalized compact-open topology, the ordinary compact-open topology τ_{CO} ([MN1]) and the Fell topology τ_F on the hyperspace of nonempty closed subsets of a topological space ([Be], [KT]). This connection and some other auxiliary material is described at the end of Section 1, while in Section 2 we list results about strong α-favorability of τ_{CO} and τ_F, respectively, needed for proving our main results; a generalization of a theorem of *Ma* on weak α-favorability of the compact-open topology is also given.

Let X and Y be Hausdorff spaces. Denote by $CL(X)$ the family of nonempty closed subsets of X and by $K(X)$ the nonempty compact subsets of X. For any $B \in CL(X)$ and a topological space Y, $C(B,Y)$ will stand for the space of all continuous functions from B to Y. A partial map is a pair (B,f) such that $B \in CL(X)$ and $f \in C(B,Y)$. Denote by $\mathcal{P} = \mathcal{P}(X,Y)$ the family of all partial maps.

1991 *Mathematics Subject Classification.* 54C35; 54B20, 90D44.

Key words and phrases. Generalized compact-open topology, Fell topology, compact-open topology, strong α-favorability, weak α-favorability.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Define the so-called *generalized compact-open topology* τ_C on $\mathcal{P}$ as the topology having subbase elements of the form

$$[U] = \{(B,f) \in \mathcal{P} : B \cap U \neq \emptyset\},$$
$$[K : I] = \{(B,f) \in \mathcal{P} : f(K \cap B) \subset I\},$$

where U is open in X, $K \subset X$ is compact and I is an open (possibly empty) subset of Y. We can assume that the I's are members of some fixed open base for Y.

The compact-open topology τ_{CO} on $C(X,Y)$ has subbase elements of the form

$$[K,I] = C(X,Y) \cap [K : I] = \{f \in C(X,Y) : f(K) \subset I\},$$

where $K \subset X$ is compact and $I \subset Y$ is open; $C_k(X)$ (see [MN1]) stands for $(C(X,Y), \tau_{CO})$ with $Y = \mathbb{R}$ (the reals). Note, that $C_k(X)$ is a topological group, so a typical basic open neighborhood of $f \in C_k(X)$ is of the form $f + [K,I] = \{f + f' \in C_k(X) : f' \in [K,I]\}$, where $K \in K(X)$ and I is a bounded open neighborhood of zero. We will also use that, if X is a Tychonoff space, then $f + [K,I] \subset f' + [K', I']$ implies $K \supset K'$.

Denote by τ_F the so-called Fell topology on $CL(X)$ having subbase elements of the form $\{A \in CL(X) : A \cap V \neq \emptyset\}$ with V open in X, plus sets of the form $\{A \in CL(X) : A \subset V\}$ with V co-compact in X. For notions not defined in the paper see [En].

In the *strong Choquet game* (cf. [Ch] or [Ke]) two players, α and β, take turns in choosing objects in the topological space X with an open base $\mathcal{B}$: β starts by picking (x_0, V_0) from $\mathcal{E}(X) = \mathcal{E}(X, \mathcal{B}) = \{(x,V) \in X \times \mathcal{B} : \ x \in V\}$ and α responds by $U_0 \in \mathcal{B}$ with $x_0 \in U_0 \subset V_0$. The next choice of β is some couple $(x_1, V_1) \in \mathcal{E}(X, \mathcal{B})$ with $V_1 \subset U_0$ and again α picks U_1 with $x_1 \in U_1 \subset V_1$ etc. Player α wins the run $(x_0, V_0), U_0, \ldots, (x_n, V_n), U_n, \ldots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$, otherwise β wins. A *winning tactic* for α (cf. [Ch]) is a function $\sigma : \mathcal{E}(X, \mathcal{B}) \to \mathcal{B}$ such that α wins every run of the game compatible with σ, i.e. such that $U_n = \sigma(x_n, V_n)$ for all n. The strong Choquet game is α-favorable if α possesses a winning tactic; in this case X is called *strongly α-favorable* (or a strong Choquet space - cf. [Ke]). We will need the following facts about the strong Choquet game:

Proposition 1.1.
- (i) *Let X be metrizable. Then X is completely metrizable if and only if X is strongly α-favorable.*
- (ii) *If X is locally compact, then X is strongly α-favorable.*
- (iii) *Let $f : X \to Y$ be continuous, open and onto. If X is strongly α-favorable, so is Y.*
- (iv) *The product of any collection of strongly α-favorable spaces is strongly α-favorable.*

Proof. It is not hard to show that (ii)-(iv) holds (cf. [Ke], Exercise 8.16); as for (i), see [Ch], Theorem 8.7 or [Ke], Theorem 8.17. $\square$

The *Banach-Mazur game* (see [HM] or the Choquet game in [Ke]) is played as the strong Choquet game except that β's choice is just a nonempty open set contained in the previous move of α. A space X is called *weakly α-favorable* if α possesses a *winning strategy* in the Banach-Mazur game (i.e. a function defined on nests of nonempty open sets of odd length picking for α the set that wins the Banach-Mazur game for α no matter what β chooses). Note that β has no winning strategy in the Banach-Mazur game if and only if X is a Baire space (i.e. countable intersections of dense open sets are dense - cf. [Ke] or [HM]), consequently, weakly α-favorable spaces are Baire spaces.

