1. Evaluate the following integrals.

   (a) Using polynomial division we obtain

   $$\int \frac{4x^2+3}{x^2+1}\,dx = \int \left(4 - \frac{1}{x^2+1}\right)\,dx = 4x - \tan^{-1}x + C$$

   (b) Using integration by parts with $u = \ln x$ and $dv = x^{1/2}dx$ we obtain

   $$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

   (c) Using integration by parts with $u = \sin^{-1}x$ and $dv = dx$ we obtain

   $$\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

   Now doing a substitution with $u = 1 - x^2$ will lead to our final answer of

   $$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

   (d) Using tabular integration (i.e. repeated integration by parts) we obtain

   $$\int 10x^3\cos x\,dx = 10x^3\sin x + 30x^2\cos x - 60x\sin x - 60\cos x + C$$

   (e) We pull off a $\sec^2 x$ with dx and write the rest of the integrand in terms of $\tan x$ to obtain

   $$\begin{aligned}
   \int \sec^6 x\,dx &= \int \sec^4 x\sec^2 x\,dx \\
   &= \int \left(\tan^2 + 1\right)^2 \sec^2 x\,dx \\
   &= \int \left(u^2+1\right)^2\,du \qquad \text{(substitution: } u = \tan x\text{)} \\
   &= \int \left(u^4 + 2u^2 + 1\right)\,du \\
   &= \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + C \\
   &= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C
   \end{aligned}$$

(f) One way is to do a careful substitution with $u = 4 + x^2$. However for this handout I have opted to do a trig substitution instead to obtain

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4+x^2}}\,dx &= \int 8\tan^3\theta\sec\theta\,d\theta && \text{(substitution: } x = 2\tan\theta\text{)}\\
&= \int 8\tan^2\theta\sec\theta\tan\theta\,d\theta\\
&= \int 8\left(\sec^2\theta - 1\right)\sec\theta\tan\theta\,d\theta\\
&= \int 8\left(u^2-1\right)\,du && \text{(substitution: } u = \sec\theta\text{)}\\
&= \frac{8}{3}u^3 - 8u + C\\
&= \frac{8}{3}\sec^3\theta - 8\sec\theta + C\\
&= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\left(\frac{\sqrt{4+x^2}}{2}\right) + C
\end{aligned}$$

(g) Using the partial fraction decomposition we obtain

$$\begin{aligned}
\int \frac{6x^2-3x+1}{(x^2+1)(x-1)}\,dx &= \left(\frac{4x+1}{x^2+1} + \frac{2}{x-1}\right)\,dx\\
&= \int\left(\frac{4x}{x^2+1} + \frac{1}{x^2+1} + \frac{2}{x-1}\right)\,dx\\
&= 2\ln\left(x^2+1\right) + \tan^{-1}x + 2\ln\left(x-1\right) + C
\end{aligned}$$

(h) Since the integrand is undefined at $x = 9$, we evaluate this integral as a limit to obtain

$$\begin{aligned}
\int_0^9 \frac{1}{\sqrt{9-x}}\,dx &= \lim_{b\to 9^-}\int_0^b (9-x)^{-1/2}\,dx\\
&= \lim_{b\to 9^-}\left[-2(9-x)^{1/2}\right]_0^b\\
&= \lim_{b\to 9^-}\left[\left(-2\sqrt{9-b}\right) - \left(-2\sqrt{9}\right)\right]\\
&= 6
\end{aligned}$$

(i) Multiplying out the numerator, using known trig identities, and rewriting everything in terms of sine and cosine we obtain

$$\int \frac{(\sin x + \cos x)^2 - 1}{\tan^4 x}\, dx = \int \frac{2\cos^5 x}{\sin^3 x}\, dx$$

Pulling off a $\cos x$ with dx and writing everything else in terms of $\sin x$ we obtain

$$\int \frac{2(1-\sin^2 x)^2}{\sin^3 x} \cos x\, dx$$

Letting $u = \sin x$ and then multiplying out the integrand we obtain

$$\int \left(2u^{-3} - 4u^{-1} + 2u\right)\, du = -u^{-2} - 4\ln u + u^2 + C$$

Writing our answer in terms of x we obtain an answer of

$$-\frac{1}{\sin^2 x} - 4\ln(\sin x) + \sin^2 x + C$$

2. Find a general formula for a_n, the nth term of the following sequence. Does this sequence converge or diverge? Explain. If the sequence converges, be sure to find its limit.

$$-\frac{\cos 1}{2}, \frac{\cos 2}{4}, -\frac{\cos 3}{6}, \frac{\cos 4}{8}, -\frac{\cos 5}{10}, \frac{\cos 6}{12}, -\frac{\cos 7}{14}, \ldots$$

$$a_n = (-1)^n \frac{\cos n}{2n}$$

This sequence converges since $\lim_{n\to\infty} (-1)^n \frac{\cos n}{2n} = 0$ (by the squeeze theorem).

3. Prove that the sequence below is either strictly increasing or strictly decreasing.

$$\frac{n^n}{n!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{(n+1)}/(n+1)!}{n^n/n!} = \frac{(n+1)^n}{n^n} > 1$$

Thus the sequence is increasing.