

MATH 550 Spring, 2008 Exam #1 Name: \_\_\_\_\_

**Note!** For full credit you must show sufficient work to support your answer. In problems requiring evaluation of an integral, the complete set-up is worth the bulk of the points and the actual integration and evaluation relatively little. There are 120 points. *Good luck!*

**Change of Variables Theorem.** In two variables,  $\iint_D f(x, y) dx dy = \iint_{D^*} f(T(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$  and in three variables  $\iiint_D f(x, y, z) dx dy dz = \iiint_{D^*} f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw$ , where  $D$  and  $D^*$  are suitable regions and  $T$  is a suitable transformation such that  $T(D^*) = D$ .

1. (14 points) Let  $x = u^2v$ ,  $y = \ln(v^2 + 4)$ , and  $z = e^{-u^3w}$ . Compute  $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ .  
If  $(x, y, z) = T(u, v, w)$  is given by the formulas above, what is the volume change factor in the vicinity of the point  $(u, v, w) = (-1, 2, 0)$ ?

2. (10 points) Let  $C$  be the curve given by  $\mathbf{c}(t) = (\sin(3t), t, \cos(3t))$  on the interval  $0 \leq t \leq \pi$ . Compute  $\int_C y^2 ds$ .

3. (15 points) Let  $\mathbf{F} = (y^2 \sin(xy^2) + z^3, 2xy \sin(xy^2), 3xz^2 + \sec^2 z)$ . This is a gradient vector field in  $D$ :  $-\infty < x < \infty$ ,  $-\infty < y < \infty$ , and  $-\pi/2 < z < \pi/2$ ; find the most general potential function  $f(x, y, z)$  so that  $\mathbf{F} = \vec{\nabla} f$  on  $D$ .

4. (12 points) Explain why  $\mathbf{F} = (e^{xy}(1 + xy), x^2 e^{xy} + \sin(2yz), \sin(2yz))$  can NOT be a gradient vector field  $\vec{\nabla} f$  for any scalar function  $f(x, y, z)$  on  $\mathbb{R}^3$ .

5. (15 points) Find a linear transformation  $(x, y) = T_A(u, v) = (au + bv, cu + dv)$ , *i.e.*, the matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , that carries the unit square  $D^* = [1, 0] \times [0, 1]$  to the parallelogram  $D$  with corners at  $(0, 0)$ ,  $(2, -2)$ ,  $(-3, 1)$ , and  $(-1, -1)$ . What is the area of  $D$ ? Does  $T_A$  preserve or reverse orientation, and how do you know?

6. (18 points) Compute  $\iiint_H \frac{1}{(x^2 + y^2 + z^2)^{7/4}} dx dy dz$  over the domain  $H$  (for “hollow shell”) between  $x^2 + y^2 + z^2 = 25$  and  $x^2 + y^2 + z^2 = b^2$ , where  $b > 5$ . Use this result to compute the same integral over the  $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \geq 25\}$ .

7. (18 points) Consider the region  $D$  in the first quadrant bounded by the curves  $y = 6 - x^2$  (or  $y + x^2 = 6$ ),  $y = 2 - x^2$ ,  $y = x^2$  (or  $y - x^2 = 0$ ), and the  $y$ -axis ( $x = 0$ ). Compute the integral  $\iint_D x \, dx \, dy$  by making a suitable change of variables. It is not so difficult, but also not so helpful, to solve for  $x$  and  $y$  individually in terms of  $u$  and  $v$ , and it is best to compute  $\frac{\partial(x,y)}{\partial(u,v)}$  indirectly.

8. (18 points) Let  $D$  be the triangle with vertices  $(0,0)$ ,  $(0,1)$ , and  $(1,0)$ . Compute  $\iint_D e^{(y-x)/(y+x)} \, dx \, dy$ . Hint: a second substitution in a single variable, say  $w$ , can help with the integration.