GENERALIZED KUREPA AND MAD FAMILIES AND TOPOLOGY

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Definition 0.1. Let S be an uncountable set. A **Kurepa family on** S is a collection $\mathcal{K}$ of countable subsets of S such that $|\mathcal{K} \upharpoonright A| \leq \omega$ for all countable sets A. A Kurepa family $\mathcal{K}$ on S is called **cofinal** if it is cofinal in the poset $[S]^\omega$.

Kurepa families can also be defined without reference to the set S. In that case, a Kurepa family is called "cofinal" if it is cofinal in the poset $([\bigcup K]^\omega, \subset)$. This is the approach taken in [7].

Cofinal Kurepa families on ω_n can be constructed in ZFC by induction for all $n < \omega$ [7]. To obtain one of cardinality $\aleph_\omega$ or of higher cardinality apparently requires more than just ZFC: there is a model constructed using a 2-huge cardinal in which there is an unattained upper bound of $\aleph_\omega$ on their size. See Theorem 2.3 of [5] and Theorem 4.2 below.

One of the main results of this paper is a necessary and sufficient topological condition for the existence of cofinal Kurepa families on arbitrarily large sets [Theorem 1.2 below].

All through this paper, "space" means "Hausdorff space."

1. A topological equivalence

Lemma 1.1. *Let ν be an uncountable cardinal. If $\mu = cf[\nu]^\omega$ and there is a cofinal Kurepa family $\mathcal{N}$ on ν then $|\mathcal{N}| = \mu = cf[\mu]^\omega$ and there is a cofinal Kurepa family (of cardinality μ) on μ.*

Proof. Suppose there is a cofinal Kurepa family $\mathcal{N} \subset [\nu]^\omega$. Let $\mathcal{C}$ by any cofinal subfamily of $[\nu]^\omega$. Then $\mathcal{N} \subset \{N \cap C : N \in \mathcal{N},\ C \in \mathcal{C}\}$, hence $|\mathcal{N}| \leq \omega \cdot |\mathcal{C}|$. Letting $|C| = \mu$ thus gives $|\mathcal{N}| \leq \mu$, and cofinality of $\mathcal{N}$ gives $|\mathcal{N}| = \mu$.

Closing a Kurepa family under finite intersection yields a Kurepa family of the same cardinality, so we may assume $\mathcal{N} = \{N_\alpha : \alpha \in \mu\}$ is closed under finite intersection. For each $N \in \mathcal{N}$ let $m(N) = \{\alpha : N_\alpha \subset N\}$. Since $\mathcal{N}$ is a Kurepa family, $m(N)$ is a countable subset of μ. Also, $m(N_1 \cap N_2) = m(N_1) \cap m(N_2)$ and so $\mathcal{K} = \{m(N) : N \in \mathcal{N}$ and $m(N)$ is infinite$\}$ is a Kurepa family of cardinality no greater than μ. To show that $\mathcal{K}$ is cofinal in $[\mu]^\omega$ and hence of cardinality $\geq \mu$, let M be a countable subset of μ and let $N(M)$ be a member of $\mathcal{N}$ containing $\bigcup\{N_\alpha : \alpha \in M\}$. Then clearly $M \subset m(N(M))$. $\square$

Can we go directly from the first hypothesis in Lemma 1.1 to the first conclusion? That is:

Problem 1. If ν is uncountable, is $cf[cf[\nu]^\omega]^\omega = cf[\nu]^\omega$?

Clearly, $cf[\omega_n]^\omega = \omega_n$ for $n \geq 1$. So Problem 1 is only interesting for $\nu \geq \aleph_\omega$, and it takes some very large cardinals to get $cf[\aleph_\omega]^\omega$ to be greater than $\aleph_{\omega+1}$ (see below). [It is easy to see that if $cf[\nu]^\omega \leq \nu^+$ then $cf[\nu^+]^\omega = \nu^+$.]

Cofinal Kurepa families on arbitrarily large sets were first constructed in [4] using GCH and $\square_\kappa$ at singular cardinals κ of cofinality ω. This was done indirectly, by constructing locally countable, ω-**bounded** [this means: every countable subset has compact closure] spaces of arbitrarily large cardinality. It is easy to see that the collection of compact, open subsets of such spaces is a cofinal Kurepa family on the underlying set. Later, the author noticed that GCH could be replaced in the construction in [4] by the axiom that $cf[\kappa]^\omega = \kappa^+$ for all singular κ of countable cofinality. This axiom and the $\square_\kappa$ axiom used in [4] hold in the Core Model, and it is easy to show that they continue to hold if the Covering Lemma holds over the Core Model. Therefore, to put an upper bound on the cardinality of cofinal Kurepa families, one must assume there is an inner model with a proper class of measurable cardinals.

Our first theorem was approximated in an article by A. Dow [1], partially correcting a handwritten note by S. Todorčević. See Theorem A near the end of this section.

Theorem 1.2. *Let λ be an infinite cardinal. The following are equivalent.*
   (a) There is a cofinal Kurepa family on λ.
   (b) There is a locally metrizable, ω-bounded 0-dimensional space of weight $cf[\lambda]^\omega$.
   (c) There is a cofinal Kurepa family on every set of cardinality $\leq cf[\lambda]^\omega$.

Proof. (a) implies (b) Let $\mathcal{K}$ be a cofinal Kurepa family on λ, which we may assume to be uncountable. Let $\mathcal{R}$ be the ring of sets generated by $\mathcal{K}$ and let $\mathcal{B}$ be the Boolean algebra generated by $\mathcal{K}$. Then $\mathcal{B} = \mathcal{R} \cup \{R^c : R \in \mathcal{R}\}$. Clearly each member of $\mathcal{R}$ is countable and has uncountable complement. Let $S(\mathcal{B})$ be the Stone space of $\mathcal{B}$. The underlying set of $S(\mathcal{B})$ is the set of $\mathcal{B}$-ultrafilters, and a base for the topology is the collection of all sets of the form $S(B) = \{p \in S(\mathcal{B}) : B \in p\}$.

Here are some fundamental facts of Stone duality: $S(\mathcal{B})$ is a compact, 0-dimensional space; all clopen subsets of $S(\mathcal{B})$ are of the form $S(B)$ for some $B \in \mathcal{B}$; and the function S taking B to $S(B)$ is an isomorphism of Boolean algebras. It follows easily that in our example, $S(R)$ is metrizable (though not necessarily countable) for all $R \in \mathcal{R}$. Since λ is uncountable, $\{R^c : R \in \mathcal{R}\}$ is an ultrafilter of $\mathcal{B}$ and is the only point of $S(\mathcal{B})$ without a countable base; call this point p_∞.

