MATH 241 Spring, 2010 Exam #2 Name: M^2

For full credit you must show sufficient work that the method of obtaining your answer is clear. There are 100 points.

1. (15 points) Suppose $z = f(x, y)$ and you have the following information: $f(1, -2) = 5\,,\ f_x(1, -2) = -3\,,\ f_y(1, -2) = 1\,.$

   a. Estimate $f(0.9, -2.5)$ as accurately as possible using this information.

$$f(0.9, -2.5) \approx f(1,-2) + df$$

$\Delta x = -0.1$ $\quad = 5 + f_x(-1,2)(-0.1) + f_y(1,2)(-0.5)$

$\Delta y = -0.5$ $\quad = 5 - 3(-0.1) + 1(-0.5)$

$$= \boxed{4.8}$$

   b. Give an equation for the tangent plane to $z = f(x, y)$ at the point $(1, -2, 5)$.

$$z = L(x,y) = 5 - 3(x-1) + 1(y+2)$$

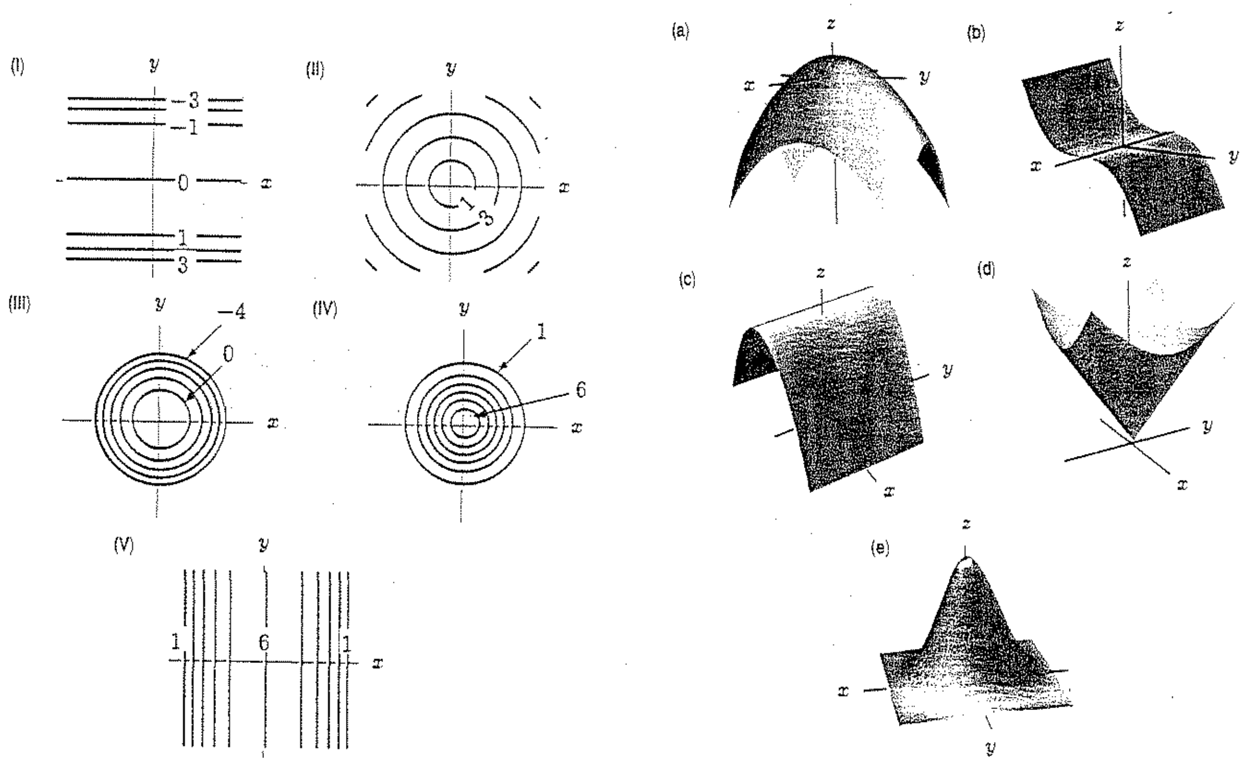
$$3x - y + z = 5 + 3 + 2 = 10$$

or use $\vec{N} = < f_x(1,-2),\ f_y(1,-2),\ -1 >$

$$-3(x-1) + 1(y+2) - 1(z-5) = 0$$

$$-3x + y - z = -10$$

2. (13 points) a. ~~Match each contour diagram with its surface. Assume that~~ adjacent contours in each diagram represent equal changes in the value of z.
I- b , II- d , III- a , IV- e , V- c .



