MATH 550 Fall, 2002 Final Exam Name:____________________

Note! For full credit you must show sufficient work to support your answer. There are 150 points. *Good luck!*

Change of Variables Theorem. In two variables, $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(T(u,v))|\frac{\partial(x,y)}{(u,v)}|\,du\,dv$ and in three variables $\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f(T(u,v,w))|\frac{\partial(x,y,z)}{(u,v,w)}|\,du\,dv\,dw$, where D and D^* are suitable regions and T is a suitable transformation such that $T(D^*) = D$.

Stokes's Theorem. Let S be a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$ and suppose that $C = \partial S$ consists of finitely many piecewise smooth simple closed curves, oriented consistently with the orientation of S. Suppose that $\mathbf{F}$ is a vector field with continuous partial derivatives defined on a domain that includes S. Then $\iint_S \vec{\nabla} \times \mathbf{F} \cdot d\mathbf{S} = \iint_S \vec{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \oint_C \mathbf{F} \cdot d\mathbf{s}$.

Divergence or Gauss's Theorem. If D is a bounded solid region in $\mathbb{R}^3$, whose boundary $S = \partial D$ consists of finitely many piecewise smooth closed oriented surfaces, oriented so that the normal vectors point out of D, and $\mathbf{F}$ is a vector field with continuous partial derivatives defined on D, then $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \iiint_D \vec{\nabla} \cdot \mathbf{F}\,dV$.

1. (5 points) On the vector field diagram below sketch the flowline $\mathbf{r}(t)$ forwards and backwards in time if if $\mathbf{r}(0) = (1, -3)$, and the flowline forwards in time if $\mathbf{r}(0) = (5, -5)$.

2. (7 points) Compute $d\mathbf{S}$ (vector) for the surface given by $x = e^{-t^2}$, $y = \ln(s^2 + t^2)$, $z = t^2 e^{-s}$.

3. (18 points) Let $\mathbf{A} = 5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, and $\mathbf{C} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. Let $f(x, y, z) = 3x \sin(yz)$, and let R be the point $(1, -1/2, \pi)$.

a. Find an equation for the plane T that contains lines parallel to each of $\mathbf{A}$ and $\mathbf{B}$ and that contains the point R.

b. Find the volume of the parallelepiped with $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ as adjacent edges.

c. The point R lies on the level surface $f(x, y, z) =$ _______ . Give a normal vector to this surface at the point R.

d. Compute the directional derivative $D_{\mathbf{A}} f$ at the point R.

4. (3 points) In the Divergence Theorem and Stokes's Theorem what does it mean to say that the surfaces in question are **smooth**?

5. (13 points) Let $P(1,3,-1)$ be a point; and S be the plane $7x - y - 5z = 6$.

   a. Give parametric equations for a line L that passes through the point P and that is **perpendicular** to the plane S.

   b. Give parametric equations for a line M that passes through the point P and that is **parallel** to the plane S.

   c. Find the distance from the point P to the plane S.

6. (5 points) The highest value of a function $z = h(x, y)$ is at point A; the lowest is at point B. The contour lines correspond to values of z from 0 to 90 in steps of 10. Draw arrows to indicate what $\vec{\nabla} h$ must look like, with attention to both length and direction, at the five heavy dots on the plot.

7. (10 points) Let L be the line given by $x - 2 = \dfrac{y-3}{3} = \dfrac{z+1}{-2}$ and M be the line given by $\dfrac{x-2}{1/2} = \dfrac{y+1}{-1/2} = z - 3$. Do L and M intersect? If so, find the point(s) of intersection; if not explain why not.

8. (7 points) Let $g(x, y, z)$ be a scalar function on $\mathbb{R}^3$, and assume that all partial derivatives are continuous. Show that $\vec{\nabla} \times (\vec{\nabla} g) = \mathbf{0}$.

9. (4 points) Assume $\mathbf{F} = M\hat{\mathbf{i}} + N\hat{\mathbf{j}}$ is defined with continuous derivatives on all of the xy-plane. Why does Green's Theorem not apply to $\int_C M\,dx + N\,dy$? How could you use Green's Theorem indirectly to compute this integral?

10. (5 points) The surface shown is found in many skateboarding parks. Indicate the normal vectors at the remaining dots, so that the surface is consistently oriented, and show how ∂S is oriented to be compatible.

11. (10 points) Let D be the solid region bounded by the surfaces $x^2 + y^2 = 100$, $z = -1$, and $z = 3$, let $S = \partial D$, and let $\hat{\mathbf{n}}$ be the unit outward pointing normal. Compute $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ for $\mathbf{F} = 5x\hat{\mathbf{i}} + 5y\hat{\mathbf{j}}$.

12. (10 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$ for $\mathbf{F} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, C is the circle $(x-1)^2 + (y-1)^2 = 1$, oriented counterclockwise.

13. (14 points) Let D be the region bounded by $y = 0$, $2x + y = 6$, and $y = 8x$.

Compute $\iint_D \sqrt{\frac{2x+y}{8x-y}}\, dA$.

14. (6 points) A vector field $\mathbf{F}$ is shown below, with a paths A, B, C. Determine if each of $\int_A \mathbf{F} \cdot d\mathbf{s}$, $\int_B \mathbf{F} \cdot d\mathbf{s}$, $\int_C \mathbf{F} \cdot d\mathbf{s}$ is negative, positive, or zero.

15. (18 points) a. Without computing a potential function for the vector field $\mathbf{G} = (3x^2z, \dfrac{2y}{1-y^2}, x^3)$, explain why $\mathbf{G}$ must be conservative in the slab $-1 < y < 1$, $-\infty < x < \infty$, $-\infty < z < \infty$.

b. Compute a potential function for $\mathbf{G}$.

c. Suppose C is any path from $P(1,1,0)$ to $Q(-1,-1,1/2)$. Compute $\int_C \mathbf{G} \cdot d\mathbf{s}$.

16. (15 points) Compute $\iint_S (\vec{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ if $\mathbf{F} = (3z^2 + yz)\,\hat{\mathbf{i}} - y^2\,\hat{\mathbf{j}} + x\,\hat{\mathbf{k}}$, S is the portion of the surface $z = 10 - (x^2 + y^2)$ that lies above the plane $z = 1$, and $\hat{\mathbf{n}}$ points away from the origin. Hint: look before you leap into calculations.

The following are extra p[roblems that did not actually appear on the exam.

17. (5 points) A small sphere S_a of radius A is inside a larger sphere S_b of radius b; the centers may not be at the same point however. Suppose $\vec{\nabla} \cdot \mathbf{F} = 0$ throughout the region between the two spheres. How are the integrals $\int_{S_a} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ and $\int_{S_b} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ related, if $\hat{\mathbf{n}}$ represents the **outward normal** to each sphere? Briefly explain.

18. (10 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$ for $\mathbf{F} = (6xz - 2y)\hat{\mathbf{i}} + (z^3 - 2x)\hat{\mathbf{j}} + (3x^2 + 3z^2 y)\hat{\mathbf{k}}$, C the line segment from $(-1, 0, 2)$ to $(4, -1, 2)$.