SCCC 411B Exam 1 Fall, 1995 Name:____________________

Instructions. Answer questions 1, 2, and 3. Then choose **two** of 4, 5, and 6. Be very clear which of these is left out (X out the problem).

1. (10 points) Give brief definitions of the following terms. Illustrative pictures might also be appropriate.

   a. equilibrium

   b. phase portrait

   c. initial conditions

   d. linearization

   e. limit cycle

2. (20 points) A system is found to have two equilibrium points, and the coefficient matrix is computed for the linearization at each of these points. At one of the points the matrix is $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, and at the other point the matrix is $B = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$.

a. Show how each matrix transforms the unit square (shaded region in the diagram)

b. Matrix A has no real eigenvalues; however, it has complex eigenvalues $\lambda_1 = 1/2 + i\sqrt{3}/2$, $\lambda_2 = 1/2 - i\sqrt{3}/2$. Show what trajectories near the corresponding equilibrium point must look like. Is the equilibrium stable or not?

c. Matrix B has eigenvalues $\mu_1 = -1$, $\mu_2 = -2$, and eigenvectors $\begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Which eigenvalue goes with which eigenvector? What do the trajectories look like near the corresponding equilibrium point? Is the equilibrium point stable or not?

3. (30 points) Consider the following model of plant-herbivore interaction.

$$\frac{dq}{dt} = K - \alpha q N(N - C)$$
$$\frac{dN}{dt} = rN(1 - \frac{\beta N}{q})$$

Here N denotes the insect (scale bug) density, q denotes the nutritive quality of the plant to the bug, and K, r, C, α and β are positive parameters.

a. At low levels of q the plant is not only not nutritious, but it is actually toxic. How is this reflected in the model?

b. What happens to the nutritive quality if bug infestation is low? When bug infestation is moderate? When it is high? What parameters or combination of parameters marks each of these transitions?

c. Is it possible for the bug density to decrease under this model? Under what conditions (in terms of values of N, q, and certain parameters) would this happen?

d. Assume that $C = 0$. Find the steady state of the system.

e. Again assuming $C = 0$, we have sketched the nullclines of the system. The equations are $q = \beta N$ and $q = K/(\alpha N^2)$. Determine which corresponds to $\frac{dq}{dt} = 0$ and which to $\frac{dN}{dt} = 0$. What happens to N if you start a trajectory in the horizontally shaded region? What happens to q if you start in the vertically shaded region? Explain!

f. Is the equilibrium stable or not? Sketch a plausible trajectory on the graph above, if the initial state is one of high q and very small N (a new infestation).

4. (20 points) Recall the standard chemostat model.

$$\frac{dN}{dt} = (K_{\max}C/(K_n + C))N - FN/V$$
$$\frac{dC}{dt} = -\alpha(K_{\max}C/(K_n + C))N - FC/V + FC_0/V$$

Suppose we add one more feature: as the organisms grow they secrete a substance that is toxic to themselves and thus inhibits their own growth.

a. If the concentration of the secretion is denoted $S(t)$, explain why $\frac{dS}{dt} = \ell N - FS/V$ is a plausible description of the rate of change of S. If N is measured in number of organisms per unit volume, what are the units of ℓ?

b. If the toxic effect is proportional to both the concentration S and the bacterial density N (that is, mortality of each organism is proportional to the amount of toxin in the environment), explain why

$$\frac{dN}{dt} = (K_{\max}C/(K_n + C))N - FN/V - bSN$$

is a plausible description of the rate of change of N.

c. Now suppose that beyond a certain point increasing S has no further toxic effect on an organism. Suggest a different mortality term in the above equation, and defend your suggestion.

5. (20 points) In his paper on human population growth, Cohen (1995) suggests that the population $P(t)$ can be described by the following equations:

$$\frac{dP(t)}{dt} = rP(t)(K(t) - P(t))$$
$$\frac{dK(t)}{dt} = c\frac{dP(t)}{dt}$$

a. How does this model differ from the logistic?

b. What is the significance of the second equation?

c. What is the meaning of the parameter c? Can it be negative, and if so, under what biological or sociological conditions?

6. (20 points) The diagram given below suggests a compartment model for the flow of water in the environment.

   a. Designate at least three compartments, and label them with the amount present in each and the amounts (in 10^{18} g/yr) that move between them.

   b. Give an abstract system of equations that describe this system. Use variables, and give a list with the meaning and numerical value of each.

   c. If Maple were to solve this system, does the data suggest what initial conditions to use? Is the system is in equilibrium or not? Explain.