MATH 550 **Final Exam, Part A** **Name:** 4^2
Spring, 2006

Note! For full credit you must show sufficient work to support your answer. In particular you must show the major steps of any integration. There are 120 points. *Good luck!*

Change of Variables Theorem. In two variables, $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(T(u,v))\left|\frac{\partial(x,y)}{(u,v)}\right|du\,dv$ and in three variables $\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(T(u,v,w))\left|\frac{\partial(x,y,z)}{(u,v,w)}\right|du\,dv\,dw$, where D and D^* are suitable regions and T is a suitable transformation such that $T(D^*) = D$.

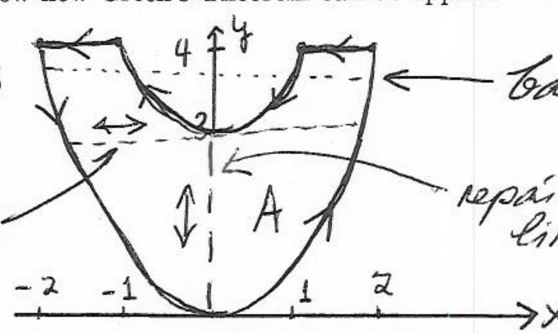
Stokes's Theorem. Let S be a bounded, piecewise regular, oriented surface in $\mathbb{R}^3$ and suppose that $C = \partial S$ consists of finitely many piecewise C^1 simple closed curves, oriented consistently with the orientation of S. Suppose that $\mathbf{F}$ is a C^1 vector field with continuous partial derivatives defined on a domain that includes S. Then $\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \oint_C \mathbf{F}\cdot d\mathbf{s}$.

Divergence or Gauss' Theorem. If W is a bounded symmetric elementary domain in $\mathbb{R}^3$, whose boundary $S = \partial W$ consists of finitely many piecewise regular closed oriented surfaces, oriented so that the normal vectors point out of W, and $\mathbf{F}$ is a C^1 vector field defined on W, then $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_W \nabla\cdot\mathbf{F}\,dV$.

These theorems, and also Green's Theorem in the plane, which they generalize, can be applied to regions that have suitable decompositions.

1. (6 points) Let A be the region in the xy-plane bounded below by $y = x^2$ for $-2 \le x \le 2$, and above by $y = x^2 + 3$ for $-1 \le x \le 1$ and $y = 4$ for $-2 \le x \le -1$, $1 \le x \le 2$. Explain why Green's Theorem cannot be used directly for this domain, and show how Green's Theorem can be applied indirectly.

2 A is not x-simple; must be both x- and y-simple (or just "simple") arrows 2



alternate repair line

bad

repair line 2

2. (10 points) a. In Stokes' Theorem what does it mean to say that the surfaces in question are **regular**?

4 $\vec{T}_u \times \vec{T}_v \neq \vec{0}$ everywhere (or) there is a tangent plane everywhere

(or) there is an $\hat{n}$ everywhere

b. ~~What~~ does it mean to say that the boundary curve(s) is (are) oriented compatibly with the surface(s)?

5 As you walk along the boundary the positive ($\hat{n}$-oriented) side is on your left. OR curve orientation with rt-hand rule gives $\hat{n}$.

3. (12 points) Compute $\iint_R (x+y)^2 e^{x-y}\, dA$ where R is the region bounded by the lines $y = 1 - x$, $y = 4 - x$, $y = x + 1$, $y = x - 1$.

2

$x+y=1 \quad x+y=4 \quad x-y=-1 \quad x-y=1$

$u=1 \quad u=4 \quad v=-1 \quad v=1$

$u = x+y$
$v = x-y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \text{ Jacobian}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2} \quad \text{abs value of } J$$

2

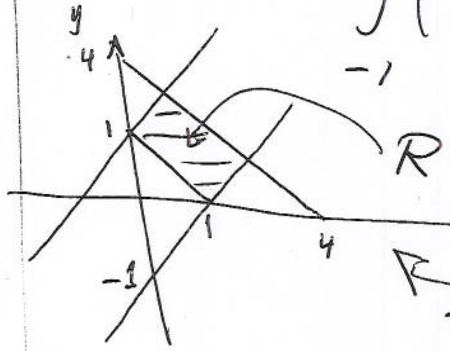
$$\iint_{R^*} u^2 e^v \tfrac{1}{2}\, du\, dv = \int_{-1}^{1}\int_{1}^{4} u^2 e^v \tfrac{1}{2}\, du\, dv$$

