MATH 700 **Final Exam** **Fall, 2008**

Honor Statement. I pledge that I have neither given nor received aid from any other person during this examination, and that the work presented here is entirely my own. **Signature and date:** ______________________________________

Printed name: ____

Instructions. For full credit you must show the essential work. If a problem restates a result that has been done in the homework, text or in class, and asks for a proof, I expect you to do it again here. Otherwise you may quote needed results from the text or lecture by name or by statement of the content of the result. **You may use an earlier part of a problem in subsequent parts, even if you cannot prove the earlier result**. Please write only on the front side of the paper. Place the exam with the signed honor statement first, then number the pages, leave room at the TOP LEFT for a staple, and put your initials at the top right hand corner of each page. There are 150 points. Assume an arbitrary coefficient field $\mathbf{F}$ and the possibility of infinite dimensionality, unless specified otherwise.

1. (10 points) Let V be the vector subspace of strictly upper triangular matrices in $\mathbf{F}^{n\times n}$. Compute the dimension of V by exhibiting (and proving that you have) a basis.

2. (10 points) Suppose V is a vector space and W_1 and W_2 are subspaces. Show that $(W_1 + W_2)^\circ = W_1^\circ \cap W_2^\circ$ in the dual space V^*. Recall that $W^\circ = \{f \in V^* \mid f(\mathbf{w}) = 0 \quad \text{for all } \mathbf{w} \in V\}$.

3. (10 points) Suppose V is an n-dimensional complex vector space and $T \in L(V)$. If 0 is not an eigenvalue for T, prove that T is invertible and show that T^{-1} is given by a polynomial in T.

4. (10 points) Assume that T is a normal operator on a complex inner product space V and that there is a cyclic vector for T. That is, $V = Z(\mathbf{v}; T) = \operatorname{span}(\langle \mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \dots \rangle)$. Show that T has n distinct eigenvalues.

5. (10 points) Suppose V is an n-dimensional vector space and $\langle f_1, \dots, f_n \rangle$ is a basis for the dual space V^*. Determine $\dim(\bigcap_{i=1}^k N(f_i))$ for each $1 \le k \le n$.

6. (10 points) Suppose V is an $\mathbf{F}$-vector space and $\{0\} = U_0 \subset U_1 \subset U_2 \subset U_3 = V$ is a chain of subspaces with $\dim U_i/U_{i-1} = i$ for $i = 1, 2, 3$. Compute $\dim V$ and prove that your answer is correct. Use the notation π_i for the natural maps $\pi\colon U_i \to U_i/U_{i-1}$ (if you make use of these).

7. (15 points) Let V be an n-dimensional vector space and S and T be in $L(V)$.
   a. (5 points) If $ST = 0$, show that $\operatorname{rank} S + \operatorname{rank} T \leq n$.
   b. (10 points) If S and T are nilpotent operators, and $ST = TS$ prove that $S + T$ is also nilpotent. Show by example that if $ST \neq TS$, then $S + T$ can fail to be nilpotent. Why does this not contradict the fact that the V in problem #1 is a vector space, and hence closed under addition?

8. (10 points) Suppose V is an n-dimensional complex vector space and $T \in L(V)$ has characteristic polynomial $\chi_T(x) = x(x-4)^3(x+2)^5$ and minimal polynomial $p_T(x) = x(x-4)^2(x+2)^3$. Compute $\dim V$. Give all possible Jordan forms for T, up to similarity, and explain how you know that the forms are not similar to one another. Instead of writing out the complete matrices, you may, if you choose, describe each of them by simply listing the Jordan blocks that appear in it.

9. (30 points) Suppose V is an n-dimensional complex inner product space and $T \in L(V)$. Define the trace of T, $\operatorname{tr}(T)$ to be $\operatorname{tr}([T])_\beta$, where β is any basis of V (you may assume this is well-defined).
   a. (5 points) Define the adjoint map T^*.
   b. (5 points) Assume that T is self-adjoint. Show that $\operatorname{tr}(T)$ is real.
   c. (10 points) Suppose that $T^2 = T$. Show that $||T\mathbf{v}|| < ||\mathbf{v}||$ if $\mathbf{v} \notin R(T)$.
   d. (10 points) Suppose once more that $T^2 = T$.Show that $\operatorname{rank} T = \operatorname{tr} T$. Explain why this result fails if $\mathbb{C}$ is replaced by a field of characteristic 2 (i.e., in which $2 = 0$).

10. (35 points) Suppose V is an n-dimensional complex vector space and T is a diagonalizable linear operator with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
   a. (5 points) If U is a T-invariant subspace, show that $S = T\big|_U$ is also diagonalizable.
   b. (12 points) Suppose V is an n-dimensional complex vector space and $T \in L(V)$. Let $U_i = E_{\lambda_i}$ be the eigenspace corresponding to λ_i and let $W_i = \oplus_{j \neq i} U_j$. Explain why $U_i \oplus W_i = V$. Let $P_i = P_{U_i, W_i}$ be the projection onto U_i along W_i. Show that P_i can be expressed as a polynomial in T. Suggestion: consider the polynomial

$$q_i(x) = \prod_{j=1, j \neq i}^{k} \frac{1}{\lambda_i - \lambda_j}(x - \lambda_j)$$

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   c. (8 points) Show that $T = \lambda_1 P_1 + \cdots + \lambda_k P_k$.
   d. (10 points) Suppose additionally that $k = n$. Explain why the assumption of diagonalizability is now redundant. If $S \in L(V)$ satisfies $ST = TS$, prove that S is a polynomial in T. Suggestion: how does S operate on eigenvectors of T?