Since $\mathcal{K}$ is cofinal, every countable subset of $S(\mathcal{B})$ that does not include p_∞ is contained in the compact set $S(K)$ for some $K \in \mathcal{K}$. Thus $S(\mathcal{B}) \setminus \{p_\infty\}$ is ω-bounded, locally metrizable, and 0-dimensional. It is also of Lindelöf degree $cf[\lambda]^\omega$ and its weight is equal to its Lindelöf degree because it is locally second countable.

(b) implies (c): Let X be a locally metrizable, ω-bounded 0-dimensional space of weight $cf[\lambda]^\omega$. Each point of X has a compact open neighborhood, and this combined with ω-boundedness implies that every countable subset of X is contained in a compact open metrizable subset.

Let $\{V_\alpha : \alpha < cf[\lambda]^\omega\}$ be a base for X and, for each $\alpha < cf[\lambda]^\omega$ let x_α be a point outside $\bigcup\{V_\beta : \beta < \alpha\}$. This gives us a subspace $Y = \{x_\alpha : \alpha < cf[\lambda]^\omega\}$ with a well-ordering such that each initial segment is open; in other words, Y is right-separated. From local second countability of X, it follows that each compact

open subset of X has only countably many compact open subsets of its own. It also follows that Y is locally countable and so each countable subset of Y is contained in a compact open subset of X. Thus $\{Y \cap C : C$ is compact open in $X\}$ is a cofinal Kurepa family on Y. Using traces and bijections, one sees that every set of cardinality $\leq |Y|$ has a cofinal Kurepa family on it. $\square$

Problem 2. Is it possible to eliminate zero-dimensionality from *(b)*?

We will return to this problem in Section 4. Here is a problem in the opposite direction.

Problem 3. Can *(b)* in Theorem 4.2 be strengthened as follows?

(b^+) There is a locally countable, ω-bounded space of weight (equivalently, of cardinalty) $cf[\lambda]^\omega$.

This is indeed a strengthening, because local compactness is an easy consequence of ω-boundedness in a locally countable space, and every locally compact, locally countable space is 0-dimensional and locally metrizable. Local metrizability follows from the elementary fact that every countable, compact space is of countable weight, and from Urysohn's metrization theorem. Theorem 1.2 makes Problem 3 equivalent to Problem 4 in [3], which asked:

Problem A. *Is $S(\kappa)$ (That is, the statement that there is a locally countable, ω-bounded space of size κ) equivalent to the existence of a cofinal Kurepa family on κ?*

The best we have on this so far is the following precursor of Theorem 1.2:

Theorem A. [1, Proposition 7.6]. *Let θ be an uncountable cardinal number. Then* $(1) \implies (2) \implies (3)$, *where:*

(1) *There is a well-founded cofinal Kurepa family of size θ which is closed under finite intersections.*

(2) *$S(\kappa)$*

(3) *There is a cofinal Kurepa family of size θ.*

The drawback of Theorem A is that no one has ever constructed a family as in (1) such that $\theta > \aleph_1$, even assuming extra axioms. It is easy to construct well-founded cofinal Kurepa families *or* cofinal Kurepa families of size θ that are closed under finite intersection, given that there is a cofinal Kurepa family of size θ, but finding one with both properties seems to be a formidable problem.

2. MAD FAMILIES OF COUNTABLE SETS

Definition 2.1. Let κ be an infinite cardinal number. Two sets of cardinality κ are said to be **almost disjoint** if their intersection is of cardinality $< \kappa$.

By a slight abuse of language, we use the term "**AD family**" to denote a family of almost disjoint sets of the same cardinality. Another slight abuse of language appears in the following definition.

Definition 2.2. A family $\mathcal{A}$ of subsets of an infinite set S is called **a MAD family on** S if it is an infinite AD family of subsets of S which is maximal among all AD families of subsets of S. Let $\mathfrak{a}(\kappa)$ denote the least cardinality of a MAD family of countable subsets of κ.

The usual notation for $\mathfrak{a}(\omega)$ is just $\mathfrak{a}$. It is easy to see that $\mathfrak{a}(\kappa) \geq \kappa$ for all κ, and that $\mathfrak{a}(\kappa) \leq \mathfrak{a}(\lambda)$ whenever $\kappa \leq \lambda$. It is also well known that $\mathfrak{a}$ is uncountable. Further information about $\mathfrak{a}$ is provided below, beginning with a trick that I learned from James Baumgartner (but which probably goes back to Sierpiński) and which uses the following concept:

Definition 2.3. A family $\mathcal{H}$ of countable subsets of a set X is $[X]^\omega$-**hitting**, if each denumerable subset of X meets some $H \in \mathcal{H}$ in an infinite set.

Lemma 2.4. *For each cardinal κ let κ^* be the least cardinality of a $[\kappa]^\omega$-hitting family of countable subsets of κ. Then $\kappa^* = cf([\kappa]^\omega)$.*

Proof. It is enough to show $\kappa^* \geq cf([\kappa]^\omega)$. Think of $T = \kappa^{<\omega}$ as the full κ-ary tree of height ω. Then $|T| = \kappa$ and so we can hit $[T]^\omega$ with a family $\mathcal{H}$ of cardinality κ^*. Each countable subset of κ is the range of some branch. So now, if we let

$$C(H) = \{\alpha \in \kappa : \exists\, n(\langle n, \alpha \rangle) \text{ is in some branch } B \text{ such that } B \cap H \text{ is infinite})\}$$

then $\{C(H) : H \in \mathcal{H}\}$ is cofinal in $[\kappa]^\omega$. $\square$

Corollary 2.5. *For each infinite cardinal κ, $\mathfrak{a}(\kappa) \geq cf[\kappa]^\omega$.*

Proof Clearly, every maximal AD family is ω-hitting, whether infinite or not. But for uncountable cardinals, maximal AD families are infinite, while $cf[\omega]^\omega = 1$. $\square$

Theorem 2.6. *If $\kappa \geq \mathfrak{c}$, then then $\mathfrak{a}(\kappa) = \kappa^\omega = cf[\kappa]^\omega$.*

proof It is an elementary exercise that $\kappa^\omega = |[\kappa]^\omega| \geq \kappa$ and also $cf[\kappa]^\omega \geq \kappa$ for all uncountable κ.

