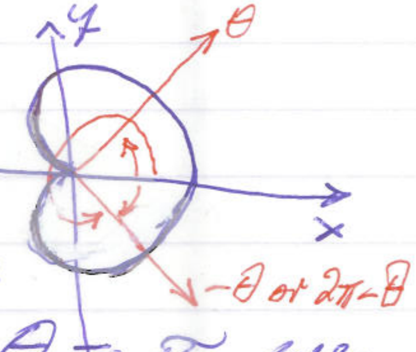
7.1 #6b We want the length of the polar curve $r = 1+\cos\theta$ for $0 \le \theta \le 2\pi$.

There is visibly a problem at the cusp where there is no well-defined tangent vector. In fact at $\theta = \pi$ we have $r = 1+\cos\pi = 0$ and $\left.\frac{dr}{d\theta}\right|_{\theta=\pi} = -\sin\pi = 0$; so



$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = 0 \text{ at } \theta = \pi.$$

Let's see what happens when we try to integrate without forethought.

$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\,d\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\cos\theta}\,d\theta = \sqrt{2}\int_0^{2\pi}\sqrt{1+\cos\theta}\,d\theta.$$

You might recall $\cos^2(\frac{1}{2}\theta) = \frac{1+\cos\theta}{2}$

so $\sqrt{1+\cos\theta} = \sqrt{2}\sqrt{\cos^2(\frac{1}{2}\theta)}$ (which is ≥ 0)

$$= \begin{cases} \sqrt{2}\cos(\frac{1}{2}\theta) & \text{if } \cos(\frac{\theta}{2}) \ge 0 \text{ i.e. } 0 \le \frac{\theta}{2} \le \frac{\pi}{2} \text{ or } 0 \le \theta \le \pi \\ -\sqrt{2}\cos(\frac{1}{2}\theta) & \text{if } \cos(\frac{\theta}{2}) \le 0 \text{ i.e. } \frac{\pi}{2} \le \frac{\theta}{2} \le \pi \text{ or } \pi \le \theta \le 2\pi \end{cases}$$

Note: $\sqrt{a^2} = a$ only if $a \ge 0$. If $a < 0$ then $\sqrt{a^2} = -a = |a|$.
e.g. $a = -3$

Keep $\sqrt{(-3)^2} = \sqrt{9} = 3 = -(-3)$ in mind!

$$\text{So } L = 2\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta + 2\int_{\pi}^{2\pi} -\cos\left(\frac{\theta}{2}\right) d\theta.$$

We can do these integrations, but it is easier to observe that the cardioid is symmetric across the x-axis (if you describe it over the interval $-\pi \le \theta \le \pi$, and use that $\cos(\theta) = \cos(-\theta)$, the conclusion follows readily); so

$$L = 2\left[\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta\right]$$

$$= 2\left[2\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta\right]$$

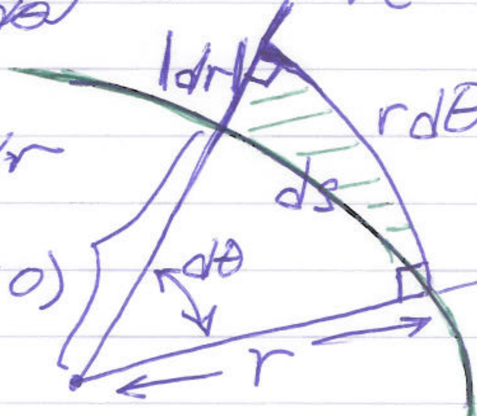
$$= 4\left(2\sin\frac{\theta}{2}\right)\Big|_0^{\pi} = 8.$$

By the way, here's a nifty pictorial illustration of

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \sqrt{(r\,d\theta)^2 + (dr)^2}$$

Look at the small (infinitesimal) right Δ.

$r + dr$ (in this case $dr < 0$)



$r\,d\theta$ = length of circular arc of radius r, angle $d\theta$

Radial segments are perpendicular to circular arcs.