Math 142 (Section 502) **Quiz 11 Solutions** **Spring 2007**

1. Determine if the following series converge or diverge. If a series converges, then you should also compute its sum.

   (a) $8 - 4 + 2 - 1 + 1/2 - 1/4 + 1/8 - 1/16 + 1/32 - 1/64 + \cdots$

   This is a geometric series with a ratio of $-\dfrac{1}{2}$. Since $\left|-\dfrac{1}{2}\right| < 1$, the series converges. The sum of the series is $\dfrac{8}{1-\left(-\frac{1}{2}\right)} = \dfrac{16}{3}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k}$

   This series is a constant multiple $\left(\frac{1}{2}\right)$ of the divergent harmonic series. Therefore the series diverges as well.

   (c) $\displaystyle\sum_{k=0}^{\infty} e^{-k}$

   This is a geometric series with a ratio of e^{-1}. Since $\left|e^{-1}\right| = \dfrac{1}{e} < 1$, the series converges. The sum of the series is $\dfrac{1}{1-e^{-1}} = \dfrac{e}{e-1}$.

   (d) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k - 1}{3^k} = \sum_{k=1}^{\infty} \frac{2^k}{3^k} - \sum_{k=1}^{\infty} \frac{1}{3^k}$

   This is the difference of two convergent geometric series with ratios of $\dfrac{2}{3}$ and $\dfrac{1}{3}$, respectively. Since each ratio is less than 1 in absolute value, we know that each series converges. Hence, the original series converges as well. The sum of the original series is $\dfrac{2/3}{1-2/3} - \dfrac{1/3}{1-1/3} = \dfrac{3}{2}$.

   (e) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{2k-1}}{3^k}$

   This is a geometric series with a ratio of $4/3$. Since $|4/3| \geq 1$, the series diverges.

2. For which values of x does the following series converge? What is the sum of the series for these values of x ?

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5 + \cdots$$

This is a geometric series with a ratio of $2x$. The series converges when $|2x| < 1$. Solving this inequality we get convergence for $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

3. Does the following series converge or diverge? If the series converges, find its exact sum.

$$\frac{5}{3\cdot 4}+\frac{5}{4\cdot 5}+\frac{5}{5\cdot 6}+\frac{5}{6\cdot 7}+\cdots$$

Rewriting the series using sigma notation, using the method of partial fractions, and taking limits we get that the series converges to $\frac{5}{3}$.

$$\begin{aligned}
\sum_{k=3}^{\infty}\frac{5}{k(k+1)} &= \sum_{k=3}^{\infty}\left(\frac{5}{k}-\frac{5}{k+1}\right)\\
&= \lim_{n\to\infty}\sum_{k=3}^{n}\left(\frac{5}{k}-\frac{5}{k+1}\right)\\
&= \lim_{n\to\infty}\left(\left(\frac{5}{3}-\frac{5}{4}\right)+\left(\frac{5}{4}-\frac{5}{5}\right)+\cdots+\left(\frac{5}{n}-\frac{5}{n+1}\right)\right)\\
&= \lim_{n\to\infty}\left(\frac{5}{3}-\frac{5}{n+1}\right)\\
&= \frac{5}{3}
\end{aligned}$$

4. Rewrite the following repeating decimal as a series and then compute its sum as a simplified fraction.

$$0.21111111111111\ldots$$

$$0.21111111111111\ldots=\frac{2}{10}+\underbrace{\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\frac{1}{10^5}+\cdots}_{\text{geometric, ratio}=1/10}$$

$$0.21111111111111\ldots=\frac{2}{10}+\frac{1/10^2}{1-1/10}=\frac{19}{90}$$