First Midterm Examination
Mathematics 700
26 September 1997
Professor George McNulty

Problem 1.
Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over the field $\mathbf{F}$ and let T be a function from V into W. Recall that such a function is a set of ordered pairs; in fact, $T \subseteq V \times W$. Prove that $T \in \mathcal{L}(\mathbf{V}, \mathbf{W})$ if and only if $T \in \mathrm{Sub}(\mathbf{V} \times \mathbf{W})$. [That is the function T is a linear transformation if and only if it is a subspace of $\mathbf{V} \times \mathbf{W}$.]

Problem 2.
Let $\mathbf{V}$ be a vector space over the field $\mathbf{F}$ and let $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ be subspaces of $\mathbf{V}$. Prove that if $\mathbf{X} \supseteq \mathbf{Z}$, then

$$\mathbf{X} \cap (\mathbf{Y} + \mathbf{Z}) = (\mathbf{X} \cap \mathbf{Y}) + \mathbf{Z}.$$

Problem 3.
Let $\mathbf{V}$ be a vector space over either $\mathbb{R}$ or $\mathbb{C}$. Let $T \in \mathcal{L}(\mathbf{V})$ such that $T^2 = I$ (that is $T \circ T$ is the identity map). Let

$$U = \{\mathfrak{u} \in V : T\mathfrak{u} = \mathfrak{u}\}$$

and

$$V = \{w \in V : Tw = -w\}.$$

a. Prove that both U and W are subspaces of $\mathbf{V}$.
b. Prove that $\mathbf{V} = \mathbf{U} \oplus \mathbf{W}$.

Problem 4.
Let $\mathbf{V}$ be a finite dimensional vector space and let $T \in \mathcal{L}(\mathbf{V})$. Prove that if $\mathrm{range}\, T + \mathrm{null}\, T = \mathbf{V}$, then $\mathrm{range}\, T \oplus \mathrm{null}\, T = \mathbf{V}$ and $\mathrm{range}\, T = \mathrm{range}\, T^2$.

Problem 5.
a. Let $\mathbf{F}$ be a field and let T be a function from F into F. Prove that $T \in \mathcal{L}(\mathbf{F})$ if and only if there is $a \in F$ such that $Tx = ax$ for all $x \in F$.
b. $\mathbb{C}$ can be construed as a vector space of dimension 2 over $\mathbb{R}$, and also as a vector space of dimension 1 over $\mathbb{C}$. Let $\mathcal{L}_{\mathbb{R}}(\mathbb{C})$ denote the set of linear operators on $\mathbb{C}$ construed as a vector space over $\mathbb{R}$, while $\mathcal{L}_{\mathbb{C}}(\mathbb{C})$ denotes the set of linear operators on $\mathbb{C}$ construed as a vector space over $\mathbb{C}$. Prove that $\mathcal{L}_{\mathbb{C}}(\mathbb{C})$ is a proper subset of $\mathcal{L}_{\mathbb{R}}(\mathbb{C})$.