7. (b) First, complete the square: $(2x^2 - 8x + 8) + y^2 + z^2 = 1 + 8$ or $2(x-2)^2 + y^2 + z^2 = 3^2$. This has the form $X^2 + y^2 + z^2 = \rho^2$, so use spherical coordinates: For the x parametrization, let $X = \sqrt{2}(x-2) = 3\cos\theta\sin\phi$, so $x = 2 + (3\cos\theta\sin\phi)/\sqrt{2}$. Thus, the parametrization is

$$\begin{aligned} x &= 2 + (3\cos\theta\sin\phi)/\sqrt{2} \\ y &= 3\sin\theta\sin\phi \\ z &= 3\cos\phi, \end{aligned}$$

with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. This is a general strategy in attacking many parametrization problems: changing cartesian coordinates into either cylindrical or spherical coordinates by completing the squares.

10. The surface area of the graph lying over D is $\iint_D \|\mathbf{T}_x \times \mathbf{T}_y\|\, dx\, dy$ and the area of D is $\iint_D dx\, dy$. The "parametrization" of the graph is $x = x$, $y = y$, $z = f(x,y)$. Thus,

$$\mathbf{T}_x = \mathbf{i} + (\partial f/\partial x)\mathbf{k} \quad \text{and} \quad \mathbf{T}_y = \mathbf{j} + (\partial f/\partial y)\mathbf{k}.$$

Therefore,

$$\mathbf{T}_x \times \mathbf{T}_y = (\partial f/\partial x)\mathbf{i} + (\partial f/\partial y)\mathbf{j} + \mathbf{k} \quad \text{and} \quad \|\mathbf{T}_x \times \mathbf{T}_y\| = [(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + 1]^{1/2}.$$

Since $(\partial f/\partial x)^2 + (\partial f/\partial y)^2 = c$, $\|\mathbf{T}_x \times \mathbf{T}_y\| = \sqrt{1+c}$. Returning to the original formula, we have $\iint_D \|\mathbf{T}_x \times \mathbf{T}_y\|\, dx\, dy = \iint_D \sqrt{1+c}\, dx\, dy$. Since c is a constant, we factor the constant from the integral to get $\sqrt{1+c}\iint_D dx\, dy = \sqrt{1+c}\cdot(\text{area of } D)$.

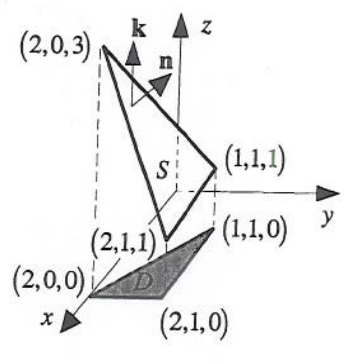
12. (b) Use cylindrical coordinates. Let $x = r\cos\theta$, $y = r\sin\theta$. In addition, $z = x = r\cos\theta$, and the intervals are $0 \le r \le 1$, $0 \le \theta \le 2\pi$ since we want to be inside the cylinder $x^2 + y^2 = 1$. We calculate

$$\begin{aligned} \mathbf{T}_r \times \mathbf{T}_\theta &= (\cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \cos\theta\mathbf{k}) \times (-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} - r\sin\theta\mathbf{k}) \\ &= (-r\mathbf{i} + r\mathbf{k}) = r(-\mathbf{i} + \mathbf{k}) \quad \text{so} \quad \|\mathbf{T}_r \times \mathbf{T}_\theta\| = \sqrt{2}r. \end{aligned}$$

Therefore,

$$\begin{aligned} \iint_S x^2\, dS &= \int_0^{2\pi}\int_0^1 (r\cos\theta)^2\sqrt{2}r\, dr\, d\theta = \left(\sqrt{2}\int_0^1 r^3\, dr\right)\left(\int_0^{2\pi} \cos^2\theta\, d\theta\right) \\ &= \left(\frac{\sqrt{2}}{4}\right)\left(\int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta\right) = \frac{\sqrt{2}}{4}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_0^{2\pi} = \frac{\sqrt{2}}{4}\pi. \end{aligned}$$