Now $[\kappa]^\omega = \bigcup\{[C]^\omega : C \in \mathcal{C}\}$ for all cofinal $\mathcal{C} \subset [\kappa]^\omega$. It follows that

$$(1) \qquad \kappa^\omega = |[\kappa]^\omega| \ \leq \ \mathfrak{c} \cdot cf[\kappa]^\omega \ \leq \ \kappa^\omega \cdot \kappa^\omega = \kappa^\omega$$

so equality holds all around. Then if $\kappa > \mathfrak{c}$ we have $\kappa^\omega = \mathfrak{c} \cdot cf[\kappa]^\omega = cf[\kappa]^\omega$ and if $\kappa = \mathfrak{c}$ then $\kappa = \kappa^\omega \geq cf[\kappa]^\omega \geq \kappa$ so again equality holds all around. This shows the second equality in Theorem 1. As for the first, we clearly have $\mathfrak{a}(\kappa) \leq \kappa^\omega$, while $\mathfrak{a}(\kappa) \geq \kappa^\omega$ follows from Corollary 2.5 and the second equality. $\square$

So at $\mathfrak{c}$ and above, the cardinalities of MAD families are all of the form λ^ω and the ones on κ are of cardinality κ^ω. Under the Singular Cardinals Hypothesis (SCH), κ, $\kappa^\omega = \kappa^+$ whenever κ is a singular cardinal of countable cofinality, otherwise $\kappa^\omega = \kappa$ for all other cardinals $\geq \mathfrak{c}$. [The SCH says that if $2^{cf(\kappa)} < \kappa$ then $\kappa^{cf(\kappa)} = \kappa^+$.] Thus if we assume SCH (which is true unless there are inner models with very large cardinals) then there are MAD families of all cardinals $\geq \mathfrak{c}$ except for singular cardinals of countable cofinality.

What about when $\kappa < \mathfrak{c}$? The following problem is open:

Problem 4. Is it a theorem of ZFC that $\mathfrak{a}(\kappa) = max\{\mathfrak{a}, cf[\kappa]^\omega\}$?

It is clear from Corollary 2.5 that $\mathfrak{a}(\kappa) \geq max\{\mathfrak{a}, cf[\kappa]^\omega\}$ for all infinite κ. Equality does hold if $\kappa \geq \mathfrak{c}$, by equation (1) and Theorem 2.6. Equality also holds whenever there is a Kurepa family on κ. To show this, we first recall the following concept.

Definition 2.7. Let $\mathcal{B} = \{B_n : n \in \omega\}$ be a collection of subsets of a countable set S, such that for each n there exists $m > n$ such that $B_m^\# = B_m \setminus \bigcup_{i=0}^{m-1} B_i$ is infinite. Let $\{Z_n : n \in \omega\}$ list all the infinite $B_m^\#$. An **RH transfer of $\mathcal{B}$ to $\omega \times \omega$** is a bijection $\psi : S \to \omega \times \omega$ which distributes the elements of $S \setminus \bigcup_{n=0}^{\infty} Z_n$ into the bottom row $\omega \times \{0\}$, and sends Z_n into the $(n+1)$st column $\{n\} \times \omega$.

Note that the image of Z_n under ψ is either $(\{n\} \times \omega) - \{0\}$ or $\{n\} \times \omega$ depending on whether or not $\langle n, 0 \rangle$ is in $\psi(a)$ for some $a \in S \setminus \bigcup_{n=0}^{\infty} Z_n$.

The following lemma is a routine application of RH transfers.

Lemma 2.8. *If S is a countably infinite set and $\{Z_n : n \in \omega\}$ is a family of infinite subsets of S, no finite subcollection of which has cofinite union, and there is a MAD family of cardinality μ on ω, then there is an AD family $\mathcal{A}$ on S such that $|\mathcal{A}| = \mu$ and such that every infinite subset of S that is almost disjoint from every Z_n meets some member of $\mathcal{A}$ in an infinite set.*

Corollary 2.9. *If S is a countably infinite set and $\mathcal{Z}$ is a countably infinite AD family on S, and there is a MAD family of cardinality μ on ω, then $\mathcal{Z}$ can be extended to a MAD family of cardinality μ on S.*

Theorem 2.10. *If there is a cofinal Kurepa family on κ, and $\lambda = cf([\kappa]^\omega)$, and there is a MAD family of cardinality μ on ω, then there is a MAD family of cardinality $\mu \cdot \lambda$ on κ.*

Proof. Let $\{K_\xi : \xi < \lambda\}$ list a cofinal Kurepa family κ. Define collections $\mathcal{A}_\xi$ of almost disjoint infinite subsets of all the K_ξ as follows. Let $\mathcal{A}_0$ be any MAD family of cardinality μ on K_0.

If $\mathcal{A}_\eta$ has been defined for all $\eta < \xi$, and K_ξ is almost contained in the union of finitely many sets of the form $K_\eta \cap K_\xi$ ($\eta < \xi$) let $\mathcal{K}_\xi = \emptyset$. If there are finitely many sets K_{η_i} that almost contain $\bigcup \{K_\eta : \eta < \xi\}$, let $\mathcal{A}_\xi = \emptyset$ if $K_\xi \setminus \bigcup \{K_\eta : \eta < \xi\}(= A)$ is finite, otherwise let $\mathcal{A}_\xi = \{A\}$.

Otherwise, define $\mathcal{A}_\xi$ by applying Lemma 2.8 with $S = K_\xi$, letting $\{Z_n : n \in \omega\}$ list all infinite sets of the form $K_\eta \cap K_\xi$ ($\eta < \xi$), and let $\mathcal{A}_\xi$ be the $\mathcal{A}$ of Lemma 2.8.

Finally, let $\mathcal{A} = \bigcup_{\xi < \lambda} \mathcal{A}_\xi$. To see that this works, let Z be any infinite subset of κ. Let ξ be the least ordinal such that $Z \cap K_\xi$ is infinite. Then clearly Z is almost disjoint from all K_η such that $\eta < \xi$, and so Z meets some member of $\mathcal{A}_\xi$ in an infinite set. Hence $\mathcal{A}$ is MAD. Since the Kurepa family $\mathcal{K}$ is of cardinality λ, $|\mathcal{A}| \leq \mu \cdot \lambda$; and now equality follows from Corollary 2.5. $\square$

Since there is a cofinal Kurepa family on ω_n, of cardinality ω_n for all finite n, we have:

Corollary 2.11. $\mathfrak{a}(\omega_n) = max\{\mathfrak{a}, \omega_n\}$ *for all finite n.*

Since $\mathfrak{a}$ is uncountable, we have:

Corollary 2.12. $\mathfrak{a} = \mathfrak{a}(\omega_1)$.