The *restriction mapping*

$$\eta : (CL(X), \tau_F) \times (C(X,Y), \tau_{CO}) \to (\mathcal{P}, \tau_C)$$

is defined as $\eta((B,f)) = (B, f\restriction_B)$.

Clearly, η is onto provided continuous partial functions with closed domain are continuously extendable over X. We can say more about η if we assume that X, Y have *property (P)*, i.e. if X, Y are such that partial continuous functions with closed domains are continuously extendable over X and there exists an open base $\mathcal{V}$ for Y closed under finite intersections such that for each nonempty $K \in \mathcal{K}(X)$ and $V \in \mathcal{V}$, every function $f \in C(K,V)$ is extendable to some $f^* \in C(X,V)$. A fundamental result about η is as follows (see [HZ1], Section 3):

Proposition 1.2.
(i) *If X, Y have property (P), then η is open, continuous and onto.*
(ii) *If X is paracompact and Y is locally convex completely metrizable or if X is T_4 and $Y \subset \mathbb{R}$ is an interval, then X, Y have property (P). In particular, η is open, continuous and onto in this case.*

2. Strong α-favorability of τ_{CO} and τ_F

As for strong α-favorability of the Fell topology, we have:

Theorem 2.1.
(i) *If X is locally compact, then $(CL(X), \tau_F)$ is locally compact (and hence strongly α-favorable).*
(ii) *If X is a strongly α-favorable space such that the countable subsets of X are closed, then $(CL(X), \tau_F)$ is strongly α-favorable.*

Proof. (i) See [Be], Corollary 5.1.4.

(ii) In our case $K(X)$ is a weakly Urysohn family, i.e. if $S \in K(X)$ and $A \subset S^c$, then there exists $T \in K(X)$ with $A \subset T^c \subset S^c$ such that $\overline{E} \subset S^c$ for all countable $E \subset T^c$ (we can choose $T = S$). Consequently, Theorem 5.1 of [Zs] yields the desired result. $\square$

Recall that a Hausdorff space X is *hemicompact* ([En], Excercise 3.4.E) provided in the family of all compact subspaces of X ordered by inclusion there exists a countable cofinal subfamily.

Theorem 2.2. *If X is locally compact paracompact and Y is completely metrizable, then $(C(X,Y), \tau_{CO})$ is strongly α-favorable.*

Proof. The proof of Theorem 5.3.1 in [MN1] can be modified to get the result: write $X = \bigoplus_{t \in T} X_t$, where each X_t is locally compact and hemicompact ([En], Theorem 5.1.27). Then $(C(X_t,Y),\tau_{CO})$ is completely metrizable for all $t\in T$ ([MN1], Exercise 5.8.1(a)). Therefore, $(C(X,Y),\tau_{CO})$ is homeomorphic to the product $\prod_{t\in T}(C(X_t,Y),\tau_{CO})$ ([MN1], Corollary 2.4.7) of completely metrizable spaces, hence, in view of Proposition 1.1 (i) and (iii), $(C(X,Y),\tau_{CO})$ is strongly α-favorable. □

A space X is a *q-space* if for each $x\in X$ there is a sequence $\{G_n\}_{n\in\omega}$ of open neighborhoods of x such that whenever $x_n\in G_n$ for all n, the set $\{x_n\}_{n\in\omega}$ has a cluster point. Notice that 1st countable or locally compact (even Čech-complete) spaces are q-spaces. The next result generalizes Theorem 1.2 of [Ma] about weak α-favorability of the compact-open topology (see also [MN2]):

Theorem 2.3. *Let X be a q-space. Then the following are equivalent:*

(i) *$C_k(X)$ is strongly α-favorable;*
(ii) *$C_k(X)$ is weakly α-favorable;*
(iii) *X is locally compact and paracompact.*

Proof. (i)⇒(ii) Clear.

(ii)⇒(iii) X is locally compact by Theorem 4.4 of [MN2], since weakly α-favorable spaces are Baire spaces. Paracompactness of X follows from Theorem 1.2 of [Ma].