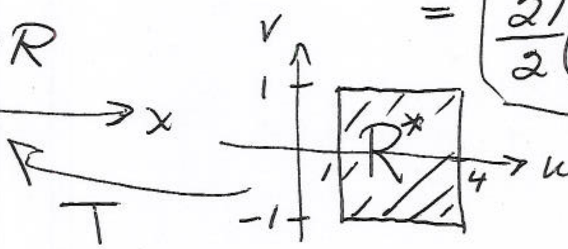
5

$$= \int_{-1}^{1}\left(\frac{u^3}{6}\Big|_1^4\right) e^v\, dv = \frac{63}{6} e^v\Big|_{-1}^{1}$$

$$= \boxed{\frac{21}{2}(e^1 - e^{-1})}$$

3





or solve $\begin{array}{l} u = x+y \\ v = x-y \end{array}$ for x, y:

$u+v = 2x \qquad x = \frac{1}{2}u + \frac{1}{2}v$

$u-v = 2y \qquad y = \frac{1}{2}u - \frac{1}{2}v$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\| \begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} \right\| = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

4. (22 points) Compute $\iiint_W \frac{dV}{(x^2+y^2+z^2)^2}$, where W is the region between the spheres $x^2+y^2+z^2=a^2$, $x^2+y^2+z^2=16$, and $0<a<4$. Use your result to evaluate the same integral for the entire region inside the sphere $x^2+y^2+z^2 \le 16$.

~~put~~ ~~into~~ spherical coords: 3

$$3 \int_a^4 \int_0^{2\pi} \int_0^{\pi} \frac{\rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho}{(\rho^2)^2} \quad 3 \qquad \frac{\rho^2}{\rho^4} = \frac{1}{\rho^2}$$

$$3$$

$$= \int_a^4 \int_0^{2\pi} \frac{1}{\rho^2}\left(-\cos\varphi\Big|_0^{\pi}\right) d\theta \, d\rho$$

$$7 \quad = 2\int_a^4 2\pi \frac{1}{\rho^2} \, d\rho$$

$$= 4\pi \left.\frac{-1}{\rho}\right|_a^4 = 4\pi\left(-\frac{1}{4}+\frac{1}{a}\right) = \frac{4\pi}{a} - \pi$$

$$\cancel{8} \int_0^4 \int_0^{2\pi} \int_0^{\pi} \frac{\rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho}{\rho^4}$$

$$= \lim_{a \to 0} \left(\frac{4\pi}{a} - \pi\right) = \infty$$

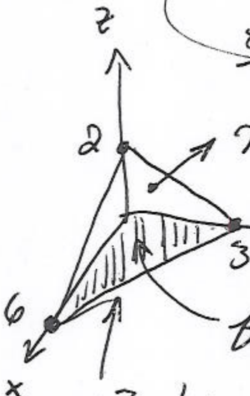
$$3 \qquad 3$$

5. (15 points) Compute the area of the portion of the plane $2x + 4y + 6z = 12$ that lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). Suggestion: orient the surface so that $\hat{\mathbf{n}}$ points away from the origin.

; 2 not penalized

Note $\hat{n} = \frac{1}{\sqrt{4+16+36}}(2,4,6) = \frac{1}{\sqrt{56}}(2,4,6)$

everywhere on S $= \frac{1}{\sqrt{14}}(1,2,3)$ 5



By the area cosine principle

Area $= \iint_S dS = \iint_{base} \frac{dA}{\hat{n}\cdot\hat{k}}$ 5

-3 if wrong

using base as rectangle -4

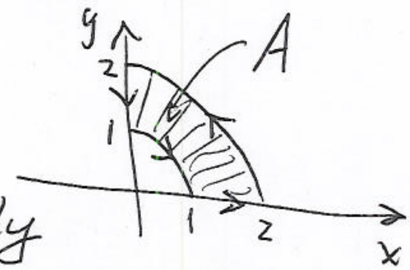
$y = 3 - \frac{1}{2}x$

$x = 6 - 2y$

$= \frac{\sqrt{14}}{3}$ Area(base) $= \frac{\sqrt{14}}{3}\frac{1}{2}(6)(3)$

$= 3\sqrt{14}$ 5

6. (22 points) Compute $\oint_C \mathbf{G} \cdot d\mathbf{s}$ for $\mathbf{G} = (3xy^2, -3x^2y)$, where C is given by the segment from $(1,0)$ to $(2,0)$, the circular arc of $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$, the segment from $(0,2)$ to $(0,1)$, and finally the circular arc of $x^2 + y^2 = 1$ from $(0,1)$ back to $(1,0)$.



