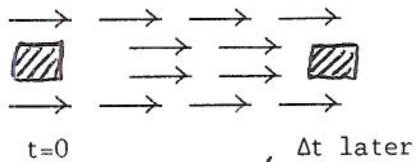
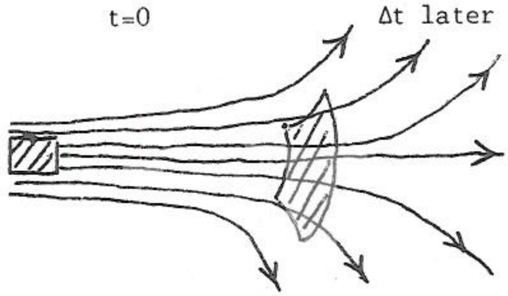
Problem #12, page 119. One way to interpret div **F** as we've seen is to fix a box about the point (x,y,z) and compute the limiting value of the flux through the surface per unit volume as the box shrinks down to a point. An alternative interpretation, easier to understand, but messier to derive, is offered in this problem. At each point div **F** measures the rate at which the flow expands or contracts volume per unit volume as the volume shrinks down to the point. Consider these 2-dimensional illustrations:



constant **F**

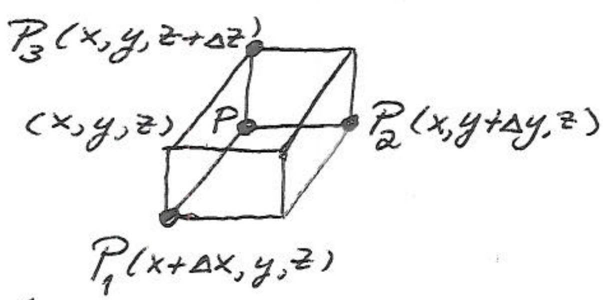
area (volume) unchanged



F "diverges"
area (volume) increasing
(follow the fluid in the box as it is carried by the flow)

V_0 = volume of original box = $\Delta x \Delta y \Delta z$
V = volume of transformed box

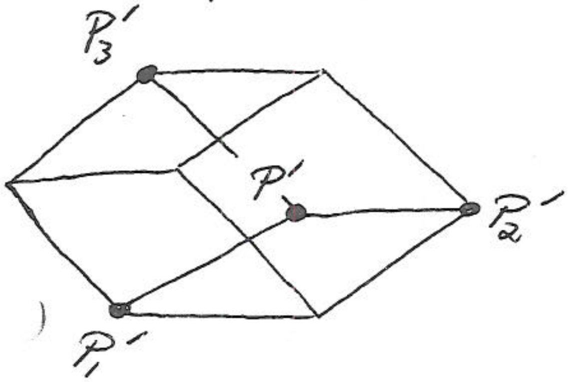
The flow will not in general preserve the box shape. But if all the dimensions are small and Δt is very small the box will not be greatly distorted and we can approximate it by forming the parallelopiped on the vectors joining one corner to its three nearest neighbors.



$\mathbf{R} = OP = x\hat{i} + y\hat{j} + z\hat{k}$

$\mathbf{R}_i = OP_i$ for $i = 1,2,3$

initial volume $= V_0$
$= \Delta x \Delta y \Delta z$



P_i is carried to P_i'

$\mathbf{R}' = OP' = \mathbf{R} + \mathbf{F}(x,y,z)\Delta t$

$\mathbf{R}_i' = OP_i' = \mathbf{R}_i + \mathbf{F}(P_i)\Delta t$

(notice that each corner flows at a different rate)

The edges of the new box are given by vectors **A**, **B**, **C**.

$$\mathbf{A} = \mathbf{R}_1' - \mathbf{R}' = \mathbf{R}_1 - \mathbf{R} + (\mathbf{F}(x+\Delta x,y,z) - \mathbf{F}(x,y,z))\Delta t$$

$$= (\Delta x + [F_1(x+\Delta x,y,z) - F_1(x,y,z)]\Delta t)\ \hat{i} + ([F_2(x+\Delta x,y,z) - F_2(x,y,z)]\Delta t)\ \hat{j} + ([F_3(x+\Delta x,y,z) - F_3(x,y,z)]\Delta t)\ \hat{k}.$$

