**BIOL 763** **Spring, 1999** **Test #1** **Name:**____________________

This exam is due on Thursday 25 February. You are to work individually, but you may use your texts, notes, or Maple. Give sufficient justification for your answers. (Don't sweat the Maple–if you know what you want it to do, but don't see how to get the existing worksheets to do it, please ask one of us for help!)

1. (20 points) A population $N(t)$ of turtles has a *per capita* growth rate given by $(0.2)(1 - (5/N) - (N/40))$ in units of (turtles/year)/turtle.

**a.** Write the equation for the *net* growth rate $\frac{dN}{dt}$. What are the units?

**b.** If $N(0) = 6$, give a good approximation for $N(20)$; explain how you did it.

**c.** This system is found to have two non-trivial equilibrium values: $N = 5.9$ and $N = 34.1$. Analyze the stability of these equilibria by selecting initial values close to them and using the differential equation. Sketch $N$ as a function of $t$ for each case (all on one graph).

**d.** This kind of model is often used in a context in which it is believed that there is a "minimal sustainable population." How can this be seen in terms of the results you obtained above?

2. (20 points) A system is found to have two equilibrium points, and the coefficient matrix is computed for the linearization at each of these points. At one of the points the matrix is $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, and at the other point the matrix is $B = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$.

**a.** Show how each matrix transforms the unit square (shaded region in the diagram).

**b.** Matrix $A$ has no real eigenvalues, but it has complex eigenvalues $\lambda = 1/2 \pm i\sqrt{3}/2$. Show what trajectories near the corresponding equilibrium point must look like. Is the equilibrium stable or not?

**c.** Matrix $B$ has eigenvalues $\lambda = -2$, $\mu = -1$, and eigenvectors $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Which eigenvalue goes with which eigenvector? Sketch the phase plane in the vicinity of the equilibrium, showing the lines of the eigenvectors, and sample trajectories along these lines and in the zones between them (use arrow heads to show the direction of forward time). Is the equilibrium stable or not?

3. (10 points) In this compartment model $X_1$ and $X_2$ represent uninjured and injured subpopulations of a prey population, and $Y$ is the predator population that only injures but does not kill the prey (e.g., flatfish consumption of bivalve siphon tips). The labels on the arrows from one box (the source) to another box (the target, or out of the system entirely) give per capita loss rates from the source population (i.e., the fraction of the source population that is lost). This simultaneously represents gain to the target population. Arrows from a box to itself represent net gain as a fraction of current population. Give the equations for $dX_1/dt$, $dX_2/dt$, and $dY/dt$.

4. (50 points) Consider the following model of plant-herbivore interaction.

$$\begin{aligned} \frac{dq}{dt} &= K - \alpha q N(N - C) \\ \frac{dN}{dt} &= rN(1 - \frac{\beta N}{q}) \end{aligned}$$

Here $N$ denotes the insect (scale bug) density, $q$ denotes the nutritive quality of the plant to the bug, and $K$, $r$, $C$, $\alpha$ and $\beta$ are positive parameters.

**a.** At low levels of $q$ the plant is not only not nutritious, but it is actually toxic. How is this reflected in the model?

**b.** What happens to the nutritive quality if bug infestation is low? When bug infestation is moderate? When it is high? What parameters or combination of parameters marks each of these transitions? (more space on next page)

**c.** Assume that $C = 0$. Find the steady state(s) $(\bar{q}, \bar{N})$ of the system in terms of the parameters. Explain why, intuitively, the parameters appear as they do.

**d.** Again assuming $C = 0$, we have sketched the nullclines of the system. The equations are $q = \beta N$ and $q = K/(\alpha N^2)$. Determine which corresponds to $\frac{dq}{dt} = 0$ and which to $\frac{dN}{dt} = 0$. What happens to $N$ if you start a trajectory in the horizontally shaded region? What happens to $q$ if you start in the vertically shaded region? Is the equilibrium stable or not? Sketch a plausible trajectory on the graph if the initial state is one of high $q$ and very small $N$ (a new infestation).

**e.** Still taking $C = 0$, compute the Jacobian matrix, the coefficient matrix for the linearization at the interesting equilibrium point, and the eigenvalues. Suggestion: make Maple do the work! Bonus: use these eigenvalues to modify or reconfirm your stability analysis.

**f.** Reformulate the model by taking the per capita birth rate of the bug population as dependent on resource $q$, with saturation taking place at high values of $q$, and a constant per capita death rate.