Math 241 **Fall, 2003**

Group Project

Due date: Tuesday, November 18, 2003 in class

You are to work in groups of your own choosing, of size two to four; three would be best. Each group will work on ONE of the following projects; each is worth 50 points in your final grade. Each group will turn in ONE REPORT, and will receive a ONE GRADE. All the group members will receive the same grade, if there has been genuine collaboration. To make sure that everyone does his/her share of the work, each of you will individually turn in a list of the group members with the percentage of effort that each group member contributed. If everyone in a 3-person group put about the same amount of work into the project, then you would rate everyone at 33.3%. I will interview groups that have large discrepancies in the percent of effort reports. Individuals who have contributed substantially less than the others will have their grade reduced. ALL MEMBERS should participate in ALL ASPECTS of the project. Do not assign one to think, one to compute, one to write up, and one to coast! In the past, projects done this way tended to be wrong, incomplete, and badly presented. It is therefore very important to start the work early enough so that a draft of the final report can be circulated to all members for correction and polishing. The quality of the presentation is an important feature of the project.

What I am looking for. First of all, I do not want to see pages and pages of computations. Reduce these to the most vital steps (which of course you should show). Instead what I do want to see is text: explanation of what you are doing, and expecially your conclusions, all written in good solid English. Here are a few pointers on writing math. Always give the cast of characters; that is, no letter should go unidentified; each should be clearly introduced before it is used, which means saying whether it is a variable or a constant, what its units are, what you know, if anything, about its range of values, and what real world quantity it represents. Computations should be done in context. Don't just write "the derivative is $2x^3y$", rather write "for the the x-partial derivative we obtain $\frac{\partial f}{\partial x} = 2x^3y$". Most of your formulas should appear in equations, in which the left hand side announces what is being computed. **Make liberal use of well-labeled diagrams.**

OPTION 1. Parameterization using a slope parameter t worked for the circle, as you may recall. It works for other curves as well. Consider $y^2 = x^2(x + 1)$.

a. Sketch this curve. You may want to use a graphing calculator, or a computer algebra system.

b. Find a parameterization. This time look at lines through $(0, 0)$ that have slope t, and where these lines meet the curve.

c. Use your parameterization to plot the curve (again, this can be done using a graphing calculator).

d. At which value(s) of t is $(0, 0)$ reached? Compute the velocity vector at each one, and use this to write the equation of each tangent line. Explain your answer in terms of the graph of the curve.

OPTION 2. Similar ideas can also be used to obtain parametric descriptions of surfaces. Since surfaces are basically 2-dimensional objects, we expect to need two parameters.

a. If $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = r$ for $0 \le \theta \le 2\pi$ and $r \ge 0$, what familiar surface is being described?

b. If $x = 3\cos(\theta)\sin(\phi)$, $y = 3\sin(\theta)\sin(\phi)$, and $z = 3\cos(\phi)$ for $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$, what familiar surface is being described?

c. Now let's try to cook up a parameterization ourselves. Take the unit sphere $x^2 + y^2 + z^2 = 1$ and label the point $(0, 0, 1)$ as N. Join each pont $(r, s, 0)$ in the plane $z = 0$ by a line to N. This line, possibly extended, meets the sphere at some point (x, y, z) other than N. Solve for these coordinates in terms of the parameters r and s. Do we obtain every point of the sphere in this way? The method described here is used in map-making (somehow one has to get the Earth's surface on to a piece of paper), and goes by the name stereographic projection.

OPTION 3. Do problems 27 and 28 on page 706 of the text; include the computation of the volume of the sphere. To completely answer this question you will have to solve problem 36 on page 279, and problem 40 on page 287. Also give the answer in more intuitive language: if the center of the earth is cored out so that the height of the remaining ring is $b = 1$ foot, and you do the same thing to a (solid) basket / beach / exercise ball, why do a and c not matter in the end? Note: many of the volume integrals involved here can be set up as single integrals. However, where possible, I expect that you will also set them up as double integrals.

