Math 550 Worksheet 1
Spring, 2008

These problems are meant to touch a lot of bases, reminding you of the material from 250/241, and informing me of items that need some propping up. We will work on some of them, and pieces of others in class, but you should get started on them before Tuesday, 22 January. You are free, in fact encouraged, to collaborate.

1. Consider the set of equations given below.
   a. Interpret this algebraic problem in terms of the geometry of certain points, lines and planes in $\mathbb{R}^3$. What does each equation represent, and what does solving them together represent? Without solving the system, what kind of solution do you expect, and why?
   b. Now determine all the solutions, or demonstrate that there are none. Sketch a diagram that illustrates the situation.

$$\begin{aligned} x + 2y + z &= -8 \\ y + z &= -6 \\ x + 4y + 3z &= -20 \end{aligned}$$

2. We are going to think about vector fields and gradient vector fields.
   a. Given a vector field $\mathbf{F} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} = (F_1, F_2)$, what test will determine that $\mathbf{F}$ is NOT a gradient vector field?
   b. Show that if $F_1 = 3x^2y^2 - 2xy\cos(x^2y)$ and $F_2 = 2x^3y - x^2\cos(x^2y)$ then $\mathbf{F} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}}$ could be a gradient vector field.
   c. Now show that $\mathbf{F} = \operatorname{grad} f$ (or ∇f) for a suitable twice continuously differentiable function f.

3. Given vectors $\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and matrix $A = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix}$, show how A transforms the unit square $SQ = \{(x, y) \mid 0 \le x, y \le 1\}$. Also show how A transforms $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 10 \\ -12 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Compute $A\mathbf{u}$ and $A\mathbf{v}$. Do you notice anything unusual about these results? Hint: if you consider all vectors to be rooted at the origin, all lie on straight lines through $(0, 0)$. What do you notice about these lines?

5. Review how to find the volume of a parallelepiped in $\mathbb{R}^3$ that is "spanned" by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Can you imagine how such a solid might be described in terms of a matrix transformation of the unit cube in $\mathbb{R}^3$?