Use Green's Theorem

$\oint_C \vec{F}\cdot d\vec{s} = \oint_C 3xy^2\,dx - 3x^2y\,dy$

3 $\quad$ ~~4~~ 6

$= \iint_A (-6xy - 6xy)\,dA = -12\iint_A xy\,dA$

Use polar coordinates $= -12\int_0^{\pi/2}\int_1^2 r\cos\theta\, r\sin\theta\, r\,dr\,d\theta$ 5

2 $\quad$ 2

6 $\quad = -12\int_0^{\pi/2}\left(\frac{r^4}{4}\Big|_1^2\right)\sin\theta\cos\theta\,d\theta = (-12)\left(\frac{15}{4}\right)\left(\frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}}\right) = (-3)(15)\left(\frac{1}{2}\right)$

$= -45/2$

Max 5/8 if they fill in P values before expanding det

7. (12 points) Let ~~$x = u^2$~~ $+ \sin(v)$, $y = \ln(1+w^2)$ ~~$+ uw$, and $z = 1/(u^2+w^2)$~~.
Compute $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$. ~~If (x,y,z)~~ $= T(u,v,w)$ is given by the formulas ~~above, what is~~ the volume change factor in the vicinity of the point P $(u,v,w) = (0,\pi,-1)$? Is this transformation orientation reversing or preserving near the point P?

$$\begin{vmatrix} 2u & \cos v & 0 \\ w & 0 & \dfrac{2w}{1+w^2}+u \\ \dfrac{-2u}{(u^2+w^2)^2} & 0 & \dfrac{-2w}{(u^2+w^2)^2} \end{vmatrix} = (-\cos v)\left(\frac{-2w^2}{(u^2+w^2)^2} + \frac{2u^2}{(u^2+w^2)^2} + \frac{4uw}{(1+w^2)(u^2+w^2)^2}\right)$$

2

At P: $\dfrac{-2}{1} = -2$

8

1

volume change factor $+2$

orientation reversing

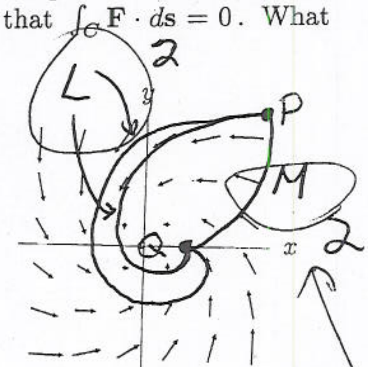
1

each sign error −1
each chain rule error −1
other deriv errors −1 each until you hit 0.

8. (6 points) For the vector field $\mathbf{F}$ shown below, show a path L from P to Q so ~~that $\int_C \mathbf{F}\cdot ds > 0$~~, and a path M from P to Q so that $\int_C \mathbf{F}\cdot ds = 0$. What does this tell you about the vector field $\mathbf{F}$?

Independence of path fails. ($\vec{F}$ is not conservative)

2



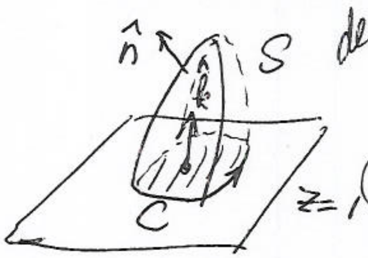
path should be $\perp$ arrow direction everywhere

9. (15 points) Compute $\iint_S \text{curl}(\mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ if $\mathbf{F} = (3z^2, -y^2, x+y)$ and S is the portion of the surface $z = 10 - x^2 - y^2$ that lies above the plane $z = 1$. Assume $\hat{\mathbf{n}}$ points away from the origin.

