MATH 700 **Exam #1** **Fall, 2005**

For full credit you must show the essential work. If a result has been done in the homework, I expect you to do it again here. Otherwise you may quote needed results from the text or lecture. Please write only on the front side of the paper and don't forget to give your name. Number the pages, and put your initials at the top right hand corner of each page.

1. (28 points) Indicate TRUE or FALSE; if true give a brief explanation; if false, give a counterexample or brief explanation.
   a. If V is a vector space of dimension $n \geq 1$, then V contains a subspace W_j of dimension j for each $j = 0, 1, \ldots, n$.
   b. If W_1 and W_2 are subspaces of a vector space V, and neither one contains the other, then $W_1 \cup W_2$ is a subspace that contains both.
   c. If S is a linearly dependent set in a vector space V, then each vector in S is a linear combination of the others.
   d. The vectors $(-1, 3, 1)$, $(2, -4, -3)$, and $(3, -7, -4)$ span $\mathbb{R}^3$.

2. (10 points) Define the following terms.
   a. a coset of a subspace W in a vector space V.
   b. a T-invariant subspace W in a vector space V, where T is a linear operator on V.

3. (10 points) Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for a vector space V. Show that each element $v \in V$ has a unique representation as a linear combination of the elements of β.

4. (6 points) State the Replacement Theorem (Exchange Lemma).

5. (14 points) Define a linear operator T on the vector space $V = \mathbb{R}^3$ using the standard basis $\beta = \{e_1, e_2, e_3\}$ by specifying the values $T(e_1) = e_2$, $T(e_2) = e_3$, and $T(e_3) = e_1$. Give the formula in general for $T(x, y, z)$. Let $W = \{v \in V \mid T(v) = v\}$. Demonstrate that W is a subspace of V, and compute its dimension.

6. (17 points) Let $D\colon P_4(\mathbb{R}) \to P_4(\mathbb{R})$ be defined by $D(f) = f'$.
   a. Compute $N(D)$.
   b. Write the matrix representation A of D with respect to the standard basis of $P_4(\mathbb{R})$.
   c. Determine the least integer k for which $A^k = 0$ (hint: try to avoid actual matrix multiplication).
   d. Briefly indicate how your answer to (a) would be different if we used the field $\mathbf{F}_2$ in which $2 = 0$ instead of $\mathbb{R}$. For 5 bonus points give a basis for $R(D)$ in this case.

7. (15 points) Let V and W be n–dimensional spaces, $T\colon V \to W$ be a linear transformation, and β be a set of n distinct elements in V, not necessarily a basis. Suppose that $T(\beta)$ is a basis for W. Prove that T is an isomorphism, and describe the inverse isomorphism $S\colon W \to V$.

8. (Bonus: 6 points) State Zorn's Lemma. If you cannot, then for partial credit state the text's less comprehensive "Maximal Principal".