6.1 #6. I'll pick up where we left off in class. We seek $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $A\vec{z} = \hat{\imath}$ and $A\vec{w} = \hat{\jmath}$, i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that by linearity $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = A(\vec{z}+\vec{w}) = A\vec{z} + A\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which takes care of the other corner (and of course $A\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$).

So $\begin{cases} 2a - b = 1 \\ -a + 3b = 0 \end{cases}$ and $\begin{cases} 2c - d = 0 \\ -c + 3d = 1 \end{cases}$. From these $5b = 1$, $b = \frac{1}{5}$, $a = \frac{3}{5}$, $5d = 2$, $d = \frac{2}{5}$, $c = \frac{1}{5}$, $A = \begin{bmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$. On the other hand to find $B(\hat{\imath}) = \vec{z} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ just note that $B\begin{bmatrix} 1 \\ 0 \end{bmatrix} = B(\hat{\imath})$ = first col of B, and $\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \vec{w} = B(\hat{\jmath}) = B\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ = second col of B. Thus $B = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ and

$$A = B^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix} \text{ as before.}$$

6.2 #29(a) $\text{Vol } E = \iiint_E dV = \iiint_{E^*} abc \, dp\,dq\,dr$

$$\frac{\partial(x,y,z)}{\partial(p,q,r)} = \frac{1}{\frac{\partial(p,q,r)}{\partial(x,y,z)}} = abc \text{ for the change of vars:}$$

$$\begin{cases} p = \frac{x}{a} \\ q = \frac{y}{b} \\ r = \frac{z}{c} \end{cases} \quad \frac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{a}\frac{1}{b}\frac{1}{c} = \frac{1}{abc}$$

This is already pos. since $a, b, c > 0$.

But what is E^*? Well $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$ has been transformed into $p^2+q^2+r^2=1$ which is the unit sphere in (p,q,r)-space. And the interior of the ellipsoid $E = \{(x,y,z) \mid \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2 < 1\}$ has been transformed into $\{(p,q,r) \mid p^2+q^2+r^2<1\}$ which is the interior of the unit ball B_1. ("sphere" or "circle" means boundary, "ball" or "disc" means interior + boundary — spheres are hollow; balls are solid). So

$$\underbrace{\iiint_{E^*} abc\, dp\,dq\,dr}_{\text{Think of this as } dV^*} = \underset{\uparrow\ \text{constant!}}{(abc)} \iiint_{B_1} dp\,dq\,dr$$

$$= abc \operatorname{vol}(B_1) = \tfrac{4}{3}\pi abc$$

(b) $$\iiint_E \left(\tfrac{x}{a}\right)^2+\left(\tfrac{y}{b}\right)^2+\left(\tfrac{z}{c}\right)^2 dV = \iiint_{B_1=E^*} (p^2+q^2+r^2)\, abc\, dp\,dq\,dr$$

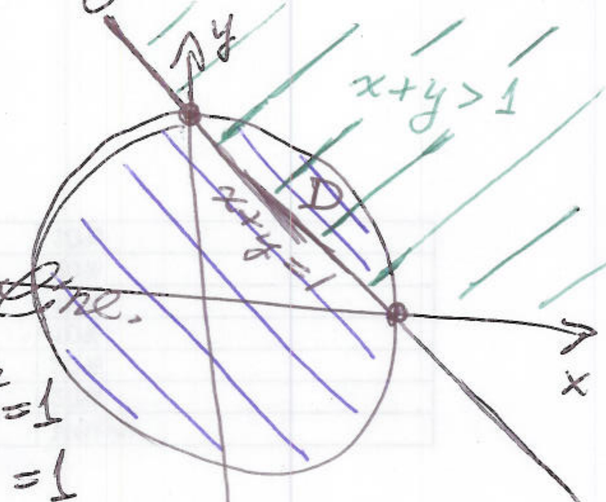
Now a point (p,q,r) in the unit ball has distance $\rho=\sqrt{p^2+q^2+r^2}$ to the origin, so let's convert to spherical coordinates and get $dp\,dq\,dr = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$,

$$abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = \left(\tfrac{2}{5}\right)(2\pi)(abc)$$

$$= 4\pi abc/5$$

(Be careful! $\int_0^\pi \sin\varphi\, d\varphi$
$= -\cos\varphi\Big|_0^\pi = [-(-1)]-[-1] = 2$)

6.2 #22 Once you decide to convert to polar coord's the only tricky spot is how to describe the line $x+y=1$. Well, don't do anything fancy. If $x = r\cos\theta$ and $y = r\sin\theta$, then $1 = x+y = r(\cos\theta+\sin\theta)$ and $r = \dfrac{1}{\cos\theta+\sin\theta}$.



The inequality $x+y>1$ describes points above and to the right of the line.

(Shade like this)

$x^2+y^2=1$

literally $r^2=1$, but simpler is better

$r=1$

The disc is described by $0 \le r \le 1$. (Shaded like this)

region D

Points (x,y) in both shaded regions (D) are in the first quadrant of the disc, so $0 \le \theta \le \frac{\pi}{2}$, are no further from $(0,0)$ than $r=1$, but are at least as far as the line $r \ge \dfrac{1}{\cos\theta+\sin\theta}$.

$$\iint_D \frac{1}{(x^2+y^2)^2}\underbrace{dx\,dy}_{r\,dr\,d\theta \text{ in polar}} = \int_0^{\frac{\pi}{2}}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} \underbrace{\frac{1}{r^4}\,r}_{r^{-3}}\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(-\frac{1}{2}\frac{1}{r^2}\Big|_{1/(\cos\theta+\sin\theta)}^{1}\right)d\theta = \int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = \frac{1}{2}$$