**MATH 700** **Exam #1** **Fall, 2008**

**Honor Statement.** I pledge that I have not used any notes, text, or any other reference materials during this examination. I pledge that I have neither given nor received any aid from any other person during this examination, and that the work presented here is entirely my own. **Signature, date, and printed name:**

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**Instructions.** For full credit you must show the essential work. If a problem restates a result that has been done in the homework, text or in class, I expect you to do it again here. Otherwise you may quote needed results from the text or lecture by name or by statement of the content of the result. I expect you to use more basic results to prove more advanced ones, and not the other way around. You may use an earlier part of a problem in subsequent parts, even if you cannot prove the earlier result. Please write only on the front side of the paper. Place the exam with the signed honor statement first, then number the pages, leave room at the TOP LEFT for a staple, and put your initials at the top right hand corner of each page. There are 100 points.

1. (16 points) Let $V$ be an $n$-dimensional vector space.
   a. Suppose $W_1$ and $W_2$ are subspaces with $W_1+W_2=V$. Prove that each of the $W_i$ is finite dimensional, and that $n=\dim W_1+\dim W_2-\dim W_1\cap W_2$.
   b. Suppose $T\in L(V,V)$ and we know that $R(T)+N(T)=V$. Prove that this sum is actually direct, *i.e.*, $R(T)\oplus N(T)=V$.

2. (14 points) Let $F$ be a field. The trace function $\mathrm{tr}\colon F^{2\times 2}\to F$ defined by $\mathrm{tr}(\begin{bmatrix} a & b \\ c & d \end{bmatrix})=a+d$ is a linear functional (no need to prove this). Determine $\dim N(\mathrm{tr})$, and give an explicit basis for $N(\mathrm{tr})$ (include an argument that proves your answer *is* a basis).

3. (14 points) Let $S\colon P_3(\mathbb{R})\to X$ be given by $S(p)=\int p(x)\,dx$ (leave off the "$+C$"). What is the most reasonable choice for the target space $X$? You may assume $S$ is linear, and that $P_3(\mathbb{R})$ has the standard basis $\beta$. Give a basis for $X$ and call it $\gamma$ (no need to prove this is a basis). Determine $A=[S]_\beta^\gamma$.

4. (16 points) Use the vector space of infinite bounded sequences of real numbers to show that an infinite dimensional space can support a linear operator $T$ that is 1-1 but not onto, and a linear operator $U$ that is onto, but not 1-1 (demonstrate these properties, but as long as the maps you give are linear you don't have to prove this). Show that this space is in fact infinite dimensional.

5. (20 points) Suppose $V$ is an $n$-dimensional vector space, and $S = \langle v_1, v_2, \ldots, v_k \rangle$ is an independent list in $V$. Show that $(\mathrm{span}(S))^\circ = S^\circ$, compute $\dim S^\circ$, and explain how to get a basis for $S^\circ$ by making use of the elements of $S$. Recall that for a subset $X$ of $V$, we define $X^\circ = \{ f \in V^* \mid f(x) = 0 \quad \text{for all } x \in X \}$.

6. (20 points) Suppose $V$ is an $n$-dimensional vector space and $S$ and $T$ are in $L(V, V)$. Suppose that $S \circ T = \mathrm{id}_V$. Prove that $T \circ S = \mathrm{id}_V$. There are many ways to do this; one approach is to show that when restricted to $R(T)$ we have $T \circ S = \mathrm{id}$, and then show that $R(T) = V$.

7. (10 points bonus) Let $F$ be a field that contains $1/2$. Assume that $V$ is a vector space over $F$ and that $T \in L(V, V)$ satisfies $T^2 = T \circ T = \mathrm{id}_V$. Let $U = \{ v \in V \mid Tv = v \}$ and let $W = \{ w \in V \mid Tw = -w \}$. You may assume that these are subspaces of $V$. Show that $V = U \oplus W$.