

1. Use any short-cut methods learned in class or in section 10.10 to find the Maclaurin series for the following functions. You should list at least the first 4 nonzero terms in each series and state the interval of convergence.

(a) $\frac{1}{1+2x}$ is equal to its Maclaurin series for $-\frac{1}{2} < x < \frac{1}{2}$.

$$\begin{aligned}\frac{1}{1+2x} &= \frac{1}{1-(-2x)} \\ &= 1 + (-2x) + (-2x)^2 + (-2x)^3 + (-2x)^4 + (-2x)^5 + \dots \\ &= 1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 + \dots\end{aligned}$$

(b) $\frac{3x}{1-x^2}$ is equal to its Maclaurin series for $-1 < x < 1$.

$$\begin{aligned}\frac{3x}{1-x^2} &= 3x \cdot \frac{1}{1-x^2} \\ &= 3x \left(1 + x^2 + (x^2)^2 + (x^2)^3 + (x^2)^4 + \dots \right) \\ &= 3x \left(1 + x^2 + x^4 + x^6 + x^8 + \dots \right) \\ &= 3x + 3x^3 + 3x^5 + 3x^7 + 3x^9 + \dots\end{aligned}$$

(c) $x \sin(x^2)$ is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned}x \sin(x^2) &= x \left((x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots \right) \\ &= x \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \right) \\ &= x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \dots\end{aligned}$$

(d) e^{-2x} is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned}e^{-2x} &= 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} + \dots \\ &= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} - \frac{32x^5}{5!} + \dots\end{aligned}$$

(e) e^{2+x} is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned}e^{2+x} &= e^2 \cdot e^x \\ &= e^2 \cdot \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \\ &= e^2 + e^2x + \frac{e^2x^2}{2!} + \frac{e^2x^3}{3!} + \frac{e^2x^4}{4!} + \dots\end{aligned}$$

(f) $\sqrt{e^x}$ is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned}\sqrt{e^x} &= e^{x/2} \\ &= 1 + (x/2) + \frac{(x/2)^2}{2!} + \frac{(x/2)^3}{3!} + \frac{(x/2)^4}{4!} + \dots \\ &= 1 + \frac{1}{2}x + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} + \dots\end{aligned}$$

(g) $\frac{6}{2+x}$ is equal to its Maclaurin series for $-2 < x < 2$.

$$\begin{aligned}\frac{6}{2+x} &= \frac{6}{2(1+x/2)} \\ &= 3 \cdot \frac{1}{1-(-x/2)} \\ &= 3 \cdot \left(1 + (-x/2) + (-x/2)^2 + (-x/2)^3 + (-x/2)^4 + (-x/2)^5 + \dots\right) \\ &= 3 - \frac{3}{2}x + \frac{3}{4}x^2 - \frac{3}{8}x^3 + \frac{3}{16}x^4 - \frac{3}{32}x^5 + \dots\end{aligned}$$

(h) $\ln((2+2x)^5)$ is equal to its Maclaurin series for $-1 < x \leq 1$.

$$\begin{aligned}\ln((2+2x)^5) &= 5 \ln(2+2x) \\ &= 5 \ln(2(1+x)) \\ &= 5(\ln 2 + \ln(1+x)) \\ &= 5 \left(\ln 2 + \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) \right) \\ &= 5 \ln 2 + 5x - \frac{5x^2}{2} + \frac{5x^3}{3} - \frac{5x^4}{4} + \dots\end{aligned}$$

(i) $\ln\left(\frac{1-x}{1+x}\right)$ is equal to its Maclaurin series for $-1 < x < 1$.

$$\begin{aligned} \ln\left(\frac{1-x}{1+x}\right) &= \ln(1-x) - \ln(1+x) \\ &= \left((-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} - \dots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right) \\ &= \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\right) \\ &= -2x - \frac{2x^3}{3} - \frac{2x^5}{5} - \dots \end{aligned}$$

(j) $\frac{5x-1}{x^2-1}$ is equal to its Maclaurin series for $-1 < x < 1$.

$$\begin{aligned} \frac{5x-1}{x^2-1} &= \frac{5x-1}{(x-1)(x+1)} \\ &= \frac{2}{x-1} + \frac{3}{x+1} \\ &= \frac{-2}{1-x} + \frac{3}{1+x} \\ &= -2(1+x+x^2+x^3+x^4+x^5+\dots) + 3(1-x+x^2-x^3+x^4-x^5+\dots) \\ &= 1-5x+x^2-5x^3+x^4-5x^5+\dots \end{aligned}$$

(k) $\frac{1}{(1-x)^2}$ is equal to its Maclaurin series for $-1 < x < 1$.

