MATH 700 **Exam #2** **Fall, 2008**

Honor Statement. I pledge that I have not used any notes, text, or any other reference materials during this examination. I pledge that I have neither given nor received aid from any other person during this examination, and that the work presented here is entirely my own.

Signature and date: ____

Printed name: ____

Instructions. For full credit you must show the essential work. If a problem restates a result that has been done in the homework, text or in class, I expect you to do it again here. Otherwise you may quote needed results from the text or lecture by name or by statement of the content of the result. I expect you to use more basic results to prove more advanced ones, and not the other way around. You may use an earlier part of a problem in subsequent parts, even if you cannot prove the earlier result. Please write only on the front side of the paper. Place the exam with the signed honor statement first, then number the pages, leave room at the TOP LEFT for a staple, and put your initials at the top right hand corner of each page. There are 100 points.

1. (10 points) Suppose that $g\colon F^{n\times n} \to F$ is a function with the property that $f(AB) = f(BA)$. Prove that if M and N are similar matrices in $F^{n\times n}$ then $f(M) = f(N)$. Now suppose V is an n-dimensional F-vector space, and $T \in L(V)$. Use the preceding result to show how to define $g(T)$ unambiguously.

2. (14 points) Let V and W be finite dimensional inner product spaces and T be in $L(V, W)$.
   a. Prove that $N(T) = (R(T^*))^\perp$.
   b. Suppose $W = V$ and U is a T-invariant subspace of V. Prove that $U^\perp$ is T^*-invariant.

3. (7 points) Suppose V is a finite dimensional F-vector space, where $2 \neq 0$ in F, and $V = U \oplus W$. Let $P = P_{U,W}$ be the projection onto U along W. Show that $I + P$ is invertible.

4. (14 points) Assume that T is a normal operator on an inner product space V.
   a. Show that $\langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle$ for all x and y in V. Deduce that $||T^*x|| = ||Tx||$ for all x in V.
   b. Suppose in addition we know that $TT^* = T^*T = I$, that is, $T^* = T^{-1}$. Show that $||Tx|| = ||x||$ for all x in V.

5. (25 points) Let T be a linear operator on a finite dimensional F-vector space V. The following items are independent of one another.
   a. Suppose there is a polynomial $p(x)$ with coefficients in F and non-zero constant term such that $p(T) = 0$ on V. Show that T is invertible, and compute T^{-1} in terms of non-negative powers of T.
   b. Suppose that T is diagonalizable and 0 is not an eigenvalue. Prove that T^{-1} exists and is also diagonalizable. Do not use a determinant argument.
   c. Suppose that $\dim R(T) = k$. Prove that T has at most $k+1$ distinct eigenvalues.

6. (20 points) Suppose that T is a normal operator on a complex finite dimensional inner product space. Prove that there is an operator S such that $S^2 = T$. Formulate and prove an analogous result (same conclusion, modified hypotheses) for an operator on a real finite dimensional inner product space.

7. (10 points) State the Structure Theorem for nilpotent operators. Define terms or notation that are not entirely basic.