15. We want to compute $\iint_S x\, dS$, where S is the triangle with vertices $(1,1,1)$, $(2,1,1)$ and $(2,0,3)$. First, we need to find the normal to the triangle: Two vectors on the triangle are $(1,0,0)$ and $(0,1,-2)$ (found by subtracting the coordinates of the vertices). Take their cross product and normalize it. We get the unit normal $\mathbf{n} = (0,2,1)/\sqrt{5}$, so $\cos\theta = \mathbf{n}\cdot\mathbf{k} = 1/\sqrt{5}$. Next, the projection of S onto the xy-plane can be described by $-y + 2 \le x \le 2$, $0 \le y \le 1$, as shown. Thus,



$$\begin{aligned} \iint_S x\, dS &= \sqrt{5}\iint_D x\, dx\, dy = \sqrt{5}\int_0^1\int_{-y+2}^2 x\, dx\, dy \\ &= \frac{\sqrt{5}}{2}\int_0^1 4 - (2-y)^2\, dy \\ &= \frac{\sqrt{5}}{2}\left[4 + \frac{(2-y)^3}{3}\Big|_0^1\right] = \frac{5\sqrt{5}}{6}. \end{aligned}$$

#12c, page 515 $\quad$ or $x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$

The surface $x^2+y^2=2x$ can be redescribed in polar coordinates as $r^2=2r\cos\theta$, or $r=2\cos\theta$ for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$.

So we have $(x,y,z)=(r\cos\theta, r\sin\theta, z)$
$=(2\cos^2\theta, 2\cos\theta\sin\theta, z)$ and $0\le z\le$
$\sqrt{x^2+y^2}=r=2\cos\theta$. Then

$\vec{T}_\theta=(4\cos\theta(-\sin\theta), 2\cos^2\theta-2\sin^2\theta, 0)$

$\vec{T}_z=(0,0,1)$

$\vec{T}_\theta\times\vec{T}_z=(2\cos^2\theta-2\sin^2\theta, 4\cos\theta\sin\theta, 0)$

Notice that $\sin^2\theta=1-\cos^2\theta$, so

$\vec{T}_\theta\times\vec{T}_z=(4\cos^2\theta-2, 4\cos\theta\sin\theta, 0)$

$=(2)(2\cos^2\theta-1, 2\cos\theta\sin\theta, 0)$

$=2(x-1, y, 0)$

Hence $dS=\|\vec{T}_\theta\times\vec{T}_z\|\,d\theta\,dz=2\sqrt{(x-1)^2+y^2}\,d\theta\,dz$

$=2\,d\theta\,dz$

Notice by the way that S can also be described by $g(x,y,z)=x^2-2x+y^2=0$ and $\vec{\nabla}g=(2x-2, 2y, 0)$

$=2(x-1, y, 0)$

so $\hat{n}=\frac{1}{2\sqrt{(x-1)^2+y^2}}\vec{\nabla}g=(x-1, y, 0)$

Then $\iint\limits_S f\,dS = \iint\limits_S x\,dS$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (2\cos^2\theta)(2\,d\theta\,dz) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta\,d\theta$$

$$= \frac{4}{3}$$

#9, p. 515 Notice $x = r\cos\theta$

$\frac{y}{2} = r\sin\theta$

$z = r$

gives $x^2 + \left(\frac{y}{2}\right)^2 = r^2 = z^2$.

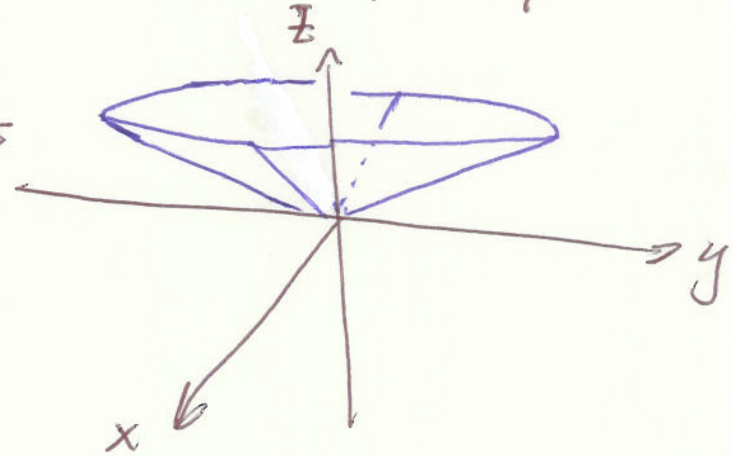
This describes an elliptic cone:

for $x = 0$ we have $y = \pm 2z$; } pairs of lines

for $y = 0$ we have $x = \pm z$;