(iii)⇒(i) See Theorem 2.2 □

Proposition 2.4. *Let X be a T_4 space with the countable subsets closed and discrete. Then $C_k(X)$ is strongly α-favorable.*

Proof. Let $(f,U)\in\mathcal{E}(C_k(X))$ with $U=f+[K,I]$ and $\mathrm{diam}(I)<\infty$ (the diameter of I). Define $\sigma(f,U)=f+[K,J]$, where J is an open neighbourhood of zero such that $\mathrm{diam}(J)=\frac{1}{2}\mathrm{diam}(I)$. Then σ is a winning strategy for α: let $(f_0,U_0),V_0,\dots,(f_n,U_n),V_n,\dots$ be a run of the strong Choquet game in $C_k(X)$, where

$$U_n=f_n+[K_n,I_n],\ V_n=\sigma(f_n,U_n),$$

$K_n\in K(X)$ and I_n is an open neighborhood of zero ($n\in\omega$). Then $U_{n+1}\subset V_n\subset U_n$ for each $n\in\omega$, so $K_{n+1}\supset K_n$ and $\mathrm{diam}(I_{n+1})\le\frac{1}{2}\mathrm{diam}(I_n)$; consequently, for each $x\in K=\bigcup_{n\in\omega}K_n$, the sequence $\{f_n(x)\}_{n\in\omega}$ converges to some $f(x)\in\mathbb{R}$. Observe that in our case the K_n's are finite and hence K is closed and discrete, so the function $f:K\to\mathbb{R}$ defined above is continuous. If we extend f to some $f^*\in C(X,\mathbb{R})$, we have $f^*\in\bigcap_{n\in\omega}U_n$ and α wins the run. □

3. Strong α-favorability of τ_C

Theorem 3.1. *Assume that X,Y have property (P). If both $(C(X,Y),\tau_{CO})$ and $(CL(X),\tau_F)$ are strongly α-favorable, so is $(\mathcal{P},\tau_C)$.*

Proof. The restriction mapping is continuous, open and onto by Proposition 1.2(i), so Proposition 1.1(iv) and (iii) applies. □

The next theorem generalizes Corollary 4.4(i) of [HZ1]:

Theorem 3.2. *Let X be a locally compact, paracompact space and Y a locally convex completely metrizable space. Then $(\mathcal{P}, \tau_C)$ is strongly α-favorable.*

Proof. $(CL(X), \tau_F)$ and $(C(X,Y), \tau_{CO})$ are strongly α-favorable by Theorem 2.1(i) and Theorem 2.2, so Proposition 1.2 and Theorem 3.1 yields the desired result. □

As a corollary of Theorem 3.2 we get the following theorem of Holá ([Ho], Theorem 3.3):

Theorem 3.3. *Let X be a Tychonoff space and Y a locally convex completely metrizable space. Then the following are equivalent:*

(i) *$(\mathcal{P}, \tau_C)$ is completely metrizable;*
(ii) *X is a locally compact 2nd countable space.*

Proof. In view of [Ho] (Theorem 2.4), $(\mathcal{P}, \tau_C)$ is metrizable if and only if X is locally compact and 2nd countable, so the implication (i)⇒(ii) immediately follows. As for (ii)⇒(i), use Theorem 3.2 and Proposition 1.1(i). □

We will now study strong α-favorability of τ_C for $Y = \mathbb{R}$ to show that Theorem 3.2 is not reversible, i.e. that local compactness plus paracompactness is not necessary for strong α-favorability of the generalized compact-open topology.

Theorem 3.4. *Let $Y = \mathbb{R}$ and X be a T_4 strongly α-favorable space with the countable subsets closed discrete. Then $(\mathcal{P}, \tau_C)$ is strongly α-favorable.*

Proof. $(CL(X), \tau_F)$ and $C_k(X)$ are strongly α-favorable by Theorem 2.1(ii) and Theorem 2.4, respectively, hence Proposition 1.2(ii) and Theorem 3.1 applies. □

To demonstrate that Theorem 3.2 is not reversible we need (by Theorem 3.4) the following:

Example 3.5. *There exists a T_4 non-paracompact, strongly α-favorable space with the countable subsets closed discrete.*

Proof. The space with the required properties is $X = \{x \in \omega_2 : \mathrm{cf}\, x > \omega\}$, which is a stationary subset of ω_2 and hence X is T_4 and, by the Pressing Down Lemma, not paracompact. Further, by the definition of X, no countable subset of X clusters, thus, countable subsets of X are closed and discrete. To show that X is strongly α-favorable, put $\sigma(x, U) = U$ for every $(x, U) \in \mathcal{E}(X)$ with $U = (a, x] \cap X$.

Then σ is a winning tactic for α, since if $(x_0, U_0), U_0, \ldots, (x_n, U_n), U_n, \ldots$ is a run of the strong Choquet game compatible with σ, then there exists some $n_0 \in \omega$ with $x = x_n = x_m$ for all $m, n \geq n_0$ (otherwise $\{x_n\}_n$ would have a subsequence of order type ω^*), whence $x \in \bigcap_{n \in \omega} U_n$. □

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Department of mathematics, University of South Carolina, Columbia, SC 29208
E-mail address: `nyikos@math.sc.edu`

Department of Mathematics and Computer Science, University of North Carolina at Pembroke, Pembroke, NC 28372
E-mail address: `laszlo@nat.uncp.edu`