

1. Determine if the following series converge or diverge. If a series converges, then you should also compute its sum.

$$(a) \sum_{k=0}^{\infty} e^{1-2k} = e + \frac{1}{e} + \frac{1}{e^3} + \frac{1}{e^5} + \frac{1}{e^7} + \cdots$$

This is a geometric series with a ratio of $\frac{1}{e^2}$. Since $|\frac{1}{e^2}| < 1$, the series converges. The sum of the series is $\frac{e}{1 - \frac{1}{e^2}} = \frac{e^3}{e^2 - 1}$.

$$(b) \sum_{k=0}^{\infty} \frac{2}{3k+3} = \sum_{k=0}^{\infty} \frac{2}{3} \cdot \frac{1}{k+1} = \sum_{k=1}^{\infty} \frac{2}{3} \cdot \frac{1}{k}$$

This series is a constant multiple ($\frac{2}{3}$) of the divergent harmonic series. Therefore the series diverges as well.

$$(c) \sum_{k=0}^{\infty} \frac{2+3^k}{2^{2k+1}} = \sum_{k=0}^{\infty} \frac{2}{2^{2k+1}} + \sum_{k=0}^{\infty} \frac{3^k}{2^{2k+1}}$$

$$\sum_{k=0}^{\infty} \frac{2}{2^{2k+1}} = \frac{2}{2} + \frac{2}{2^3} + \frac{2}{2^5} + \frac{2}{2^7} + \frac{2}{2^9} + \cdots$$

This is a geometric series with a ratio of $\frac{1}{4}$. Since $|\frac{1}{4}| < 1$, the series converges. The sum of the series is $\frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$.

$$\sum_{k=0}^{\infty} \frac{3^k}{2^{2k+1}} = \frac{1}{2} + \frac{3}{2^3} + \frac{3^2}{2^5} + \frac{3^3}{2^7} + \frac{3^4}{2^9} + \cdots$$

This is a geometric series with a ratio of $\frac{3}{4}$. Since $|\frac{3}{4}| < 1$, the series converges. The sum of the series is $\frac{1/2}{1 - \frac{3}{4}} = 2$.

Therefore the original series converges and its sum is $\frac{4}{3} + 2 = \frac{10}{3}$.

$$(d) \sum_{k=1}^{\infty} \left(\frac{\pi}{e}\right)^k = \frac{\pi}{e} + \left(\frac{\pi}{e}\right)^2 + \left(\frac{\pi}{e}\right)^3 + \left(\frac{\pi}{e}\right)^4 + \left(\frac{\pi}{e}\right)^5 + \cdots$$

This is a geometric series with a ratio of $\frac{\pi}{e}$. Since $|\frac{\pi}{e}| \geq 1$, the series diverges.

2. For which values of x does the following series converge? What is the sum of the series for these values of x ?

$$\frac{3}{4} - \frac{3(x-3)}{8} + \frac{3(x-3)^2}{16} - \frac{3(x-3)^3}{32} + \frac{3(x-3)^4}{64} - \frac{3(x-3)^5}{128} + \dots$$

This is a geometric series with a ratio of $-\frac{1}{2}(x-3)$. The series converges when $|\frac{1}{2}(x-3)| < 1$. Solving this inequality we get convergence for $1 < x < 5$.

3. Rewrite the following repeating decimal as a series and then compute its sum as a simplified fraction.

$$0.19\overline{0} = 0.1909090909090\dots$$

$$0.1909090909090\dots = \frac{1}{10} + \underbrace{\frac{90}{10^3} + \frac{90}{10^5} + \frac{90}{10^7} + \frac{90}{10^9} + \dots}_{\text{geometric, } |ratio| = 1/100 < 1, \text{ converges}}$$

$$0.1909090909090\dots = \frac{1}{10} + \frac{90/10^3}{1 - 1/100} = \frac{1}{10} + \frac{1}{11} = \frac{21}{110}$$

4. Does the following series converge or diverge? If the series converges, find its exact sum.

$$\frac{3}{2 \cdot 4} + \frac{3}{3 \cdot 5} + \frac{3}{4 \cdot 6} + \frac{3}{5 \cdot 7} + \frac{3}{6 \cdot 8} + \frac{3}{7 \cdot 9} + \dots$$

Rewriting the series using sigma notation, using the method of partial fractions, and taking limits we get that the series converges to $\frac{5}{4}$.

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{3}{k(k+2)} &= \sum_{k=2}^{\infty} \left(\frac{3/2}{k} - \frac{3/2}{k+2} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{k=2}^n \left(\frac{3/2}{k} - \frac{3/2}{k+2} \right) \\ &= \frac{3}{2} \cdot \lim_{n \rightarrow \infty} \sum_{k=2}^n \left(\frac{1}{k} - \frac{1}{k+2} \right) \\ &= \frac{3}{2} \cdot \lim_{n \rightarrow \infty} \left(\left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) + \left(\frac{1}{n} - \frac{1}{n+2} \right) \right) \\ &= \frac{3}{2} \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{3}{2} \cdot \left(\frac{1}{2} + \frac{1}{3} \right) \\ &= \frac{5}{4} \end{aligned}$$