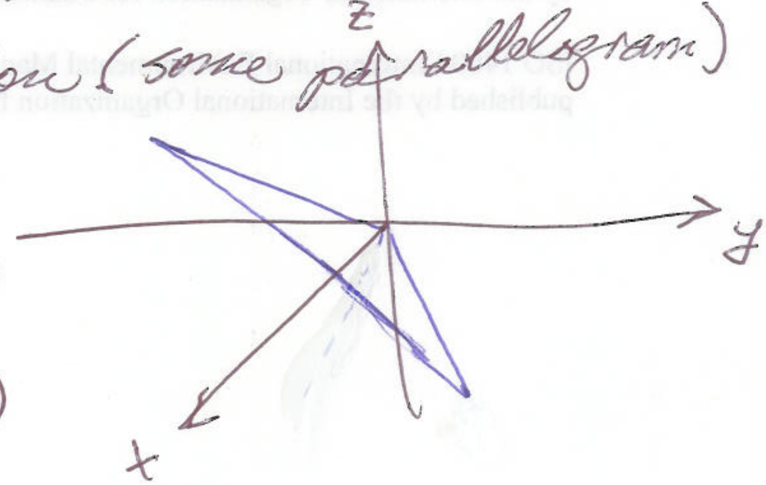
and for $z = c$ we have $\frac{x^2}{1} + \frac{y^2}{4} = c^2$ (ellipse).

The surface rises to $z = r = 1$.



#8, p-515 $\quad x = u + v \qquad$
$y = u \qquad 0 \le u \le 1$
$z = v \qquad 0 \le v \le 1$

$x = y + z$ eliminates u, v.
$x - y - z = 0$ is the equation of a
plane and the restrictions on u, v
cut out a portion (some parallelogram)



Compute $\vec{T}_u = (1, 1, 0)$
$\vec{T}_v = (1, 0, 1)$

$\vec{T}_u \times \vec{T}_v = (1, -1, -1)$

$dS = \| \vec{T}_u \times \vec{T}_v \| du dv = \sqrt{3}\, du dv$

<u>or</u> use $\vec{N} = (1, -1, -1)$ is the normal to
the plane, which has the form of $z = f(x, y)$,
and we use area cosine principle with
$\hat{n} = \frac{-1}{\sqrt{3}}(1, -1, -1) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ upward
pointing normal. In this case $dA = dy\,dx$
and we need to find the correct
limits for x and y.

#16, p. 515

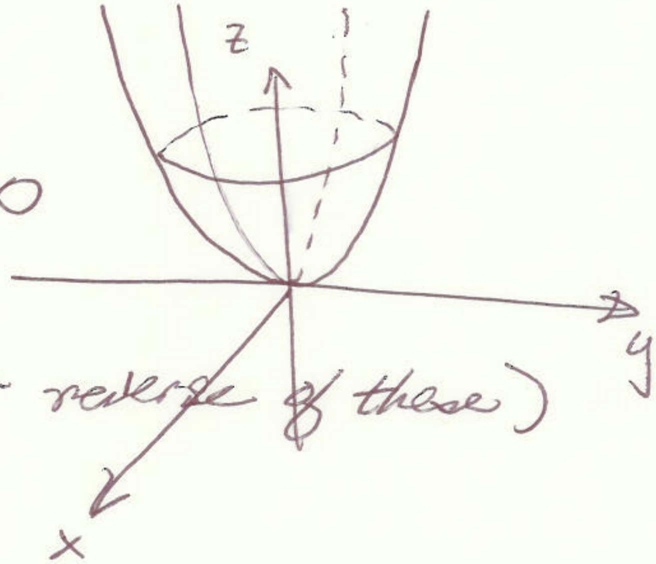
$x = u\cos v$
$y = u \sin v$
$z = u^2$

$2 \geq u \geq 0$
$0 \leq v \leq 2\pi$

$x^2 + y^2 = u^2 = z$

or $g(x,y,z) = x^2 + y^2 - z = 0$

Get a normal from $\vec{\nabla} g$ or $\vec{T}_u \times \vec{T}_v$ (or reverse of these)