OPTION 4. Captain Ralph is in trouble on the sunny side of Mercury. When at location (x, y, z), the temperature (in degrees Celsius) of the ship's titanium hull is $H(x, y, z) = 2500e^{-0.02x^2-0.01y^2-0.03z^2}$, where distances are measured in millions of km (10^6 km) from the origin at the sun, the x-axis points towards Mercury, the y-axis points in the direction of the motion of Mercury (assuming a nearly circular orbit), and the z-axis is then out of the plane of the sun and the planets given by the right hand rule (technically the planets don't all lie in the same plane with one another and the sun, but except for Pluto, it's not a bad approximation). We assume the ship can move much faster than Mercury does (600 km/sec as opposed to approximately 48 km/sec) so the coordinate system may assumed to be standing still. The ship is located at $(5, 0, 1/2)$.

In this problem it is very important to keep track of units, and to do precise calculations. Keep at least 3, preferably 4, *significant digits* in your caclulations; you will find it helpful to write a number such as 0.000005406 as 5.406×10^{-6}, or in calculator notation 5.406E-6, because rounding it off to 0 will lead to totally incorrect answers.

a. Mercury is approximately 60 million km from the sun. Where is the ship relative to the sun and Mercury? How hot is the hull? (By the way, the melting point of titanium is 1668°C.)

b. In which direction should Ralph direct the ship so as to decrease the temperature most rapidly? Rewrite your answer as a unit vector $\hat{\mathbf{u}}$.

c. If the ship travels at 600 km/sec (6×10^{-4} million km/sec) in this direction, what is the velocity vector $\mathbf{v}$? Be sure to give the units! How fast will the temperature be decreasing in °C/sec? Suggestion: use the chain rule.

d. Suppose that the hull will crack if it cools at a rate more than 0.16 °C/sec. Assuming the ship maintains its speed of 600 km/sec, in which possible directions should Captain Ralph steer the ship so that the hull cools off as fast as is safely possible?

OPTION 5. You are running a small printing business and decide to expand because you have more orders than you can handle. Should you start a night shift and increase your total labor force L (which might be measured in terms of number of workers, or in terms of payroll and fringe benefit dollars)? Or should you buy more expensive, but faster, computers and printers, which will enable the current personnel to keep up with the increased production; we could call your total capital investment K, which obviously would be measured in dollars. Or should you do some combination of the two? Obviously, the way such a decision is made in practice depends on lots of other thing– such as whether you could get as many trained workers as you would need, whether such improved equipment is available at a cost you can afford, or if you need to get a loan from the bank, what interest rate they will charge, and so on. But for simplicity let's say that monthly production P depends only on the independent variables L and K.

How would you expect such a production function to behave? In general, having more workers and better or more equipment will enable you to increase P. But improving on the equipment side will not increase production beyond a certain point; if equipment is idle or under-used because you don't have enough or skilled enough workers, having more of it won't help. Similarly increasing the number of workers or their skill level (both of which are included in L if that is measured in payroll dollars) won't increase production past a certain point; if the equipment isn't there for them to use, hiring more or better workers won't help.

a. Which of the following contour diagrams best models the situation described above, and why?

b. In 1928 Cobb and Douglas used government estimates for total yearly production P, K and L from 1899 to 1922 and found that $P = 1.01L^{0.75}K^{0.25}$. Is this formula consistent with your answer to part a. above? Approximate the error in measurement of P if the error in measurement of L is as much as 4% and the error in measurement of K is at most 2%.

c. If the measurements are revised so that L is now exactly 4% more than its previous value, and K is now exactly 2% less than its previous value, what is the exact value of P in terms of its previous value?

d. If we try to extrapolate to the future and imagine that L doubles and K also doubles, what should happen to P (note: these are not small changes in L and K, so you will have to compute the exact new value of P).

e. Suppose that besides $L \geq 0$ and $K \geq 0$ there is an additional constraint that $L + K \leq 1,000,000$. At what values of L and K is production maximized, and what is this maximum value? If the constraint is instead $L + K \leq 3,000,000$, how does your answer change?

f. Now suppose that Cobb and Douglas were around at a different point in history, when $P = 1.01L^{0.5}K^{0.2}$. How would your answer to item d. change?

g. If Cobb and Douglas were around at a yet another point in history, when $P = 1.01L^{0.8}K^{0.4}$, how would your answer to item d. change? Suggest a pattern to these changes, based on the exponents that are used.