Mathematics 700 Third Midterm Examination
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Problem 1.
Let $\mathbf{V}$ and $\mathbf{W}$ be a finite dimensional vector spaces over the field $\mathbf{F}$. Let $\mathbf{V}^* = \mathcal{L}(\mathbf{V}, \mathbf{F})$ and let $\mathbf{W}^* = \mathcal{L}(\mathbf{W}, \mathbf{F})$. Let $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$. Define $T^* : \mathbf{W}^* \to \mathbf{V}^*$ by

$$T^*(f) = f \circ T$$

for every $f \in \mathbf{W}^*$. Prove that $T^* \in \mathcal{L}(\mathbf{W}^*, \mathbf{V}^*)$.

Problem 2.
Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$. Let $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ be subspaces of $\mathbf{V}$ such that $\mathbf{X} \subseteq \mathbf{Y}$, $\mathbf{Y} + \mathbf{Z} = \mathbf{V}$, and $\mathbf{Y} \cap \mathbf{Z} = \{0\}$. Prove that $\mathbf{X} = \mathbf{Y} \cap (\mathbf{X} + \mathbf{Z})$.

Problem 3.
Let $\mathbf{V}$ be a vector space over $\mathbf{F}$ and let $S, T \in \mathcal{L}(\mathbf{V})$. Prove that if

(1) $S^2 = S$,
(2) $T^2 = T$,
(3) $ST = \mathsf{O}$, and
(4) $S + T = I$,

then $V = \operatorname{range} S \oplus \operatorname{range} T$.

Problem 4.
Let $\mathbf{V}$ be a finite dimensional vector space, and let $S \in \mathcal{L}(\mathbf{V})$. Prove each of the following:

(1) If $T \in \mathcal{L}(\mathbf{V})$ and $ST = \mathsf{O}$, then $\dim \operatorname{range} S + \dim \operatorname{range} T \leq \dim \mathbf{V}$.
(2) There is $T \in \mathcal{L}(\mathbf{V})$ such that $ST = \mathsf{O}$ and $\dim \operatorname{range} S + \dim \operatorname{range} T = \dim \mathbf{V}$.

Problem 5.
Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbf{F}$, let $T \in \mathcal{L}(\mathbf{V})$, and let λ an eigenvalue of T. Prove that there is $f \in \mathcal{L}(\mathbf{V}, \mathbf{F})$ so that $fT = \lambda f$.

Problem 6.
Let $T \in \mathcal{L}(\mathbb{R}^4)$ be defined by

$$T(x_0, x_1, x_2, x_3) = (x_0 - x_3, x_0, -2x_1 - x_2 - 4x_3, 4x_2 + x_3).$$

(1) Display the matrix of T with respect to the standard basis.
(2) Display the matrix of T with respect to the basis which is the result of reversing the order of the standard basis. This basis is displayed below:

$$(0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0).$$

Problem 7.
Let $\mathbf{V}$ be a vector space and let $S, T \in \mathcal{L}(\mathbf{V})$ such that $ST = TS$. Prove each of the following:

(1) If λ is an eigenvalue of S and $U = \{\mathfrak{u} \in V : S\mathfrak{u} = \lambda \mathfrak{u}\}$, then U is T-invariant.
(2) S and T have a common eigenvector.

Problem 8.
Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbf{F}$, and let $T \in \mathcal{L}(\mathbf{V})$. Prove that T is invertible if and only if the constant term of the minimal polynomial of T is not 0.

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PROBLEM 9.
Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbf{F}$, and let $T \in \mathcal{L}(\mathbf{V})$ be diagonalizable. Prove that if $\mathbf{U}$ is a T-invariant subspace of $\mathbf{V}$, then the restriction of T to U is also diagonalizable.

PROBLEM 10.
Consider $n \times n$ matrices over a field $\mathbf{F}$. Prove each of the following:

(1) If A and B are such matrices and at least one of them is invertible, then AB and BA are similar.
(2) Provide an example of matrices A and B such that AB and BA are *not* similar.

PROBLEM 11.
Suppose that T is a linear operator on the n-dimensional vector space $\mathbf{V}$. A vector ν is said to be T-cyclic if and only if $(\nu, T\nu, T^2\nu, \ldots, T^{n-1}\nu)$ is a basis for $\mathbf{V}$. Suppose that ν is T-cyclic. Prove that if $S \in \mathcal{L}(\mathbf{V})$ such that $ST = TS$, then there is a polynomial $p(x)$ such that $S = p(T)$.

PROBLEM 12.
Provide an example of a matrix A with real entries whose characteristic polynomial is $x^4 - 1$. What is the Jordan form of the matrix A?