Exam 2 – Solution Key

1. $u'' + 0.25u' + 4u = 2\cos(3t)$.

Let $x_1 = u$, $x_2 = u'$. Then: $x_1' = u' = x_2$

$x_2' = u'' = -0.25u' - 4u + 2\cos(3t)$
$\quad = -0.25x_2 - 4x_1 + 2\cos(3t)$

So $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -0.25x_2 - 4x_1 + 2\cos(3t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & -0.25 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2\cos(3t) \end{pmatrix}$.

2. $\vec{x}' = A\vec{x}$ with $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$.

$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$.

$\lambda_1 = 2$: $(A - 2I)\vec{v} = \vec{0}$: $\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -v_1 + v_2 = 0$, $v_1 = v_2$ $\therefore \vec{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\lambda_2 = -3$: $(A + 3I)\vec{v} = \vec{0}$: $\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 4v_1 + v_2 = 0$, $v_2 = -4v_1$ $\therefore \vec{v}^{(2)} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

a) Fundamental solutions are: $\vec{x}^{(1)} = \vec{v}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

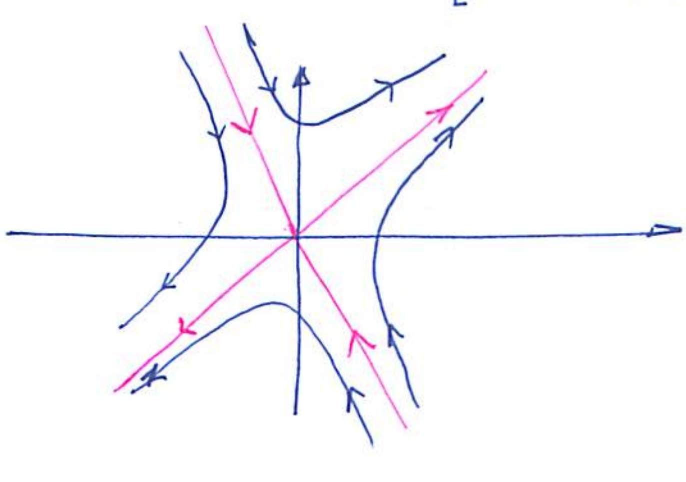
$\vec{x}^{(2)} = \vec{v}^{(2)} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$.

b) General solution: $\vec{x}(t) = c_1\vec{x}^{(1)}(t) + c_2\vec{x}^{(2)}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$

c) To show independence of these solutions:

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \det\begin{bmatrix} e^{2t} & e^{-3t} \\ e^{2t} & -4e^{-3t} \end{bmatrix} = -4e^{-t} - e^{-t} = -5e^{-t} \neq 0.$$

d).



3. We are given $\lambda_1 = 2$, $\vec{v}^{(1)} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\lambda_2 = 4$, $\vec{v}^{(2)} = \begin{bmatrix}1\\1\end{bmatrix}$.

(a) The general solution to $\vec{x}\,' = A\vec{x}$ is

$$\vec{x}(t) = c_1 \begin{bmatrix}1\\3\end{bmatrix} e^{2t} + c_2 \begin{bmatrix}1\\1\end{bmatrix} e^{4t}$$

(b) $\vec{x}(0) = c_1 \begin{bmatrix}1\\3\end{bmatrix} + c_2 \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}c_1+c_2\\3c_1+c_2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$

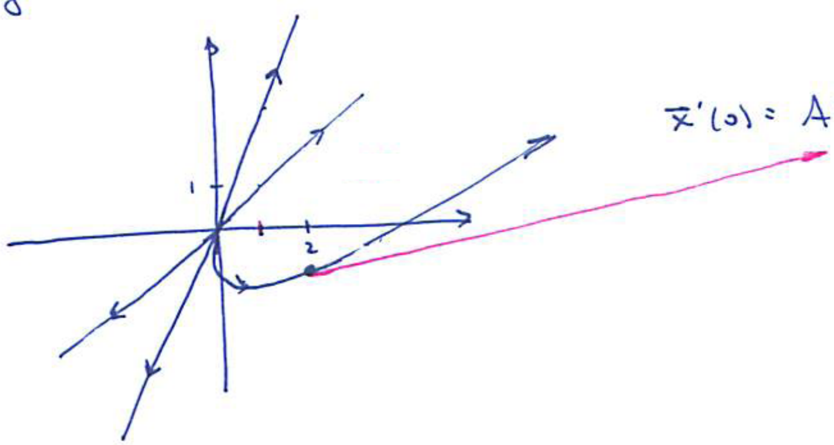
$\left[\begin{array}{cc|c}1 & 1 & 2\\3 & 1 & -1\end{array}\right] \longrightarrow \left[\begin{array}{ccc}1 & 1 & 2\\0 & -2 & -7\end{array}\right]$ so $c_2 = \frac{-7}{-2} = \frac{7}{2}$

$c_1 = 2 - c_2 = 2 - \frac{7}{2} = -\frac{3}{2}$.

(c) As $t \to \infty$, solutions become unbounded by moving along a trajectory that eventually becomes parallel to $\begin{bmatrix}1\\1\end{bmatrix}$, the eigenvector for the dominant eigenvalue.

(d) As $t \to -\infty$, solutions approach the origin along a trajectory that is parallel to $\begin{bmatrix}1\\3\end{bmatrix}$, the "dominant" eigenvalue when you think about t running backwards ($t \to -\infty$).

(e)



$\vec{x}\,'(0) = A\vec{x}(0) = \begin{bmatrix}5 & -1\\3 & 1\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}11\\5\end{bmatrix}$

4. $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \quad \lambda_1 = i, \ \vec{v}^{(1)} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}.$

(a) $A\vec{v}^{(1)} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 4+2i-5 \\ 2+i-2 \end{bmatrix} = \begin{bmatrix} -1+2i \\ i \end{bmatrix}$

$\lambda_1 \vec{v}^{(1)} = i\begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2i-1 \\ i \end{bmatrix} = \begin{bmatrix} -1+2i \\ i \end{bmatrix}$

(b) $\lambda_2 = \bar{\lambda}_1 = -i \qquad \vec{v}^{(2)} = \overline{\vec{v}^{(1)}} = \begin{bmatrix} 2-i \\ 1 \end{bmatrix}.$

(c) $\vec{x} = \vec{v}^{(1)} e^{\lambda_1 t} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix} e^{it} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}(\cos t + i\sin t)$

$= \begin{bmatrix} 2\cos t + i\cos t + 2i\sin t - \sin t \\ \cos t + i\sin t \end{bmatrix} = \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + i\begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

$\therefore \ \vec{x}^{(1)}(t) = \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix}, \quad \vec{x}^{(2)}(t) = \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}$

is a fundamental set of solutions to $\vec{x}' = A\vec{x}$.

(d) It can't be. This new eigenvector must be a multiple of the one we've been using up to now. To check this:

$\begin{bmatrix} 5 \\ 2-i \end{bmatrix} = a\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ requires (2^{nd} components) $a = 2-i$.

Checking the first component: $(2-i)(2+i) = 4 - i^2 = 4+1 = 5$.

so $\begin{bmatrix} 5 \\ 2-i \end{bmatrix}$ is just a multiple of $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ which makes this eigenvector dependent on $\vec{v}^{(1)}$.