Similarly, the new edge $\mathbf{B} = \mathbf{R}_2' - \mathbf{R}'$ is $([F_1(x,y+\Delta y,z) - F_1(x,y,z)]\Delta t)\ \hat{i}$

$$+ (\Delta y + [F_2(x,y+\Delta y,z) - F_2(x,y,z)]\Delta t)\hat{j} + ([F_3(x,y+\Delta y,z) - F_3(x,y,z)]\Delta t)\hat{k}.$$

and $\mathbf{C} = ([F_1(x,y,z+\Delta z) - F_1(x,y,z)]\Delta t)\ \hat{i} + ([F_2(x,y,z+\Delta z) - F_2(x,y,z)]\Delta t)\ \hat{j}$

$$+ (\Delta z + [F_3(x,y,z+\Delta z) - F_3(x,y,z)]\Delta t)\ \hat{k}.$$

Now simplify these differences by using the Mean Value Theorem (for typographical convenience I shall use $(F_3)_y$ for $\frac{\partial F_3}{\partial y}$).

$$\mathbf{A} = (\Delta x + (F_1)_x(\xi,y,z)\Delta x\Delta t)\hat{i} + ((F_2)_x(\xi,y,z)\Delta x\Delta t)\hat{j} + ((F_3)_x(\xi,y,z)\Delta x\Delta t)\hat{k}$$

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$$\lim_{\Delta t\to 0}\frac{V-V_0}{(\Delta t)V_0} = \lim_{\Delta t\to 0}\frac{\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z\,\Delta t}\cdot\left(p + q(\Delta t)^2 - 1\right)$$

(left when V_0 factors out)

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$$q = [(F_2)_x(\xi,y,z)][(F_3)_y(x,\upsilon,z)(F_1)_z(x,y,\zeta)\Delta t - (F_1)_y(x,\upsilon,z)(\Delta z + (F_3)_z(x,y,\zeta)\Delta t)]$$
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As Δx, Δy, $\Delta z \to 0$ we have $\xi \to x$, $\upsilon \to y$, $\zeta \to z$. If the partial derivatives of the F_i are continuous, then it follows that

$$\text{div}\ \mathbf{F} = (F_1)_x(x,y,z) + (F_2)_y(x,y,z) + (F_3)_z(x,y,z)$$

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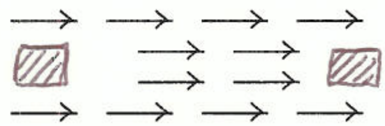
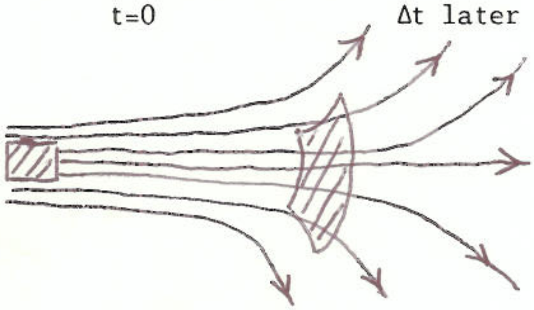


constant **F**

area (volume) unchanged

t=0 $\qquad$ Δt later



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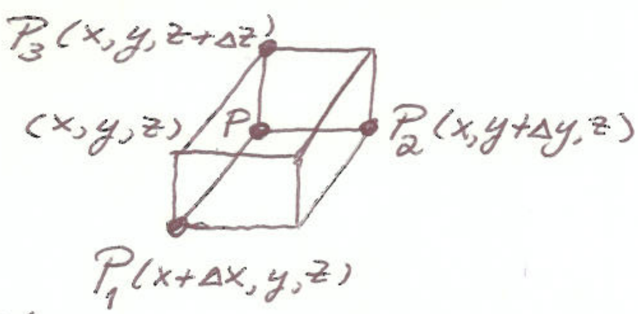

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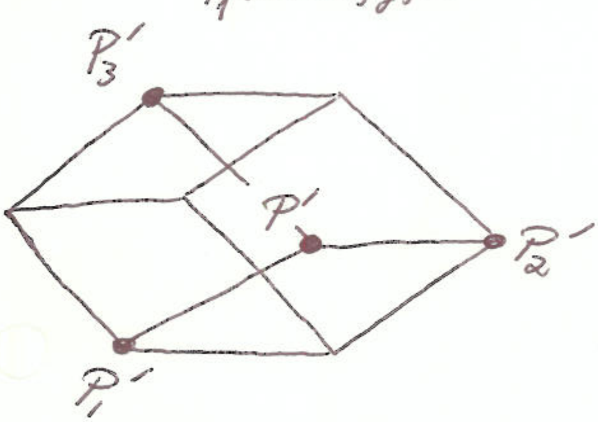

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