Second Midterm Examination
Mathematics 700
29 October 1997

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Problem 1.
Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbf{F}$. Let $\mathcal{F}(V,F)$ be the set of all functions from V into F. We know that $\mathcal{F}(V,F)$ is a vector space over $\mathbf{F}$. Prove each of the following:

(1) $\mathcal{L}(\mathbf{V},\mathbf{F})$ is a subspace of $\mathcal{F}(V,F)$.
(2) $\dim \mathcal{L}(\mathbf{V},\mathbf{F}) = \dim \mathbf{V}$.

Problem 2.
Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$. Can $\mathbf{V}$ have three distinct proper subspaces $\mathbf{X},\mathbf{Y}$, and $\mathbf{Z}$ such that $\mathbf{X} \subseteq \mathbf{Y}$, $\mathbf{X}+\mathbf{Z} = \mathbf{V}$, and $\mathbf{Y} \cap \mathbf{Z} = \{0\}$?

Problem 3.
Let $\mathbf{V}$ be a vector space over $\mathbf{F}$ and let $T \in \mathcal{L}(\mathbf{V})$. Prove that if $T^2 = T$, then $V = \operatorname{null} T \oplus \operatorname{range} T$.

Problem 4.
Let $\mathbf{V}$ be a finite dimensional vector space, and let $T \in \mathcal{L}(\mathbf{V})$. Prove each of the following:

(1) There is a positive integer k such that $\operatorname{null} T^k = \operatorname{null} T^j$ for all $j \geq k$.
(2) There is a positive integer k such that $\operatorname{null} T^k \oplus \operatorname{range} T^k = V$.

Problem 5.
Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$. Let $S,T \in \mathcal{L}(\mathbf{V},\mathbf{F})$. Prove that if $\operatorname{null} S = \operatorname{null} T$, then there is $\lambda \in F$ so that $T = \lambda S$.

Problem 6.
Let $\mathbf{V}$ and $\mathbf{W}$ be finite dimensional vector spaces over the field $\mathbf{F}$. Let $T \in \mathcal{L}(\mathbf{V},\mathbf{W})$ with $\dim \operatorname{range} T = r$. Prove that there are bases B_V of $\mathbf{V}$ and B_W of $\mathbf{W}$ so that the matrix of T with respect to these two bases is

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

where I denotes the $r \times r$ identity matrix and the various 0's in this matrix displayed above represent matrices of the correct sizes and entries all 0.

Problem 7.
Let $\mathbf{V}$ be a vector space and let $S,T \in \mathcal{L}(\mathbf{V})$. Prove that ST and TS have the same eigenvalues.

Problem 8.
Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbf{F}$, and let $T \in \mathcal{L}(\mathbf{V})$. Prove that if T is invertible, then there is a polynomial $q(x) \in \mathbf{F}[x]$ so that $T^{-1} = q(T)$.

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Problem 9.

Let $\mathbf{V}$ be a finite dimensional vector space over the field $\mathbf{F}$, and let $S, T \in \mathcal{L}(\mathbf{V})$. Prove that if T has n distinct eigenvalues, where $n = \dim \mathbf{V}$ and $ST = TS$, then there is a polynomial $p(x) \in \mathbf{F}[x]$ so that $S = p(T)$.