Putting Theorems 2.6 and 2.10 together, we get:

Theorem 2.13. *For all cardinals κ,*

$$max\{\mathfrak{a}, cf[\kappa]^\omega\} \ \leq \ \mathfrak{a}(\kappa) \ \leq \kappa^\omega = \ max\{\mathfrak{c}, cf[\kappa]^\omega\}$$

and the first $\leq$ is an equality if there is a cofinal Kurepa family on κ. Moreover, if $\kappa \geq \mathfrak{c}$ then $\mathfrak{a}(\kappa) = \kappa^\omega$.

3. Mrówka families

Notation 3.1. Let $\mathcal{A}$ be an infinite AD family of countably infinite subsets of a set S. $\Psi(S, \mathcal{A})$ denotes the space whose underlying set is the union of S with a set $\{p_A : A \in \mathcal{A}\}$ disjoint from S, with a set being open iff it contains a cofinite subset of A for every p_A that it contains.

If S is not specified, $\Psi(\mathcal{A})$ will denote $\Psi(\bigcup \mathcal{A}, \mathcal{A})$ and it will be understood that $\mathcal{A}$ is an AD family of countable subsets of $\bigcup \mathcal{A}$.

It is easy to show that $\Psi(S, \mathcal{A})$ is 0-dimensional and locally metrizable; that it is pseudocompact iff $\mathcal{A}$ is a MAD family on S; that S is a dense set of isolated points, while $\{p_A : A \in \mathcal{A}\}$ is closed discrete; and that $A \cup \{p_A\}$ is a clopen copy of $\omega + 1$ for all $A \in \mathcal{A}$.

Definition 3.2. Let S be an infinite set. A **Mrówka family on** S [resp. **fully Mrówka family on** S] is a MAD family $\mathcal{M}$ of countable subsets of S, such that for every continuous $f : \Psi(S, \mathcal{M}) \to \mathcal{R}$, there exists $r \in \mathcal{R}$ such that $|\Psi(S, \mathcal{M}) \setminus f^{\leftarrow}\{r\}| < |\mathcal{M}|$ [resp. such that $\Psi(S, \mathcal{M}) \setminus f^{\leftarrow}\{r\}$ is countable]. We call $\Psi(S, \mathcal{M})$ a [**fully**] **Mrówka space**.

Mrówka showed that there is a fully Mrówka family on ω, of cardinality $\mathfrak{c}$. The other ZFC possibility is:

Theorem 3.3. *There is a fully Mrówka space of cardinality $\mathfrak{a}$.*

Proof. If $\mathfrak{a} = \mathfrak{c}$ we use Mrówka's construction, or a simpler construction below which takes advantage of $\mathfrak{a} = \mathfrak{c}$. Otherwise, if $\mathcal{A}$ is a MAD family of cardinality $\mathfrak{a}$, every real-valued function on $\Psi(\mathcal{A})$ has range $< \mathfrak{c}$. [Also, if $f : \Psi(\mathcal{A}) \to \mathcal{R}$ is continuous, its range (being pseudocompact, hence compact) is countable.] Consequently, the Stone-Čech compactification $\beta\Psi(\mathcal{A})$ cannot have a dense-in-itself subspace: if it did, then it would admit a continuous function onto $[0, 1]$, but the range of any real-valued continuous function has to be (the closure of) the range of its dense subspace $\Psi(\mathcal{A})$, of size $< \mathfrak{c}$. [It even has to be countable.]

So now $\beta\Psi(\mathcal{A})$ is scattered and so, of course, is every subspace. Since $\beta\Psi(\mathcal{A})$ is compact and totally disconnected, it is 0-dimensional. Let p be an isolated point of the Stone-Čech remainder $\beta\Psi(\mathcal{A}) \setminus \Psi(\mathcal{A})$, and let C be a clopen (in particular, compact) subset of $\beta\Psi(\mathcal{A})$ containing p and missing the rest of $\beta\Psi(\mathcal{A}) \setminus \Psi(\mathcal{A})$.

Claim. The trace of C on $\Psi(\mathcal{A})$ is a fully Mrówka space of cardinality $\mathfrak{a}$.

⊢ *Proof of Claim.* The trace $C \setminus \{p\}$ meets the subspace $\Psi(\mathcal{A}) \setminus \omega$ of nonisolated points in an infinite set, otherwise $C \setminus \{p\}$ would be compact, being a clopen subspace of $\Psi(\mathcal{A})$; and this would contradict the fact that $\Psi(\mathcal{A})$ is dense in $\beta\Psi(\mathcal{A})$. Thus the trace is a version of Ψ and is clearly of cardinality $\mathfrak{a}$. To show that it is fully Mrówka, let $f : C \to \mathcal{R}$ be a continuous function, and let $r = f(p)$. Every open neighborhood of p in C has compact, hence countable complement.

Thus the complement of $f^{\leftarrow}\{r\}$ is countable, being the union of the subspaces $(f^{\leftarrow}(r - \frac{1}{n}, r + \frac{1}{n}))^c$ of $\Psi(\mathcal{A})$. ⊣

To construct a fully Mrówka space when $\mathfrak{a} = \mathfrak{c}$, begin with any version $X = \Psi(\omega, \mathcal{M})$ of Ψ. We will build quotient spaces $X_\alpha (\alpha < \mathfrak{c})$ by induction, each time identifying at most two points of X, neither of which was identified with any other point at any earlier stage.

Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ list all functions $f : \omega \to \mathcal{R}$. Let $X_0 = X$. If X_α and the quotient map $\phi_\alpha : X_0 \to X_\alpha$ have been defined, and f_α cannot be extended to a (unique!) continuous function on all of X_α, or if it can be so extended to $f_\alpha^\# : X_\alpha \to \mathcal{R}$, and the preimage under $f_\alpha^\#$ of some point has countable complement (equivalently, complement of cardinality $< \mathfrak{c}$,) let $X_{\alpha+1} = X_\alpha$.