b. How can you be certain that (I) is NOT the contour diagram of a plane? (Hint: if $z = ax + by + c$, with a, b, and c constants, what are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?)

$\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$ are constant.

In (I) $\frac{\partial z}{\partial y}$ is not constant, the spacing of the contours varies.

3. (35 points) Suppose $w = h(x, y, z) = z^2 + \ln(1 + xy)$, $x = s^3 t$, $y = s^2 \sin(st)$, and $z = 4 - t^2$.

a. Compute $\operatorname{grad} h = \vec{\nabla} h$ in terms of x, y and z.

$$\vec{\nabla} h = \langle h_x, h_y, h_z \rangle = \left\langle \frac{y}{1+xy}, \frac{x}{1+xy}, 2z \right\rangle$$

vector!

b. Compute the directional derivative $D_{\hat{\mathbf{u}}} h(P)$ at the point $P(1, 3, -1/2)$ in the direction of $\mathbf{a} = \langle 2, 2, -1 \rangle$.

$$\hat{u} = \frac{1}{\|\vec{a}\|} \vec{a} = \frac{1}{\sqrt{9}} \langle 2, 2, -1 \rangle = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$$

$$D_{\vec{u}} h(P) = \vec{\nabla} h(P) \cdot \hat{u}$$

$$= \left\langle \frac{3}{4}, \frac{1}{4}, -1 \right\rangle \cdot \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle = \frac{1}{2} + \frac{1}{6} + \frac{1}{3} \boxed{= 1}$$

c. Give the maximum value for any directional derivative of h at P.

$$\|\vec{\nabla} h(P)\| = \sqrt{\frac{9}{16} + \frac{1}{16} + 1} = \boxed{\frac{\sqrt{26}}{4}}$$

d. Which level surface for w (or h) is the point $Q(1, 1, -1)$ on? Give an equation for the tangent plane to this level surface at Q.

$w = 1 + \ln(2)$

Use $\vec{N} = \vec{\nabla} h(P) = \left\langle \frac{3}{4}, \frac{1}{4}, -1 \right\rangle$

$$\frac{3}{4}(x-1) + \frac{1}{4}(y-1) - 1(z+1) = 0$$

$$\boxed{\frac{3}{4}x + \frac{1}{4}y - z = 2}$$

e. Compute $\frac{\partial w}{\partial s}$ in terms of x, y, z, s, and t, using the multivariable chain rule.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\underbrace{\frac{\partial y}{\partial s}}_{\text{product rule for } \frac{\partial}{\partial s}} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= \frac{y}{1+xy}(3s^2 t) + \frac{x}{1+xy}\left(s^2 \cos(st)\, t + 2s \sin(st)\right) + 2z(0)$$

4. (25 points) Let $f(x,y) = xy^2 - 6x^2 - 3y^2$.

a. Find all relative (local) maximum, minimum and saddle points (and say which is which) that lie **inside** the circle $x^2 + y^2 = 64$. You may use the $D = f_{xx}f_{yy} - f_{xy}^2$.