Exercise 16. *A paraboloid of revolution S is parameterized by the mapping $\Phi(u,v) = (u\cos v, u\sin v, u^2), 0 \leq u \leq 2, 0 \leq v \leq 2\pi$.*

(a) *Find an equation in x, y, and z describing the surface.*

(b) *What are the geometric meanings of the parameters u and v?*

(c) *Find a unit vector orthogonal to the surface at $\Phi(u,v)$.*

(d) *Find the equation for the tangent plane at $\Phi(u_0, v_0) = (1,1,2)$ and express your answer in the following two ways:*

  i. *parameterized by u and v; and*

  ii. *in terms of x, y, and z.*

(e) *Find the area of S.*

Solution. The paraboloid S is parameterized by $\Phi(u,v) = (u\cos v, u\sin v, u^2), 0 \leq u \leq 2, 0 \leq v \leq 2\pi$.

(a) Since $u^2 = u^2\cos^2 v + u^2 \sin^2 v$, we have $z = x^2 + y^2$.

(b) u and v correspond to r and θ of the cylindrical coordinates.

(c)

$$\begin{aligned}\Phi_u \times \Phi_v &= (\cos v, \sin v, 2u) \times (-u\sin v, u\cos v, 0)\\ &= (-2u^2\cos v, -2u^2 \sin v, u)\end{aligned}$$

is normal to the surface at $\Phi(u,v)$, so $\Phi_u \times \Phi_v / \sqrt{4u^4 + u^2}$ is a unit normal.

(d) Notice first that $\Phi(\sqrt{2}, \frac{\pi}{4}) = (1, 1, 2)$.

(i) Simple computation shows that

$$\Phi_u(\sqrt{2}, \frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\sqrt{2}), \quad \Phi_v(\sqrt{2}, \frac{\pi}{4}) = (-1, 1, 0).$$

Since both Φ_u and Φ_v are tangent to the surface at $(1, 1, 2)$, the tangent plane at $(1, 1, 2)$ is parameterized by

$$(1, 1, 2) + \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 2\sqrt{2}\right) u + (-1, 1, 0)v.$$

(ii) $(\Phi_u \times \Phi_v)(\sqrt{2}, \frac{\pi}{4}) = (-2\sqrt{2}, -2\sqrt{2}, \sqrt{2})$, so $(2, 2, -1)$ is normal to the surface at $(1, 1, 2)$ and hence $2(x-1)+2(y-1)-(z-2) = 0$ is the equation of the tangent plane.

(e) Since $\|\Phi_u \times \Phi_v\| = \sqrt{4u^4 + u^2}$, we have

$$\iint_S dS = \int_0^2 \int_0^{2\pi} \sqrt{4u^4 + u^2}\, dv\, du = 2\pi \int_0^2 u\sqrt{4u^2 + 1}\, du$$
$$= \frac{\pi}{4} \int_1^{17} \sqrt{w}\, dw = \frac{\pi}{6}(\sqrt{17^3} - 1). \quad \diamond$$

Exercise 26. *Calculate*

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = (x, y, -y)$ *and* S *is the cylindrical surface defined by* $x^2 + y^2 = 1, 0 \le z \le 1$, *with normal pointing out of the cylinder.*

Solution. Since S is the cylindrical surface defined by $x^2 + y^2 = 1, 0 \le z \le 1$ with outward normal, we know that $\mathbf{n} = (x, y, 0)$. Hence

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (x, y, -y) \cdot (x, y, 0)\, dS$$
$$= \iint_S (x^2 + y^2) dS = \iint_S dS = 2\pi. \quad \diamond$$