MATH 700 Fall, 2005 Exam #2 Name:____________________

For full credit you must show the essential work. If a result has been done in the homework, I expect you to do it again here. Otherwise you may quote needed results from the text or lecture. Please write only on the front side of the paper and don't forget to give your name. Number the pages, and put your initials at the top right hand corner of each page. There are 100 points.

1. (26 points) Short answers; no proof required.
   a. If a square matrix M can be partitioned into submatrices $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ in which A and D are square, what is the determinant of M in terms of the determinants of the submatrices?
   b. Assume you have a 3×3 matrix. Give the elementary matrix corresponding to the row operation cR_2; do the same for the row operation $cR_1 + R_3$ (this changes row 3 but leaves row 1 unchanged).
   c. Let $\beta = \{e_1, e_2\}$ be the standard ordered basis for $\mathbb{R}^2$ and γ be the ordered basis given by $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Give the matrix that gives γ in terms of β; and give the matrix that gives β in terms of γ.
   d. Define similar matrices. How is this important for defining the determinant of a linear operator on a finite dimensional space?
   e. A system $AX = 0$ of m linear equations in n unknowns $x_1, \ldots, x_n$ always has the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. What condition on A will guarantee that there exist non-trivial solutions?

2. (14 points) Let T be a linear operator on a finite dimensional vector space V whose characteristic polynomial splits (that is, factors into a product of linear factors, possibly repeated). Let W be a nontrivial T-invariant subspace. Prove that W contains at least one eigenvector for T.

3. (16 points) Let A be an upper triangular matrix $n \times n$ matrix, and suppose A has distinct eigenvalues $\lambda_1, \ldots, \lambda_t$ with corresponding algebraic multiplicities $k_1, \ldots, k_t$.
   a. Compute the characteristic polynomial χ_A from the definition, and show that it splits.
   b. How are the diagonal entries of A, namely the a_{ii}'s, related to the eigenvalues of A?
   c. Prove that the trace of A, defined as $\sum_{i=1}^{n} a_{ii}$, is equal to $\sum_{i=1}^{t} k_i\lambda_i$, and and that $\sum_{i=1}^{t} k_i = n$. How is $\mathrm{tr}(A)$ related to one of the coefficients of χ_A?

4. (16 points) A linear operator T on $V = \mathbb{R}^4$ has characteristic polynomial $x^4 - 1$. Determine whether T is diagonalizable or not. Determine whether T is invertible or not, and if so, give T^{-1} in terms of non-negative powers of T. How would your answers to these questions change if $V = \mathbb{C}^4$ (as a vector space over $\mathbb{C}$) instead?

5. (14 points) Suppose V is an n-dimensional vector space and the non-zero linear operator T satisfies $T^{\ell} = 0$ for some $\ell > 0$. What conclusions can you draw about the minimal polynomial $p_T(x)$, the characteristic polynomial $\chi_T(x)$, and the operator T^n? Explain.

If W is a subspace of V, recall that $W^{\circ} = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}$. In this problem T^* represents the dual map on V^* if T is a linear operator on V, not the adjoint.

6. (14 points) Let V be a finite dimensional vector space and T be a linear operator on V. Suppose W is a T-invariant subspace of V. Prove that W° is T^*-invariant.