7.4

10. Let $x = u\cos v$, $y = f(u)$, $z = u\sin v$, $a \le u \le b$, $0 \le v \le 2\pi$. The reader should verify that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = -uf'(u)\sin v, \quad \left|\frac{\partial(y,z)}{\partial(u,v)}\right| = uf'(u)\cos v, \quad \text{and} \quad \left|\frac{\partial(x,z)}{\partial(u,v)}\right| = u.$$

Thus, the surface area is

$$A(S) = \iint_D \sqrt{u^2 + u^2(f'(u))^2}\,du\,dv.$$

Since the integrand does not depend on v, the v integral can be performed, and we get the desired formula:

$$A(S) = 2\pi \int_a^b |u|\sqrt{1 + (f'(u))^2}\,du.$$

We are rotating a curve about the y axis, so consider the distance from the y-axis to the curve as the "height," which is $|x|$. Thus, a cross-sectional circumference of the surface at a fixed y_0 is $2\pi|x|$. Next, describe the curve $y = f(x)$, $a \le x \le b$ as a path $\mathbf{c}(t) = (t, f(t))$. Then an infinitesimal arc length can be expressed as $\sqrt{1 + (f'(t))^2}\,dt$ or simply ds. The surface area is obtained by integrating the cross-sectional circumferences along the path $\mathbf{c}$ and the above formula reduce to $A(S) = \int_{\mathbf{c}} 2\pi|x|\,ds$.

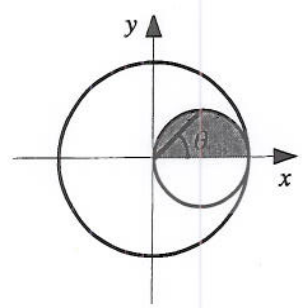
13. We are interested in the area of the surface $z(x,y) = f(x,y) = 1 - x - y$, inside $x^2 + 2y^2 \le 1$. First, compute

$$\sqrt{1 + f_x^2 + f_y^2}\,dx\,dy = \sqrt{3}\,dx\,dy.$$

To compute the surface area, we need to parametrize the disc $z = 0$, $x^2 + 2y^2 \le 1$ using polar coordinates: $x = r\cos\theta$, $y = (r/\sqrt{2})\sin\theta$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$, and the Jacobian is $r/\sqrt{2}$. Our integral then becomes

$$\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{2}} r \cdot \sqrt{3}\,dr\,d\theta = \frac{\pi\sqrt{6}}{2}.$$

17. Completing squares, the equation $x^2 + y^2 = x$ becomes $(x^2 - x + \frac{1}{4}) + y^2 = \frac{1}{4}$, *i.e.*, $(x - \frac{1}{2})^2 + y^2 = \left(\frac{1}{2}\right)^2$. This equation represents a cylinder whose base circle is centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$, as shown. To find the surface area of S_1, we need to consider where the cylinder "sticks out" of the sphere. Consider the positive octant. The surface area is $\iint_D \sqrt{1 + f_x^2 + f_y^2}\,dx\,dy$, where D is half of the base circle (shaded), and $z = f(x,y) = \sqrt{1 - x^2 - y^2}$ is the sphere. Since we will be integrating over a circular region, we



can use polar coordinates: $x^2 + y^2 = x$ is the same as $r^2 = r\cos\theta$ or $r = \cos\theta$. From the figure, one can see that D is described by $0 \le r \le \cos\theta$ and $0 \le \theta \le \pi/2$. Also, we compute $f_x = -x/\sqrt{1 - x^2 - y^2}$ and by symmetry, $f_y = -y/\sqrt{1 - x^2 - y^2}$. So $\sqrt{1 + f_x^2 + f_y^2} = 1/\sqrt{1 - x^2 - y^2}$, which becomes $1/\sqrt{1 - r^2}$ in polar coordinates. Remembering that the Jacobian is r and that S_1 consists of four equal surfaces, we get

$$\begin{aligned} A(S_1) &= 4\int_0^{\pi/2}\int_0^{\cos\theta} \frac{r}{\sqrt{1 - r^2}}\,dr\,d\theta = 4\int_0^{\pi/2} \left(-\sqrt{1 - r^2}\Big|_{r=0}^{\cos\theta} \right) d\theta \\ &= 4\int_0^{\pi/2} (1 - \sin\theta)\,d\theta = 4(\theta + \cos\theta)\Big|_0^{\pi/2} = 2\pi - 4. \end{aligned}$$

By high school geometry, we know that $A(S_2) = 4\pi - (2\pi - 4) = 2\pi + 4$, so $A(S_2)/A(S_1) = (\pi + 2)/(\pi - 2)$.