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx} \left(\frac{1}{1-x} \right) \\ &= \frac{d}{dx} (1+x+x^2+x^3+x^4+x^5+\dots) \\ &= 1+2x+3x^2+4x^3+5x^4+\dots \end{aligned}$$

(l) $e^x \sin x$ is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned} e^x \sin x &= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right) \cdot \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right) \\ &= x+x^2+\left(-\frac{1}{3!}+\frac{1}{2!}\right)x^3+\left(-\frac{1}{3!}+\frac{1}{3!}\right)x^4+\left(\frac{1}{5!}-\frac{1}{2!3!}+\frac{1}{4!}\right)x^5+\dots \\ &= x+x^2+\frac{1}{3}x^3-\frac{1}{30}x^5+\dots \end{aligned}$$

(m) $e^{x^2} \cos x$ is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned} e^{x^2} \cos x &= \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots\right) \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{31}{720}x^6 + \dots \end{aligned}$$

(n) $6 \sin x \cos x$ is equal to its Maclaurin series for $-\infty < x < \infty$.

$$\begin{aligned} 6 \sin x \cos x &= 6 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) \\ &= 6x - 4x^3 + \frac{4}{5}x^5 - \frac{8}{105}x^7 + \dots \end{aligned}$$

(o) $\frac{\tan^{-1} x}{1+x}$ is equal to its Maclaurin series for $-1 < x < 1$.

$$\begin{aligned} \frac{\tan^{-1} x}{1+x} &= \tan^{-1} x \cdot \frac{1}{1+x} \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) \left(1 - x + x^2 - x^3 + x^4 - x^5 + \dots\right) \\ &= x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + \dots \end{aligned}$$

(p) $\frac{\ln(1+x)}{1-x}$ is equal to its Maclaurin series for $-1 < x < 1$.

$$\begin{aligned} \frac{\ln(1+x)}{1-x} &= \ln(1+x) \cdot \frac{1}{1-x} \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) \left(1 + x + x^2 + x^3 + x^4 + \dots\right) \\ &= x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \dots \end{aligned}$$

2. Use Maclaurin series to approximate the following definite integrals. Give an upper bound on the error in your approximation.

$$(a) \int_0^1 e^{-x^2} dx$$

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \right) dx \\ &= \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots \right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \cdots \end{aligned}$$

$$(b) \int_0^1 \sin(x^2) dx$$

$$\begin{aligned} \int_0^1 \sin(x^2) dx &= \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots \right) dx \\ &= \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \cdots \end{aligned}$$

$$(c) \int_0^1 \frac{8}{8+x^3} dx$$

$$\begin{aligned} \int_0^1 \frac{8}{8+x^3} dx &= \int_0^1 \frac{8}{8(1+x^3/8)} dx \\ &= \int_0^1 \frac{1}{1+x^3/8} dx \\ &= \int_0^1 \left(1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + \cdots \right) dx \\ &= \left[x - \frac{x^4}{4 \cdot 8} + \frac{x^7}{7 \cdot 64} - \frac{x^{10}}{10 \cdot 512} + \cdots \right]_0^1 \\ &= 1 - \frac{1}{4 \cdot 8} + \frac{1}{7 \cdot 64} - \frac{1}{10 \cdot 512} + \cdots \end{aligned}$$

3. Use Maclaurin series to evaluate the following limits.

$$(a) \lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} &= \lim_{x \rightarrow 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x} \\ &= \lim_{x \rightarrow 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right) \\ &= 1 \end{aligned}$$

$$(b) \lim_{x \rightarrow 0} \frac{\sin x - x + x^3/6}{x^5}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x - x + x^3/6}{x^5} &= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) - x + x^3/6}{x^5} \\ &= \lim_{x \rightarrow 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5} \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \dots \right) \\ &= \frac{1}{120} \end{aligned}$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan^{-1} x - x}{x^3}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan^{-1} x - x}{x^3} &= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) - x}{x^3} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots}{x^3} \\ &= \lim_{x \rightarrow 0} \left(-\frac{1}{3} + \frac{x^2}{5} - \frac{x^4}{7} + \dots \right) \\ &= -\frac{1}{6} \end{aligned}$$