Math 142 (Section 501) **Quiz 12 Solutions** **Spring 2008**

1. The following infinite series converges to a sum S.

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{6}} + \cdots$$

   (a) There are two conditions that are necessary for an alternating series to converge. Are those conditions met here? Explain.

      i. The quantity $\frac{1}{\sqrt{k}}$ is decreasing on the interval $[1, \infty)$. This is pretty obvious but you should also be able to show that the derivative of $f(x) = \frac{1}{\sqrt{x}}$ is negative on the interval $[1, \infty)$ to fully justify this claim.

      ii. $\lim_{k \to \infty} \frac{1}{\sqrt{k}} = 0$

   (b) If we approximate the sum of the infinite series with

$$S \approx \sum_{k=1}^{3} (-1)^{k+1} \frac{1}{\sqrt{k}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}},$$

   then how can the error in our approximation be bounded? (fill in the blank below)

   $|\text{error}| \leq \frac{1}{\sqrt{4}} = 0.5$

   (c) How many of the beginning terms of the infinite series could you add together to get an estimate for its sum S that was within 0.01 of the correct sum?

   By adding the first 9999 terms, we have that

   $|\text{error}| \leq \frac{1}{\sqrt{10000}} = 0.01$

2. A function $f(x)$ and its derivatives have the following values at $x = 0$.

$$f(0) = 5, f'(0) = -7, f''(0) = 6, f'''(0) = 12$$

   What is the 3rd Maclaurin polynomial for $f(x)$?

$P_3(x) = 5 - 7x + \frac{6}{2!}x^2 + \frac{12}{3!}x^3$

$P_3(x) = 5 - 7x + 3x^2 + 2x^3$

3. Find the 4th Taylor polynomial for $f(x) = \dfrac{1}{x}$ about $x = 1$.

$$\begin{aligned} f(x) &= x^{-1} & f(1) &= 1 \\ f'(x) &= -x^{-2} & f'(1) &= -1 \\ f''(x) &= 2x^{-3} & f''(1) &= 2 \\ f'''(x) &= -6x^{-4} & f'''(1) &= -6 \\ f^{(4)}(x) &= 24x^{-5} & f^{(4)}(1) &= 24 \end{aligned}$$

$$P_4(x) = 1 - (x-1) + \frac{2}{2!}(x-1)^2 + \frac{-6}{3!}(x-1)^3 + \frac{24}{4!}(x-1)^4$$

$$P_4(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$$

4. Find the interval of convergence for the following power series. You must thoroughly justify your claim.

   (a) $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dfrac{x^8}{8!} + \cdots$

   We apply the absolute ratio test to $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$

$$\begin{aligned} \rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\ &= \lim_{n\to\infty} \left| \frac{x^{2n+2}/(2n+2)!}{x^{2n}/(2n)!} \right| \\ &= \lim_{n\to\infty} \left| \frac{(2n)!x^{2n+2}}{(2n+2)!x^{2n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| \\ &= x^2 \cdot \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \right| \\ &= x^2 \cdot 0 \\ &= 0 \quad \text{for all values of} \quad x \end{aligned}$$

   Since $\rho < 1$ for all x, the series converges absolutely and therefore converges on the interval $(-\infty, \infty)$.

(b) $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k+2}$

We apply the absolute ratio test.

$$
\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}/(n+3)}{(x-5)^n/(n+2)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}(n+2)}{(x-5)^n(n+3)} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-5)(n+2)}{n+3} \right| \\
&= |x-5| \lim_{n\to\infty} \left| \frac{n+2}{n+3} \right| \\
&= |x-5|
\end{aligned}
$$

The series converges absolutely and therefore converges when $\rho < 1$. This occurs when $4 < x < 6$, but we still need to check the endpoints.

When $x = 4$ the series becomes $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+2}$ which converges (alternating series test).

When $x = 6$ the series becomes $\sum_{k=0}^{\infty} \frac{1}{k+2}$ which diverges (harmonic series).

Thus the series converges for $4 \le x < 6$ giving the interval of convergence $[4, 6)$.