Otherwise, there exist points x_0^α, x_1^α of X_0 such that neither has been identified with any other point of X_0 at any earlier stage, and such that if g_α is the composition of $f_\alpha^\#$ with ϕ_α, then $g_\alpha(x_0^\alpha) \neq g_\alpha(x_1^\alpha)$. Let $X_{\alpha+1}$ be the quotient of X_α obtained by identifying x_0^α and x_1^α, and let $\phi_{\alpha+1} : X_0 \to X_{\alpha+1}$ be the composition of ϕ_α with the quotient map. Then f_α cannot be extended to a continuous function from $X_{\alpha+1}$ because it approaches two different limits, $g_\alpha(x_0^\alpha)$ and $g_\alpha(x_1^\alpha)$.

If α is a limit ordinal, and X_β has been defined for all $\beta < \alpha$, let X_α be the direct limit of the X_β. In other words, X_α is the quotient space obtained by identifying to single points all pairs $\{x_0^\xi, x_1^\xi\}$ that have been defined for some $\xi < \alpha$.

This applies even to $\alpha = \mathfrak{c}$, and the resulting space $X_\mathfrak{c}$ is fully Mrówka, as can be seen by considering any continuous $g : X_\mathfrak{c} \to \mathcal{R}$ and finding the α such that $g \restriction \omega = f_\alpha$. Continuity of $f_\alpha^\#$ was not destroyed at stage α, because once it is destroyed, f_α can no longer be extended to a continuous function on any later X_γ. So g sends all but countably many points of $X_\mathfrak{c}$ to a single real number r, as desired. □

Theorem 3.3, and Mrówka's result that there is always a fully Mrówka space of cardinality $\mathfrak{c}$, are the best that can be done in ZFC. Arnold Miller has shown that adding any number of Cohen reals to a model of CH produces a model where the only cardinalities of MAD families on ω are $\mathfrak{a} = \aleph_1$ and $\mathfrak{c}$, but $\mathfrak{c}$ can be arbitrarily large. Hence these are also the only cardinalities of fully Mrówka families, whether on ω or elsewhere. (The "or elsewhere" follows from Lemma 6 and the comments following it.)

On the other hand, Hechler [2] showed that one can produce models where $\mathfrak{c}$ is arbitrarily large and there are MAD families on ω of all uncountable cardinalities $\leq \mathfrak{c}$. The MAD families this forcing produces are all fully Mrówka.

For the sake of brevity, we refer to a Mrówka family of cardinality κ as a κ-**Mrówka family**.

Theorem 3.4. *If there is a κ-Mrówka family on ω, then there is a κ-Mrówka family on ν for all $\nu \leq \kappa$. Moreover, if the original family is fully Mrówka, then so is the resulting family.*

Here are some recent results of A. Dow and J. Vaughan. The proof of the first is similar to that of the $\mathfrak{a} = \mathfrak{c}$ case of Theorem 3.3.

Theorem B. *If $\kappa^\omega = 2^\kappa$, then there is a Mrówka family on κ.*

Theorem C. *Let $\kappa = \kappa^\omega$. If there is a Mrówka family on κ, then there is a Mrówka family on κ^+.*

The proof of Theorem 3.6 below is similar.

Using Theorem B at cardinals of countable cofinality, Theorem C for successor cardinals, and an inductive construction for all cardinals $< \kappa$ when κ is singular of uncountable cofinality, Dow and Vaughan showed:

Theorem D. Assume GCH. Then there is a Mrówka family on κ, of cardinality κ^ω, for all infinite cardinals κ, and there is a fully Mrówka family on κ iff $\kappa \in \{\mathcal{A}_0, \mathcal{A}_1\}$.

In the first part of Theorem D, GCH can be replaced with the hypothesis "SCH + every singular cardinal is a strong limit cardinal," thanks to the well-known fact that $\kappa^{cf(\kappa)} = 2^\kappa$ for all singular strong limit cardinal of countable cofinality.

The proof of the following theorem is very similar to that of the $\mathfrak{a} < \mathfrak{c}$ part of Theorem 3.3.

Theorem 3.5. If $\kappa \leq \mathfrak{c}$, and κ is of uncountable cofinality, there is a Mrówka family of cardinality κ if, and only if, there is a MAD family of cardinality κ.

Proof of "if". If $\kappa = \mathfrak{c}$ then Mrówka's original construction works, and when $\kappa = \alpha$ there is Theorem 3.3. Otherwise, if $\mathcal{A}$ is a MAD family of cardinality κ, then $\beta\Psi(\mathcal{A})$ is scattered and there exists a point $y \in \beta\Psi(\mathcal{A})$ with the following properties:

(1) There is a compact neighborhood C of y in which y is the unique point of maximal Cantor-Bendixson level, and such that $C \cap \Psi(\mathcal{A})$ has cardinality κ.

(2) If p is a point of lower Cantor-Bendixson level than that of y, then for every compact neighborhood D of p in which p is the one point of maximal level, $D \cap \Psi(\mathcal{A})$ has cardinality $< \kappa$.

Assuming (1) and (2), the subspace $Y = C \cap \Psi(\mathcal{A})$ is a version of Ψ which is a Mrówka space. Indeed, every neighborhood of y in Y has a complement of cardinality $< \kappa$ in Y, and since κ is not of countable cofinality, and $f : Y \to \mathcal{R}$ is continuous, and $r = f(y)$, then $|Y \setminus f^\leftarrow(r)| < \kappa$.

To see that there is at least one point y as in (1), let $\mathcal{W}$ be a clopen cover of $\beta\Psi(\mathcal{A})$ by sets of the form $C(x)$ where x is the unique point of maximal Cantor-Bendixson level in $C(x)$. Then use compactness to get a finite subcover, at least one member of which must satisfy (1). Then take a point satisfying (1) of minimal level to get y. $\square$

A similar construction to that in the proof of Theorem C is used in the following proof, substituting a Kurepa family and Lemma 2.4 and Claim 1 for the hypothesis that $\kappa = \kappa^\omega$.

Theorem 3.6. *If there is a cofinal Kurepa family on κ, and there is a κ-Mrówka family on κ, then there is a κ^+-Mrówka family on κ^+.*

Proof. Let $\mathcal{K}$ and $\mathcal{M}$ be a cofinal Kurepa family and a κ-Mrówka family on $[0, \kappa)$, respectively. From Lemma 1.1 and Lemma 2.4 it follows that $|\mathcal{K}| = \kappa$: every MAD family of countable subsets of $[0, \kappa)$ is $[\kappa]^{\omega}$-hitting. Let $\lambda_{\alpha} = \kappa \cdot \alpha$; in particular, $\lambda_0 = 0$. Divide up κ^+ into chunks $[\lambda_{\alpha}, \lambda_{\alpha+1})$ that are natural copies of κ. For each $\alpha < \kappa^+$ let $f_{\alpha} : [0, \kappa) \to [\lambda_{\alpha}, \lambda_{\alpha+1})$ be the unique order-preserving bijection and let $\mathcal{M}_{\alpha} = \{f^{\to}M : M \in \mathcal{M}\}$.

