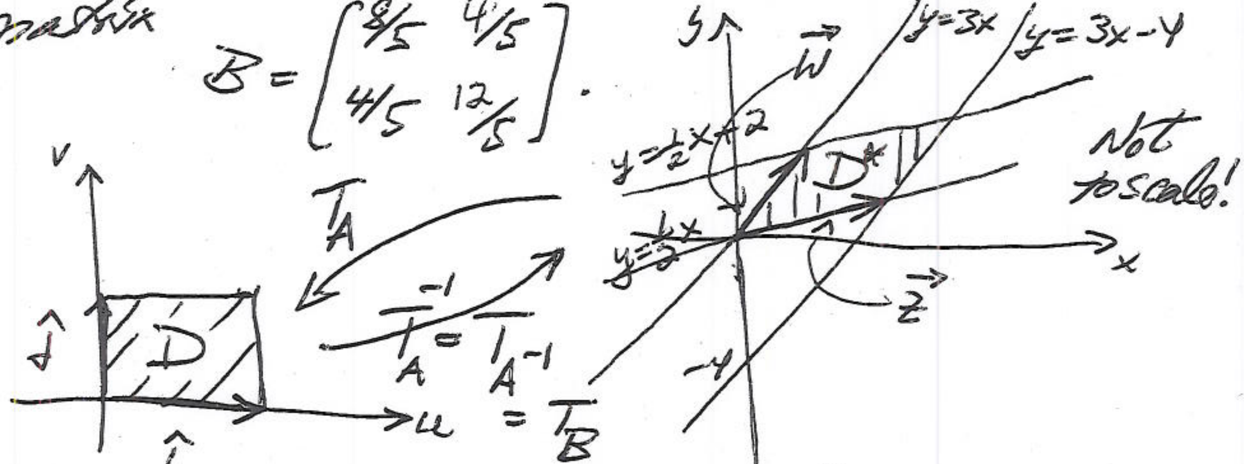
6.1 #4 It is actually easier to find $B = A^{-1}$ and $T_B = T_A^{-1}$ by using the observation that $B\vec{e}_i = i^{th}$ column of B. Now D^* is spanned by the vectors $\vec{z} = \begin{bmatrix} 8/5 \\ 4/5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4/5 \\ 12/5 \end{bmatrix}$. Take $T_B :$ $[0,1]\times[0,1] \longrightarrow D^*$ to be given by the matrix

$$B = \begin{bmatrix} 8/5 & 4/5 \\ 4/5 & 12/5 \end{bmatrix}.$$



It is a tedious calculation* then to compute $A = B^{-1} = \begin{bmatrix} 3/4 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$. You can then check that $T_A(\vec{z}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{i}$ and $T_A(\vec{w}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \hat{j}$ by using left mult. by A.

Notice $T_B(u,v) = \begin{bmatrix} 8/5 & 4/5 \\ 4/5 & 12/5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 8/5\,u + 4/5\,v \\ 4/5\,u + 12/5\,v \end{bmatrix}$

$= (x,y)$, so what have essentially done is to solve $\begin{cases} x = 8/5\,u + 4/5\,v \\ y = 4/5\,u + 12/5\,v \end{cases}$ for (u,v) in terms of (x,y). It is worth comparing $\dfrac{\partial(x,y)}{\partial(u,v)}$ with $\dfrac{\partial(u,v)}{\partial(x,y)}$.

* but straightforward & available on demand

Of course it is a bit of a leap of faith to expect T to be linear, but in retrospect it worked.

6.1 #7 We discussed this one in class.

6.1 #8 You can use some more sophisticated linear algebra, but bare Gauss also works. Assume $T = T_A$ where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

I. <u>Claim</u> If $\det A \neq 0$ then T_A is one-to-one.

Suppose $A\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} x' \\ y' \end{bmatrix}$, a seeming failure of 1-1. We aim to show that actually $x = x'$ and $y = y'$. Put $z = x - x'$ and $w = y - y'$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix} = A\begin{bmatrix} z \\ w \end{bmatrix} = A\begin{bmatrix} x - x' \\ y - y' \end{bmatrix} = A\begin{bmatrix} x \\ y \end{bmatrix} - A\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\begin{cases} az + bw = 0 \\ cz + dw = 0 \end{cases}$. Now $\det A \neq 0$

means $ad - bc \neq 0$; thus not both $a = 0$ and $c = 0$. By exchanging the equations the argument using $c \neq 0$ runs just like $a \neq 0$, so without loss of generality we may assume $a \neq 0$. Then we can divide by a and solve $z = -\frac{b}{a}w$. The second equation becomes

$$c\left(-\frac{b}{a}w\right) + dw = 0 \text{ or } \left(-\frac{cb}{a} + d\right)w = 0.$$

If we get a common denominator then we have $\left(\frac{-cb+da}{a}\right) w = 0$ and since the quantity in parentheses is not zero (why not?), we conclude $w = 0$. It follows that $z = 0$, and then $x = x'$, $y = y'$, showing that T_A is 1-1.

II. <u>Claim</u> If T_A is 1-1, then $\det A \neq 0$.

Suppose, to the contrary, that $ad - bc = \det A = 0$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} ad-bc \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But T_A is 1-1, and now we have

$$A\begin{bmatrix} d \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ So it must be that $d = c = 0$. Similarly

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ cb - da \end{bmatrix} = \begin{bmatrix} 0 \\ -\det A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

forces us to conclude that $b = a = 0$. But $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all x, y is as far from being 1-1 as a linear transformation can be! So our supposition was false, and in fact $\det A \neq 0$.