$4x$

deduct 1 if C is not described anywhere

$x^2 + y^2 = 9,\ z = 1$



$C = \partial S$ oriented CCW

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{s} \quad ⊛$$

Stokes

$$= \iint_D \vec{\nabla} \times \vec{F} \cdot d\vec{S} \qquad 3$$

diagram OK

Let D = interior of C in $z = 1$ plane, oriented by $\hat{k}$. $d\vec{S} = \hat{k}\, dA$ — 2

These might be implicit in other words but awareness "D" somewhere, somehow is required as is $\hat{k}$

$$\vec{\nabla} \times \vec{F} = (1, 6z-1, 4) = (1, 5, 4) \text{ in } D \qquad 2$$

This far is OK

$$\iint_D (1,5,4) \cdot \hat{k}\, dA = 4\, \text{Area}(D) = 4(9\pi) = 36\pi \qquad 3$$

if actual integration is done, it probably should be in polar coords

unless it was already taken off for wrong F

deduct 1 for wrong ans

⊛ Alternate approach from here:

$$\oint_C \vec{F} \cdot d\vec{s} = \int_0^{2\pi} (3, +12\cos\theta - 9\sin^2\theta, 3\cos\theta + 3\sin\theta) \cdot (-3\sin\theta, 3\cos\theta, 0)\, dt \quad \text{all setup}$$

$$= \int_0^{2\pi} (-9\sin\theta + 36\cos^2\theta - 27\sin^2\theta\cos\theta)\, d\theta \qquad 2$$

C: $z = 1$
$x = 3\cos\theta$
$y = 3\sin\theta$
$0 \le \theta \le 2\pi$ — 2

$\vec{C}'(t) = (-3\sin\theta, 3\cos\theta, 0)$
$d\vec{s} = \vec{C}'(t)\, dt$ — 3

Integration

3 $= 0 + \int_0^{2\pi} 36\left(\frac{1+\cos 2\theta}{2}\right)d\theta + 0$

from $\cos\theta\Big|_0^{2\pi}$ from $\frac{\sin^3\theta}{3}\Big|_0^{2\pi}$

1 $= 18(2\pi) = 36\pi$ (from $\sin(2\theta)\Big|_0^{2\pi}$)

Alternate

#5

$\vec{A} = (-6, 0, 2)$ 4

$\vec{B} = (-6, 3, 0)$

$\cancel{5}$ 5

Area $= \frac{1}{2}\|\vec{A}\times\vec{B}\| = \frac{1}{2}\|(-6, -12, -18)\|$

$\cancel{3}$ 4

$= 3\|(1, 2, 3)\|$

$= 3\sqrt{1+4+9} = 3\sqrt{14}$

2

Attempt #6

$$\int_1^2 (0,0)\cdot(1,0)\,dt = 0$$ 2

$x = t$
$y = 0$

$x = 2\cos t$
$y = 2\sin t$
$0 \le t \le 2\pi$

$$\int_0^{\frac{\pi}{2}} \left(3(2\cos t)(4\sin^2 t), -3(4\cos^2 t)(2\sin t)\right)\cdot (-2\sin t, 2\cos t)\,dt$$ 10

$$= \int_0^{\frac{\pi}{2}} \left(-48\cos t\sin^3 t - 48\cos^3 t\sin t\right)dt$$

$$= -48\frac{\sin^4 t}{4} + 48\frac{\cos^4 t}{4}\Big|_0^{\frac{\pi}{2}} = -12-12$$
$$= -24$$

$y = t$
$x = 0$

$$\int_2^1 (0,0)\cdot(0,1)\,dt = 0$$ 2

$$\int_{\frac{\pi}{2}}^{0} (3\cos t \sin^2 t, -3\cos^2 t \sin t)(-\sin t, \cos t)\, dt$$

$$= \int_{\frac{\pi}{2}}^{0} (-3\cos t \sin^3 t - 3\cos^3 t \sin t)\, dt$$

6

$$= \left(-3\frac{\sin^4 t}{4} + 3\frac{\cos^4 t}{4}\right)\Bigg|_{\frac{\pi}{2}}^{0}$$

$$= \frac{3}{4} - \left(-\frac{3}{4}\right) = \frac{6}{4}$$

$$\boxed{\text{Total} = -24 + \frac{6}{4} = -24 + \frac{3}{2} = \frac{-48+3}{2} = \frac{-45}{2}}$$

2

MATH 550 **Final Exam, Part B** **Name:** M^2
Spring, 2006

Note! For full credit you must show sufficient work to support your answer. In particular you must show the major steps of any integration. There are 60 points. Unless otherwise stated you may assume that curves, surfaces, and regions of $\mathbb{R}^3$ are suitable for application of the major theorems.