$f_x = y^2 - 12x = 0$

$f_y = 2xy - 6y = 0$ → $2y(x-3) = 0$

$y = 0$, $x = 0$; $x = 3$, $y^2 = 36$, $y = \pm 6$

$f_{xx} = -12 < 0$

$f_{yy} = 2x - 6$, $f_{xy} = 2y$

Note: $3^2 + (\pm 6)^2 = 9 + 36 = 45 < 64$

$D = (-12)(2x-6) - 4y^2$

At $(0,0)$ $D = (-12)(-6) - 0 > 0$
$f_{xx} < 0$ relative max

At $(3,6)$ $D = 0 - 4(36) < 0$
saddle point

At $(3,-6)$, $D = (-12)(0) - 4(36) < 0$
saddle points

All three pts are <u>inside</u> the circle.

b. ~~SET UP the Lagrange multiplier equations (there are three of them) to find the maximum and minimum values of $f(x,y)$ along the boundary~~ $x^2 + y^2 = 64$. DO NOT solve.

$g(x,y) = x^2 + y^2 - 64 = 0$

$\vec{\nabla} g = \langle 2x, 2y \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

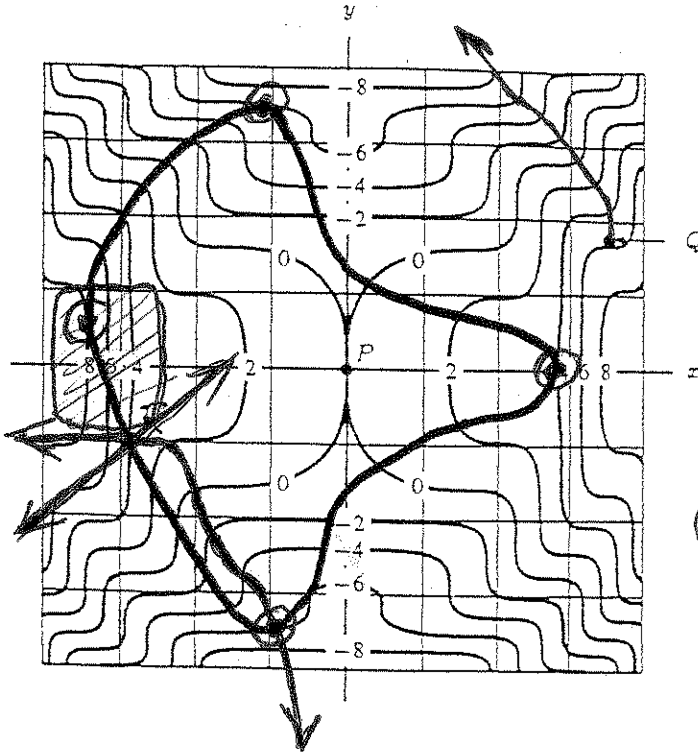
$$\begin{cases} y^2 - 12x = 2\lambda x \\ 2xy - 6y = 2\lambda y \\ x^2 + y^2 = 64 \end{cases}$$

Lagrange multiplier equations tell about the gradients <u>and</u> the constraint.

5. (12 points) The diagram shows level curves for the height of a surface $z = f(x, y)$ above the xy-plane.

   a. If a small, smooth bead is gently placed on the surface over each point P, Q, and R, sketch the path it will roll, until it exits the scene (or state that it does not roll anywhere).

P is a saddle point.
$\vec{\nabla} f(P) = \vec{0}$
so bead does not move. (Near P, f is constant = 0 along y-axis.)
From Q, R bead follows $-\vec{\nabla} f$ downhill, ⊥ contour lines.



⊙ S

   b. If the heavy curve represents a constraint $g(x, y) = 0$, explain why neither the absolute maximum nor minimum value of $z = f(x, y)$ on or inside the curve is found at R. Mark the point(s) S where these values might be found according to the method of Lagrange multipliers.

Or state that contours with $z > 4$ and $z < 4$ cross the heavy curve.

At R, $\vec{\nabla} f$ is ⊥ contour line and points left. $\vec{\nabla} g$ is ⊥ heavy curve and points SW or NE. So they are not lined up by $\vec{\nabla} f = \lambda \vec{\nabla} g$. They do line up at 4 points marked S ⊙

   c. Box in a small portion of the diagram where it appears that the magnitude of the gradient vector of f is the largest. In which direction does it point?

The ascent seems to be steepest in the shaded boxed region. At points in this region $\vec{\nabla} f$ points directly to the left (west).