We will define AD families $\mathcal{A}_{\alpha}$ on $[0, \lambda_{\alpha})$ by induction starting with $\mathcal{A}_1 = \mathcal{M}_0 = \mathcal{M}$. If $\mathcal{A}_{\beta}$ has been defined for all $\beta < \alpha$ let $\mathcal{A}_{\alpha} = \bigcup_{\beta < \alpha} \mathcal{A}_{\beta}$. If $\alpha = \beta + 1$ and β is either a successor or a limit ordinal of uncountable cofinality, let $\mathcal{A}_{\alpha} = \mathcal{A}_{\beta} \cup \mathcal{M}_{\beta}$. On the other hand, if β is a limit ordinal of countable cofinality, we define $\mathcal{A}_{\alpha}$ as follows. Let $\xi_n \nearrow \beta$, with $\xi_0 = 0$.

Claim 1. There is an AD family $\mathcal{A}$ of κ countable subsets of $[0, \beta)$ such that every member of $\mathcal{A}$ is almost disjoint from every $[0, \lambda_{\xi_n})$, and such that every infinite subset of $[0, \beta)$ that is almost disjoint from every $[0, \lambda_{\xi_n})$ meets some member of $\mathcal{A}$ in an infinite set.

$\vdash$ *Proof of Claim 1.* Let $\{K_{\xi} : \xi < \kappa\}$ list $\mathcal{K}$. There is a MAD family $\mathcal{D}$ on ω of cardinality $\leq \kappa$: take the trace of $\mathcal{M}$ on a countable subset B of κ that contains infinitely many members of $\mathcal{M}$ and transfer the trace to ω with a bijection $f : \omega \to B$. Define collections $\mathcal{K}_{\xi}$ of almost disjoint infinite subsets of K_{ξ} as follows. If $K_0 \subset [0, \gamma)$ for some $\gamma < \beta$, let $\mathcal{K}_0 = \emptyset$, otherwise use Lemma 2.8 with $A = K_0$ and $\mathcal{D}$ as the MAD family on ω and

$$Z_n = [\lambda_{\xi_n}, \lambda_{\xi_{n+1}}) \cap K_0$$

to obtain $\mathcal{K}_0$ that behaves like $\mathcal{A}$ of Lemma 2.8, with $\mu \leq \kappa$. That is, $|\mathcal{K}_0| = \mu$; $\mathcal{K}_0$ is AD; every member of $\mathcal{K}_0$ is almost disjoint from every Z_n; and every infinite subset of K_0 that is almost disjoint from every Z_n meets some member of $\mathcal{K}_0$ in an infinite set.

If $\mathcal{K}_{\eta}$ has been defined for all $\eta < \xi$, and K_{ξ} is contained in the union of some $[0, \lambda_{\xi_n})$ with finitely many sets of the form $K_{\eta} \cap K_{\xi}$ ($\eta < \xi$) let $\mathcal{K}_{\xi} = \emptyset$. Otherwise, define $\mathcal{K}_{\xi}$ by applying Lemma 2.8 with $S = K_{\xi}$ as we defined $\mathcal{K}_0$ with $S = K_0$, but this time letting $\{Z_n : n \in \omega\}$ list not only the sets $[\lambda_{\xi_j}, \lambda_{\xi_{j+1}}) \cap K_{\xi}$ but also all infinite sets of the form $K_{\eta} \cap K_{\xi}$ ($\eta < \xi$). Each member of $\mathcal{K}_{\xi}$ will then be almost disjoint from each $[0, \gamma)$ such that $\gamma < \beta$, because $\gamma < \lambda_{\xi_n}$ for some finite n.

Finally, let $\mathcal{A} = \bigcup_{\xi < \kappa} \mathcal{K}_{\xi}$. To see that this works, suppose Z is almost disjoint from each $[0, \gamma)$ such that $\gamma < \beta$. Let ξ be the least ordinal such that $Z \cap K_{\xi}$ is infinite. Then clearly Z is almost disjoint from all K_{η} such that $\eta < \xi$, and so Z meets some member of $\mathcal{K}_{\xi}$ in an infinite set. Since the Kurepa family $\mathcal{K}$ is of cardinality κ, $|\mathcal{A}| \leq \kappa$; to show equality, note that if we let Q_{θ} be the set of all θth members of the chunks $[\lambda_{\xi_j}, \lambda_{\xi_{j+1}})$ then $\{Q_{\theta} : \theta < \kappa\}$ is a family of κ disjoint countably infinite subsets of $[0, \beta)$ each of which is almost disjoint from each $[0, \gamma)$ ($\gamma < \beta$); and the union of fewer than κ members of $\mathcal{A}$ cannot meet them all. $\dashv$

To conclude the construction of $\mathcal{A}_{\alpha}$, we define a bijection ϕ from $\mathcal{A}$ to $\mathcal{M}_{\beta}$. Then let

$$\mathcal{A}_{\beta}^1 = \{A \cup \phi(A) : A \in \mathcal{A}\} \quad \text{and let} \quad \mathcal{A}_{\alpha} = \mathcal{A}_{\beta}^1 \cup \mathcal{A}_{\beta}.$$

Then $\mathcal{M}' = \bigcup_{\alpha < \kappa^+} \mathcal{A}_\alpha$ is the desired family. An easy induction shows that $\mathcal{A}_\alpha$ is a MAD family on $[0, \lambda_\alpha)$ for all $\alpha < \kappa^+$, except when α is a limit ordinal of countable cofinality, and so $\mathcal{M}'$ is MAD.

Corollary 3.7. *If there is a κ-Mrówka family on ω, there is a $max\{\kappa, \omega_n\}$-Mrówka family on ω_n.*

Lemma 3.8. *If X is a first countable T_2 space, in which every point has a neighborhood of cardinality $\leq \mathfrak{c}$, then every subset of cardinality $\leq \mathfrak{c}$ is contained in a clopen subset of cardinality $\leq \mathfrak{c}$.*

Proof. Let A be a subset of X of cardinality $\leq \mathfrak{c}$ and let F_0 be the closure of A. Since every point of F_0 has a sequence from A converging to it, and limits of nets in T_2 spaces are unique, $|F_0| \leq \mathfrak{c}$. Let G_0 be an open subset of X, of cardiality $\leq \mathfrak{c}$, containing F_0.