1. (6 points) Let $\mathbf{F}$ be defined on a domain D in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Give three different, but equivalent, statements that $\mathbf{F}$ be a conservative vector field, without mentioning curl.

each all or nothing

2 1. $\vec{F} = \nabla f$ for some (C^2) function f or $\vec{F}$ is a gradient v.f.

2 2. Independence of path holds ($\int_C \vec{F}\cdot d\vec{s} = f(Q) - f(P)$ for any path from P to Q) or

2 3. Integrals around closed loops are 0 ($\oint_C \vec{F}\cdot d\vec{s} = 0$)

2. (5 points) Suppose that $\mathbf{F}$ is everywhere **tangent** to the closed regular surface $S = \partial W$ for a suitable domain W in $\mathbb{R}^3$. Demonstrate that $\iiint_W \operatorname{div}(\mathbf{F})\,dV = 0$.

Div. Thm. 2

$$\iiint_W \vec{\nabla}\cdot\vec{F}\,dV = \iint_{S=\partial W} \vec{F}\cdot\underbrace{\hat{n}\,dS}_{d\vec{S}} = 0$$

deduct 1 if no label

given!

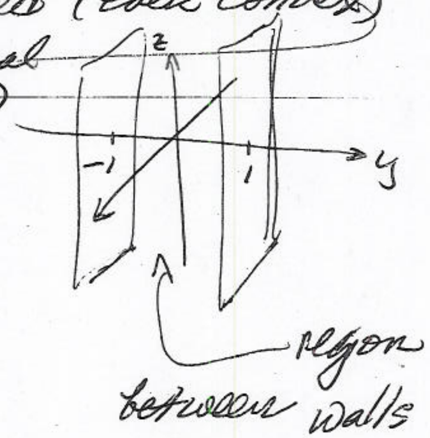
since $\vec{F}\cdot\hat{n} = \vec{F}\cdot d\vec{S} \overset{2}{=} 0$ (F is $\parallel$ surface, $\hat{n}$ is $\perp$ surface) 1

3. (8 points) Computing only derivatives (no integrals), explain why $\mathbf{G} = (3x^2 z, \frac{2y}{1-y^2}, x^3)$ must be conservative in the slab $-1 < y < 1$, $-\infty < x < \infty$, $-\infty < z < \infty$.

4 $\vec{\nabla}\times\vec{G} = (0, 3x^2 - 3x^2, 0) = \vec{0}$

optional

4 The slab is simply connected (even convex) domain. (or has no exceptional points)



Together these imply $\vec{G}$ is conservative.

RB

4. (12 points) Compute the flux $\iint_S \mathbf{F}\cdot d\mathbf{S}$ of $\mathbf{F} = (xy^2, x^2y, z)$ through the closed cylinder S (oriented by outward normals) given by $x^2+y^2=4$, $z=-2$, $z=3$.

2 Use Div. Thm! $\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV$ 3

$= \iiint_W (y^2+x^2+1)\, dV$

W = interior of cylinder

use cylindrical coords:

$= \int_{-2}^{3}\int_0^{2\pi}\int_0^{2} (r^2+1)\, r\, dr\, d\theta\, dz$ 1

$= \int_{-2}^{3}\int_0^{2\pi} \left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Big|_0^2 \, d\theta\, dz$

$= 6(2\pi)(5) = 60\pi$ 3

If missing r don't penalize the integration

If they don't use the D.T., or they don't use polar, mark it for me to grade.

5. (18 points) In each of the following parts determine if $\mathbf{F}$ is conservative or not, and give your reason. The compute $\int_C \mathbf{F}\cdot d\mathbf{s}$ by whatever method seems easiest and most approrpiate.

part (a) 4

must be explicit 1

a. $\mathbf{F} = (z, 0, -x)$, C is the line segment from $(0,-1,3)$ to $(1,1,4)$.

2 $\vec{\nabla}\times\vec{F} = (0, 1-(-1), 0) = (0,2,0) \neq \vec{0}$

1 $\vec{F}$ is NOT conservative.

$\vec{c}(t) = (t, -1+2t, 3+t)$

3 C: $x = 0+t$, $y = -1+2t$, $z = 3+t$, $0 \le t \le 1$

$d\vec{s} = \vec{c}\,'(t)\,dt = (1,2,1)\,dt$

2 $\int_C \vec{F}\cdot d\vec{s} = \int_0^1 (3+t, 0, -t)\cdot(1,2,1)\, dt$

$= \int_0^1 (3+t-t)\, dt = 3$

21

part 6. Q

b. $\mathbf{F} = (2xy, x^2 + 4y^3/(y^4+z^4), 4z^3/(y^4+z^4))$, C is given by $x = \theta$, $y = 4\cos\theta$, $z = 3\sin\theta$, ~~0≤θ≤π/2~~, $0 \le \theta \le \pi/2$.