7.6

6. First, we compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix} = -2z\mathbf{j} + (3y-1)\mathbf{k}.$$

Use spherical coordinates to parametrize S: $x = 4\cos\theta\sin\phi$, $y = 4\sin\theta\sin\phi$, $z = 4\cos\phi$ with $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. Then $\mathbf{T}_\theta \times \mathbf{T}_\phi = 16(-\sin^2\phi\cos\theta\mathbf{i} - \sin^2\phi\sin\theta\mathbf{j} - \sin\phi\cos\phi\mathbf{k})$. Thus

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_S (0, -2z, 3y-1) \cdot (\mathbf{T}_\theta \times \mathbf{T}_\phi)\, d\theta\, d\phi \\
&= -16 \iint_S [(0, -8\cos\phi, 12\sin\theta\sin\phi - 1) \\
&\quad \cdot (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\theta)]\, d\theta\, d\phi \\
&= -16 \int_0^{2\pi}\int_0^{\pi/2} (4\sin\theta\sin^2\phi\cos\phi - \sin\phi\cos\phi)\, d\phi\, d\theta \\
&= -16 \int_0^{2\pi} \left[\left(\frac{4}{3}\sin^3\phi\sin\theta - \frac{1}{2}\sin^2\phi \right)\Big|_{\phi=0}^{\pi/2} \right] d\theta \\
&= -16 \int_0^{2\pi} \left(\frac{4}{3}\sin\theta - \frac{1}{2} \right) d\theta \\
&= -16 \left[\frac{-4}{3}\cos\theta - \frac{\theta}{2} \right]\Big|_0^{2\pi} = 16\pi.
\end{aligned}$$

8. (a) The wall lies under the circle $z = 4R^2$, $x^2 + (y-R)^2 = R^2$ and above the mountain $x^2 + y^2 + z = 4R^2$. From the top view, we may parametrize the circle by

$$x = R\cos\theta, \quad y - R = R\sin\theta, \quad 0 \le \theta \le 2\pi.$$

Then the mountain becomes

$$\begin{aligned}
z &= 4R^2 - (x^2 + y^2) = 4R^2 - [(R\cos\theta)^2 + (R + R\sin\theta)^2] \\
&= 4R^2 - [2R^2 + 2R^2\sin\theta] = 2R^2 - 2R^2\sin\theta.
\end{aligned}$$

To find the surface area of the "cylindrical" wall of the restaurant, we parametrize the wall by

$$\begin{aligned}
&x = R\cos\theta, \quad y = R + R\sin\theta, \quad z = z \\
&0 \le \theta \le 2\pi, \quad 2R^2 - 2R^2\sin\theta \le z \le 4R^2,
\end{aligned}$$

and the surface area becomes

$$\int_0^{2\pi}\int_{2R^2-2R^2\sin\theta}^{4R^2} \|\mathbf{T}_\theta \times \mathbf{T}_z\|\, dz\, d\theta.$$

The reader should verify that $\|\mathbf{T}_\theta \times \mathbf{T}_z\| = R$. Thus, the integral becomes

$$\int_0^{2\pi}\int_{2R^2-2R^2\sin\theta}^{4R^2} R\, dz\, d\theta = \int_0^{2\pi} R(4R^2 - 2R^2 + 2R^2\sin\theta)\, d\theta = 4\pi R^3.$$

(b) Parametrize the restaurant interior by

$$x = r\cos\theta, \quad y = r + r\sin\theta, \quad z = z,$$

where $0 \le r \le R$, $0 \le \theta \le 2\pi$, $4R^2 - (2r^2 + 2r^2\sin\theta) \le z \le 4R^2$. The Jacobian is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ 1+\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r + r\sin\theta.$$