If $\alpha \leq \omega_1$ and F_β and $G_\beta \supset F_\beta$ have been defined for all $\beta < \alpha$, with $|G_\beta| \leq \mathfrak{c}$, suppose first that $\alpha = \beta+1$. Let F_α be the closure of G_β and let G_α be an open set of cardinality $\leq \mathfrak{c}$ containing F_α. If α is a limit ordinal, let $F_\alpha = G_\alpha = \bigcup\{G_\beta : \beta < \alpha\}$.

Then $F_{\omega_1} = G_{\omega_1}$ is clearly open, but it is also closed, because every point in its closure is in the closure of some countable subset B, and $B \subset G_\beta$ for some $\beta < \omega_1$ since the G_ξ form an ascending chain; so $\overline{B} \subset F_{\beta+1}$. $\square$

Corollary 3.9. *Every fully Mrówka family is of cardinality $\leq \mathfrak{c}$.*

Proof. Let $\kappa > \mathfrak{c}$, let X be a κ-Mrówka space, and let C be an uncountable clopen subspace of cardinality $\leq \mathfrak{c}$. Let f be the characteristic function of C. Because C is clopen, f is continuous; and the complement of every point preimage has uncountable cardinality. $\square$

What if $\kappa < \mathfrak{c}$? Well, if $\kappa^+ = \mathfrak{c}$ then we can use Mrówka's original construction to get a fully Mrówka space of cardinality $\mathfrak{c}$. Otherwise, an argument almost identical to that of Lemma 3.8 shows:

Lemma 3.10. *If a space X is locally countable and every countable subset of X has closure of cardinality $\leq \kappa$, and κ is uncountable, then every countable subset of X is contained in a clopen set of cardinality $\leq \kappa$.*

From this and an argument similar to the last corollary, it follows that if κ is regular, and there is a fully Mrówka space of cardinality κ, it must contain a separable subspace of cardinality κ. In particular, there must be a MAD family on ω of cardinality κ. When is the converse true? Even the simplest case is unsolved:

Problem 4. If $\mathfrak{a} = \aleph_1$ and there is a MAD family on ω of cardinality $\aleph_2$, is there a fully Mrówka space of cardinality $\aleph_2$?

4. Generalized Kurepa families and applications

A modification of "Kurepa" is easily seen to be satisfied if zero-dimensionality is dropped from Theorem 1.2:

Definition 4.1. Let κ, λ and μ be cardinal numbers, with $\kappa \leq \lambda$. A $(\kappa, \lambda; \mu)$-*Kurepa family* is a collection $\mathcal{K}$ of sets of cardinality λ such that $|\mathcal{K} \restriction A| < \mu$ for all $A \in [\bigcup \mathcal{K}]^\kappa$. It is *cofinal* if it is cofinal in $[\bigcup \mathcal{K}]^\lambda$.

The following theorem generalizes Theorem 1.2, and the proof only requires trivial modifications:

Theorem 4.2. *Let κ, λ and μ be infinite cardinals, with $\kappa < \lambda$. The following are equivalent.*
   (a) There is a cofinal $(\kappa, \kappa; \mu)$-Kurepa family on λ.
   (b) There is a κ-bounded, 0-dimensional space of local weight $< \mu$ and weight $cf[\lambda]^\kappa$.
   (c) There is a cofinal $(\kappa, \kappa; \mu)$-Kurepa family on every set of cardinality $\leq cf[\lambda]^\kappa$.
$\square$

For some choices of κ and μ, the conditions in Theorem 4.2 are satisfied for all $\lambda > \kappa$ in ZFC. For example, there is clearly a cofinal $(\kappa, \kappa; (2^\kappa)^+)$-Kurepa family on such λ. In the case $\kappa = \omega$ we cannot substitute $\mathfrak{c} = 2^\omega$ for $\mathfrak{c}^+$, because the Chang Conjecture variant is compatible with CH, and under CH a $(\omega, \omega; \mathfrak{c})$-Kurepa family is an ordinary Kurepa family. But the following problems are open:

Problem 5. Do there exist in ZFC infinite numbers κ such that there is a cofinal $(\kappa, \kappa; 2^\kappa)$-Kurepa family on every set?

Problem 6. Does $\neg$CH imply that there is a cofinal $(\omega, \omega; \mathfrak{c})$-Kurepa family on every infinite set?

For the next theorem, we will eliminate the last parameter in Definition 4.1.

Definition 4.3. *A $(\kappa, \lambda, \kappa^+)$-Kurepa family is called a (κ, λ)-Kurepa family, while a (κ, κ)-Kurepa family is called a κ-Kurepa family.*

In particular, a Kurepa family (Definition 0.1) is an ω-Kurepa family.

Caution. The use of the prefix κ here is different from that in the definition of a κ-Mrówka family. There, the κ referred to the cardinality of the family. Here, it says nothing about the size of the κ-Kurepa family, instead referring to the cardinality of its members and the cardinality of the traces on each set of cardinality κ.

Our next theorem and problem address the question of what happens if zero-dimensionality is dropped from Theorem 1.2:

Theorem 4.4. *If there is a locally metrizable, ω-bounded (hence locally compact, hence Tychonoff) space of weight λ, then there is a cofinal (ω, ω_1)-Kurepa family on every set of cardinality $\leq cf[\lambda]^\omega$.*

Problem 7. Does the existence of a cofinal (ω, ω_1)-Kurepa family on λ imply that of:
   (a) a cofinal Kurepa family on λ?
   (b) a cofinal ω_1-Kurepa family on λ?

If the answer to **(a)** is Yes, then so is the answer to Problem 2, by Theorems 1.2 and 4.4. An affirmative answer to **(a)** in turn would be implied by one for the following problem, since compact point-preimages in a map with metrizable domain are of countable weight.

Problem 8. Is every locally compact, locally metrizable space the perfect image of a locally compact, locally metrizable, 0-dimensional space? What if the space is ω-bounded as well?

Definition 4.5. *A $(\kappa, \lambda; \mu)$-Kurepa family is* **weakly cofinal** *if every member of* $[\bigcup \mathcal{K}]^{\kappa}$ *is contained in some* $K \in \mathcal{K}$.

