MATH 700 **Final Exam** **Fall, 2005**

For full credit you must show the essential work. If a result has been done in the homework, I expect you to do it again here. Otherwise you may quote needed results from the text or lecture, but not by number, rather by statement of the content of the result. Further, I expect you to use more basic results to prove more advanced ones, and not the other way around. Please write only on the front side of the paper and don't forget to give your name. Number the pages, leave room at the TOP LEFT for a staple, and put your initials at the top right hand corner of each page. There are 150 points.

1. (20 points) Only very short answers are required in this problem.
   a. If $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$ is there a real 2×2 matrix B so that $B^2 = A$? Briefly explain why or why not.
   b. If W_1 and W_2 are subspaces of a vector space V, is $W_1 \cup W_2$ always a subspace of V? Briefly give a proof or disprove by counterexample.
   c. Show that if T satisfies $TT^{\text{adj}} = T^{\text{adj}}T = I$ on an inner product space V, then $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x and y in V.
   d. If T is a normal operator on an inner product space V, λ_1 is an eigenvalue with eigenvector v, λ_2 is an eigenvalue with eigenvector w, and $\lambda_1 \neq \lambda_2$, show that v and w are orthogonal.
   e. Suppose $V = W_1 + W_2 + W_3$, where W_i are subspaces, and suppose $W_i \cap W_j = \{0\}$ for $i \neq j$. Prove, or disprove by a counterexample, that $V = W_1 \oplus W_2 \oplus W_3$.

2. (14 points) Suppose that V is an n-dimensional vector space, and W_1 and W_2 are subspaces with $W_1 + W_2 = V$. Prove that $n = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.

3. (16 points) Let T and U be non-zero linear operators on a finite dimensional complex vector space V. Assume that $TU = UT$. Prove that T and U have a common eigenvector. Suggestion: how does T act on an eigenspace of U?

4. (20 points) Suppose that V is a T-cyclic subspace of itself, where T is a linear operator on V and $\dim V = n$.
   a. Prove the minimal polynomial $p_T(x)$ of T is equal to $(-1)^n \chi_T(x)$, where $\chi_T(x)$ is the characteristic polynomial of T.
   b. Give the matrix representation of T with respect to a T-cyclic basis of V.
   c. Assume, in addition, that T is diagonalizable. Prove that each eigenspace is one dimensional.

5. (16 points) Let T and U be non-zero linear transformations $V \to W$ such that $R(T) \cap R(U) = \{0\}$.
   a. Prove that $\{T, U\}$ is an independent set in $L(V, W)$.
   b. Prove that $\{T^*, U^*\}$ is an independent set in $L(W^*, V^*)$.

6. (10 points) Use the vector space of infinite bounded sequences of real numbers to show that an infinite dimensional space can support a linear operator T that is 1-1 but not onto, and a linear operator U that is onto, but not 1-1. Show that this space is in fact infinite dimensional.

7. (12 points) Suppose V is an n-dimensional vector space, T is a linear operator, and $V = R(T) \oplus W$ for some T-invariant subspace W. Prove that $W = N(T)$.

8. (14 points) Suppose V is an n-dimensional inner product space and T is a linear operator. Prove that $R(T^{\mathrm{adj}}) = N(T)^{\perp}$.

9. (12 points) Suppose a finite dimensional vector space V decomposes as a direct sum of two non-trivial T-invariant subspaces W_1 and W_2.
   a. Prove that the characteristic polynomial of T is the product of the characteristic polynomials of T_{W_1} and T_{W_2}.
   b. Is the corresponding result true for the minimal polynomials? Prove it, or give a counterexample.

10. (16 points) If $A = \begin{bmatrix} 2 & -2 \\ -2 & -1 \end{bmatrix}$, prove that A is similar to a diagonal matrix D, and give an orthogonal matrix P so that $D = P^{\mathrm{adj}} A P$.