Compute potential function f: F is conservative 1

$\vec\nabla \times \vec F = \vec 0$ up to 2

4
$$f = \int 2xy\,dx = x^2y + A(y,z)$$
$$f = \int \left(x^2 + \frac{4y^3}{y^4+z^4}\right)dy = x^2y + \ln(y^4+z^4) + B(x,z)$$
$$f = \int \frac{4z^3}{y^4+z^4}\,dz = \ln(y^4+z^4) + C(x,y)$$

1 Compatible: $f(x,y,z) = x^2y + \ln(y^4+z^4)$

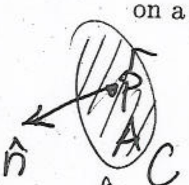
3
P at $\theta = 0$: $(0,4,0)$

Q at $\theta = \frac{\pi}{2}$: $(\frac{\pi}{2}, 0, 3)$

$$\int_C \vec F \cdot d\vec s = f(Q) - f(P) = \ln(81) - \ln(256)$$

16 × 16 = 96, 16

6. (5 points) We found for a vector field $\mathbf{F}$ at a point P, and an axis given by a unit vector $\hat{\mathbf{n}}$, that $(\operatorname{curl}\mathbf{F})(P)\cdot\hat{\mathbf{n}}$ is equal to the "circulation" of $\mathbf{F}$ about P on a surface with normal vector $\hat{\mathbf{n}}$. How is circulation defined? 1

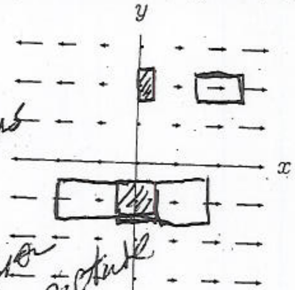


$$(\operatorname{curl}\vec F)(P)\cdot\hat n = \lim_{A\to 0} \frac{1}{A} \oint_{C=\partial A} \vec F \cdot d\vec s$$
1 1

diagram or words → 2 A = area of small piece of plane with normal $\hat n$ at P

7. (6 points) Using "flowboxes", determine if each of the following vector fields has positive, negative, or zero divergence in general.

(a)

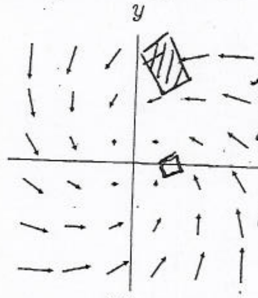


box lengthens

each conclusion 1; each picture and/or words 1

div > 0

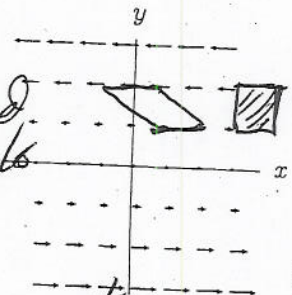
(b)



box spirals around & shrinks

div < 0

(c)



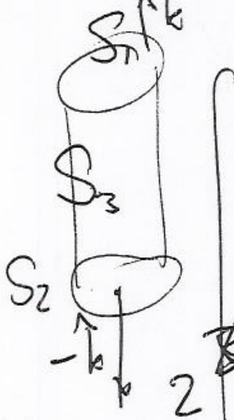
same base, same height, but top moves faster

div = 0

shaded = start, open = finish

26

#4



Top

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_S (*,*,3)\cdot \hat{k}\,dA = 3(4\pi) = 12\pi$$

Bottom

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint (*,*,-2)\cdot(-\hat{k})\,dA = 2(4\pi) = 8\pi$$

3

$$\int_{-2}^{3}\int_{0}^{2\pi} 32\cos^2\theta\sin^2\theta\,d\theta\,dz$$

$$= 5\cdot 32\int_0^{2\pi}\frac{1}{2}\frac{1}{2}(1+\cos 2\theta)(1-\cos 2\theta)\,d\theta$$

$$= 5\cdot 8(2\pi) + 0 - 0 - 5\cdot 8\int_0^{2\pi}\cos^2 2\theta\,d\theta$$

$$= 80\pi - 5\cdot 4(2\pi) = 40\pi$$