Obviously, weak cofinality and cofinality are equivalent if $\lambda = \kappa$. Moreover, the existence of a weakly (κ, λ)-cofinal family always implies that of a cofinal family if $\lambda = cf[\lambda]^{\kappa}$. This is a consequence of the following lemma.

Lemma 4.6. *Let $\mathcal{C}$ and $\mathcal{K}$ be collections of sets such that every member of $\mathcal{C}$ is a subset of some member of $\mathcal{K}$. Let $\kappa = sup\{|C| : C \in \mathcal{C}\}$. If $|K| \leq \lambda$ for all $K \in \mathcal{K}$ and $\kappa < \lambda = cf[\lambda]^{\kappa}$ then there exists a cofinal subcollection $\mathcal{L}$ of $[\bigcup \mathcal{K}]^{\lambda}$ such that $\mathcal{L} \restriction C \subset \mathcal{K} \restriction C$ for all $C \in \mathcal{C}$.*

Proof: Let $W \in [\bigcup \mathcal{K}]^{\lambda}$. Let $\nu = \kappa^{+}[\leq \lambda]$. Define $L_{\alpha}(W)$ for all $\alpha \leq \nu$ as follows. Let $L_0(W) = W$. If α is a limit ordinal let $L_{\alpha}(W) = \bigcup_{\beta < \alpha} L_{\beta}(W)$. If $\alpha = \beta + 1 \leq \nu$ let $\mathcal{K}_{\beta}(W)$ be a subcollection of $\mathcal{K}$ of minimum cardinality such that each set of the form $C \cap L_{\beta}(W)$ $(C \in \mathcal{C})$ is a subset of some $K \in \mathcal{K}_{\beta}(W)$. Let $L_{\alpha}(W) = L_{\beta}(W) \cup \bigcup \mathcal{K}_{\beta}(W)$.

By induction, $|L_{\alpha}(W)| = \lambda$ for all $\alpha \leq \nu$: at limit ordinals we are taking the supremum of $\leq \lambda$ sets while if $|L_{\beta}| = \lambda$ then no more than $cf[\lambda]^{\kappa} = \lambda$ sets are needed to cover all sets of the form $C \cap L_{\beta}(W)$.

Let $\mathcal{L} = \{L_{\nu}(W) : W \in [\bigcup \mathcal{K}]^{\lambda}\}$. Clearly $\mathcal{L}$ is cofinal in $[\bigcup \mathcal{K}]^{\lambda}$. The $L_{\alpha}(W)$ form a monotone nondecreasing sequence for each W, so that if $C \in \mathcal{C}$, there exists $\beta < \nu$ such that $C \cap L_{\nu}(W) = C \cap L_{\beta}(W)$. Then there exists $K \in \mathcal{K}_{\beta}(W)$ such that $C \cap L_{\beta}(W) \subset K$, while $K \subset L_{\nu}(W)$, so that $C \cap L_{\nu}(W) = C \cap K$. $\square$

If κ happens to be a regular limit cardinal (= a weakly inaccessible cardinal) and there is no $C \in \mathcal{C}$ of cardinality κ, then it is not necessary to have $\kappa < \lambda$ in Lemma 4.6, and we can also get by with letting $\nu = \kappa$. Of course, $\kappa \leq \lambda$ since every member of $\mathcal{C}$ is a subset of some member of $\mathcal{K}$.

Theorem 4.7. *If there is a weakly cofinal $(\kappa, \lambda; \mu)$-Kurepa family on a set Y, then there is a cofinal $(\kappa, \lambda'; \mu)$-Kurepa family for all $\lambda' \leq |Y|$ such that $\lambda' = cf[\lambda']^{\kappa}$.*

Proof: In Lemma 4.6 let $\mathcal{C} = [Y]^{\kappa}$, let $\mathcal{K}$ be a weakly cofinal $(\kappa, \lambda; \mu)$-Kurepa family. Then $\mathcal{L}$, with λ' in place of λ, is the desired family. $\square$

Corollary 4.8. *If there is a cofinal Kurepa family on λ, then there is a cofinal $(\omega, cf[\kappa]^{\omega})$-Kurepa family on $cf[\lambda]^{\omega}$ for all infinite $\kappa \leq \lambda$.* $\square$

In the topological setting which inspired Lemma 4.6, Theorem 4.7, and Corollary 4.8, $\mathcal{C}$ is $[Y]^{\omega}$ where Y is a locally countable subset Y of a locally metrizable, ω-bounded space X, while $\mathcal{K}$ is the set of traces on Y of the clopen subsets of X of Lindelöf degree ω_1. Theorem 4.9 below implies that every member of $\mathcal{C}$ is a subset of some member of $\mathcal{K}$, and we we can choose the $\mathcal{L}$ of Lemma 4.6 so that $\mathcal{L} \restriction C = \mathcal{K} \restriction C$ for all $C \in \mathcal{C}$. This we do by simply letting $\mathcal{L}$ be the set of traces of clopen sets of Lindelöf degree λ' on Y. This works for the following reason: if D is a clopen set of Lindelöf degree ω_1 and E is a clopen subset of D, let F be any clopen set of Lindelöf degree $\lambda > \omega_1$; then $F' = (F \setminus D) \cup E$ meets D exactly in E and is also of Lindelöf degree λ. If X is 0-dimensional, then we can begin with $\mathcal{K}$ being the set of traces on Y of compact open subsets of X and we can let λ be any infinite cardinal number.

Theorem 4.9. *Let X be an ω-bounded, first countable space and let $\omega < \lambda \leq |X|$. Then every subset of X of cardinality λ is contained in a clopen subset of X of Lindelöf degree $cf[\lambda]^\omega$. If X is 0-dimensional, then also every countable subset is contained in a compact open subset.*

There are some cardinal invariants of the space X of Theorem 1.2 which are still unsettled. One is its *density,* the least cardinality of a dense subspace.

Theorem 4.10. *Every locally separable topological space has a locally countable dense subspace. Moreover the subspace can be chosen to admit a partition into countable clopen subsets.*

Proof: Let X be locally separable, and let $\mathcal{D}$ be a maximal family of disjoint separable open subsets of X. Picking a countable dense subset of each $D \in \mathcal{D}$ and taking the union of these subsets gives us a locally countable dense subspace Y of X, and $\{Y \cap D : D \in \mathcal{D}\}$ is a partition as described. $\square$

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